



Research Article

Solution and stability analysis of homogeneous piecewise fractional systems

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ABSTRACT

This paper studies linear piecewise fractional delay systems motivated by models with switching dynamics and memory effects. An explicit interval-wise solution is constructed via a generating-function approach, and its continuity at all junction points is established. Based on this representation, a computable finite-time stability condition is derived in the form of an explicit inequality involving the system parameters, delay, and fractional order. Several scalar and matrix-valued examples illustrate the theoretical results and confirm the effectiveness of the proposed stability criterion. The novelty of this work lies in providing, to the best of our knowledge, the first explicit interval-wise solution framework for piecewise fractional delay systems together with a directly verifiable finite-time stability condition, thereby extending existing results that rely on implicit formulations or restrictive structural assumptions.

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INTRODUCTION

Finite-time stability constitutes a critical characteristic of dynamical systems, describing the ability of a system to reach a desired state or remain within a prescribed bound over a finite time interval. This notion has been widely investigated in nonlinear systems, fractional-order systems [1, 2], and systems with delays [3-10]. Its importance stems from applications such as synchronization in complex networks, stabilization of robotic systems [11], and regulation of time-sensitive processes, where transient performance is often more relevant than asymptotic behavior.

The increasing use of fractional differential equations (FDEs) in engineering, economics, physics, and related fields has further motivated the study of finite-time stability in fractional-order settings [12-17]. Fractional models are particularly effective in capturing memory and hereditary effects that cannot be adequately described by classical integer-order dynamics. A wide range of analytical and numerical methods has been developed for FDEs, including transform-based techniques and numerical schemes [18-23]. More recently, explicit analytical solution techniques and stability analyses for fractional and conformable systems have attracted increasing attention in the applied mathematics and engineering literature. For example,

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explicit solution formulas for linear conformable systems with variable coefficients were proposed in [24], while fractional epidemic models illustrating real-world applications were studied in [25]. In addition, existence, uniqueness, and generalized stability notions for fractional operators were addressed in [26].

For classical linear systems of the form $v'(\zeta) = Bv(\zeta)$, the solution is given by the matrix exponential $v(\zeta) = e^{B\zeta}v(0)$. However, the introduction of a delay term $v(\zeta - \tau)$ significantly complicates the analysis. Khusainov and Shuklin [27] investigated linear autonomous delay systems using delay exponential functions under restrictive commutativity assumptions, while Li and Wang [28] studied the corresponding fractional analogues. In a related direction, Huiping and Wang [29] analyzed piecewise linear state-delay systems with more complex structures, focusing primarily on controllability properties.

From an application viewpoint, piecewise fractional delay systems naturally arise in models with switching dynamics and memory effects. Typical examples include networked control systems with mode-dependent communication delays, fractional-order viscoelastic models subject to regime changes, and hybrid engineering or biological processes in which system parameters vary in a piecewise manner. For such systems, understanding transient behavior and guaranteeing performance within a finite time horizon are of practical importance.

In parallel, several studies have addressed finite-time stability of fractional delay differential equations. Li and Wang [30] derived quantitative finite-time stability criteria, while delayed Mittag-Leffler type matrix functions were introduced in [28] to obtain more constructive solution representations and stability bounds. Despite these advances, the existing literature largely relies on implicit solution formulations, numerical approaches, or restrictive structural assumptions. In particular, explicit interval-wise analytical solutions for piecewise fractional delay systems, together with rigorous continuity analysis at switching points and directly verifiable finite-time stability conditions, remain relatively scarce.

Classical approaches for fractional delay differential equations are typically based on the fractional method of steps, where the solution is constructed recursively on successive delay intervals by solving a sequence of fractional initial-value problems; see, for instance [28, 30]. Within this framework, explicit representations are usually expressed in terms of delayed Mittag-Leffler matrix functions and are mainly developed for systems with constant coefficients or under restrictive structural assumptions. Similarly, linear delay systems with special matrix structures are treated via delay exponential functions under permutation or commutativity conditions [27]. More recently, piecewise state-delay systems have been investigated primarily from the controllability viewpoint, without providing explicit analytical solution formulas or finite-time stability guarantees [29].

In contrast to these studies, the present work introduces a generating-function-based construction that encapsulates the stepwise procedure into a single explicit interval-wise representation. This approach yields closed-form expressions on each delay subinterval with interval-dependent coefficient matrices, enables a transparent continuity analysis at all junction points, and leads to a directly computable finite-time stability condition expressed as an explicit inequality involving the system matrices, delay length, and fractional order. Hence, beyond the existing results in [27–30], the proposed framework provides an explicit, unified, and verifiable solution-stability methodology for piecewise fractional delay systems.

Motivated by these gaps, the present work develops a generating-function-based framework for linear piecewise fractional delay systems. The proposed approach yields explicit closed-form solutions on successive delay intervals, proves continuity at all junction points, and leads to a computable finite-time stability criterion expressed as an explicit inequality involving the system matrices, delay parameter, and fractional order. In contrast to existing methods, the results do not rely on restrictive commutativity assumptions and provide transparent analytical tools suitable for both theoretical analysis and engineering applications.

Inspired by the studies [28] and [29], we dedicated this paper to investigate the following homogeneous fractional piecewise delayed systems,

$$\begin{cases} {}^C D_{(m-1)\tau+}^\alpha v(\zeta) = \Omega_m v(\zeta - \tau), & \zeta \in [(m-1)\tau, m\tau), \\ \vdots \\ {}^C D_{\tau+}^\alpha v(\zeta) = \Omega_2 v(\zeta - \tau), & \zeta \in [\tau, 2\tau), \\ {}^C D_{0+}^\alpha v(\zeta) = \Omega_1 v(\zeta - \tau), & \zeta \in [0, \tau), \\ v(\zeta) = \varphi(\zeta), & \zeta \in [-\tau, 0), \end{cases} \quad (1)$$

where $\varphi : [-\tau, 0] \rightarrow \mathbb{R}^n$ is an initial function, ${}^C D_{i\tau+}^\alpha$, $i = 0, 1, \dots, m-1$ stand for the well-known Caputo fractional derivatives, $\tau > 0$ is a delay parameter, and $\Omega_i \in \mathbb{R}^{n \times n}$, $i = 1, \dots, m$.

This paper is organized as follows. Section 1 emphasises the importance of finite-time stability and fractional differential equations and their usages addition to introducing our system by touching on the existing studies in the literature, while Section 2 provides the necessary symbols and definitions of fractional derivatives and integrals and some their properties. In Section 3, a solution to the system by proposing the generating function is constructed and its continuity is proved, and in Section 4 a finite-time stability analysis is carried out. Section 5 presents illustrative examples, and Section 6 concludes the paper with final remarks.

QUICK PRELIMINARIES

In this section, we briefly recall the basic definitions, notations, and results that will be used throughout the paper.

Let $\mathbb{R}^+$ and $\mathbb{R}$ denote the sets of all positive real numbers and all real numbers, respectively. For $-\infty < a < b < \infty$, we denote by $C([a, b], \mathbb{R})$ the Banach space of

continuous real-valued functions on $[a, b]$ endowed with the maximum norm $\|\cdot\|_\infty$, is defined by,

$$\|\psi\|_\infty = \max_{t \in [a, b]} |\psi(t)|,$$

where $|\cdot|$ denotes an arbitrary norm on $\mathbb{R}$. The Riemann-Liouville integral $I_{a+}^\beta v(\zeta)$ of fractional order $\beta > 0$ is given by

$$I_{a+}^\beta v(\zeta) = \frac{1}{\Gamma(\beta)} \int_a^\zeta (\zeta - s)^{\beta-1} v(s) ds, \quad \zeta > a > -\infty,$$

the Caputo derivative ${}^C D_{a+}^\beta$ of fractional order $0 < \beta < 1$ is given by

$${}^C D_{a+}^\beta v(\zeta) = \frac{1}{\Gamma(1-\beta)} \int_a^\zeta (\zeta - s)^{-\beta} v'(s) ds, \quad \zeta > a > -\infty,$$

where $\Gamma(\cdot)$ is the well-known Gamma function. Their compositions satisfy

$$I_{a+}^\beta {}^C D_{a+}^\beta v(\zeta) = v(\zeta) - v(a), \quad I_{a+}^\beta {}^C D_{a+}^\beta v(\zeta) = v(\zeta).$$

Moreover, ${}^C D_{a+}^\beta c = 0$ for any constant $c \in \mathbb{R}$. For more details, we recommend the readers to look at the references [31, 32]. The methodological structure of this paper is shown by Figure 1.

Glossary of Symbols

- $\alpha \in (0, 1)$: Order of the fractional operators.
- $\tau > 0$: Constant time delay.
- $m \in \mathbb{N}$: Number of subintervals.
- $v(\zeta)$: State vector (or scalar state) of the system.
- $\varphi(\zeta)$: Initial history function defined on $[-\tau, 0]$.
- Ω_k : Constant system matrix (or scalar coefficient) acting on the k -th subinterval
- I_{a+}^α : Riemann-Liouville fractional integral of order α .

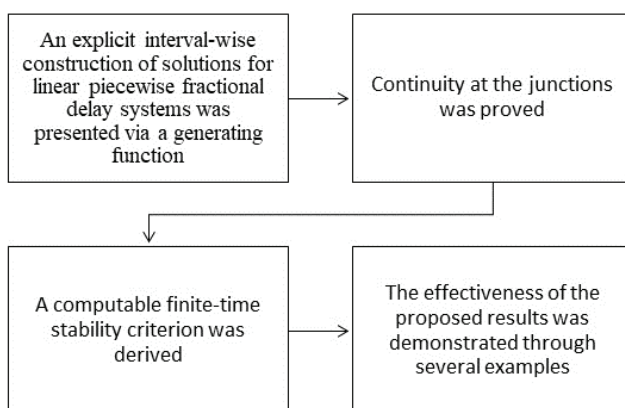


Figure 1. The owchart of methodological structure in this paper.

- ${}^C D_{a+}^\alpha$: Caputo fractional derivative of order α .
- $D_m(\zeta, \varphi)$: Generating function used to construct the solution recursively on each subinterval.
- $\|\cdot\|$: Vector or matrix norm.
- M : Uniform bound such that $\|\Omega_k\| \leq M$ for all k .

A Solution to Linear System (1)

In this section, we construct a solution to the linear system (1) by introducing an appropriate generating function. We then verify that the proposed solution satisfies the system, provide alternative proofs including an induction-based argument, and establish the continuity of the solution.

Let's understand the root of the linear piecewise fractional delayed systems more and discover the representations of their explicit solutions. In the interval $\zeta \in [0, \tau]$, the equation acquired from system (1) is given by:

$${}^C D_{0+}^\alpha v(\zeta) = \Omega_1 v(\zeta - \tau), \quad \zeta \in [0, \tau]. \quad (2)$$

By applying the definition of the Riemann-Liouville fractional integral I_{0+}^α , this equation can be rewritten in terms of the Riemann-Liouville fractional integral as:

$$v(\zeta) - v(0) = \Omega_1 I_{0+}^\alpha v(\zeta - \tau) = \Omega_1 \frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta - x)^{\alpha-1} v(x - \tau) dx. \quad (3)$$

Employing the substitution $\lambda_1 = x - \tau$, one has

$$v(\zeta) = v(0) + \frac{\Omega_1}{\Gamma(\alpha)} \int_{-\tau}^{\zeta-\tau} \frac{(\zeta - \tau - \lambda_1)^{\alpha-1}}{\Gamma(\alpha)} v(\lambda_1) d\lambda_1.$$

Since $\lambda_1 \in [-\tau, \zeta - \tau]$, and $\zeta \in [0, \tau]$, and we recall that the initial condition satisfies $v(\zeta) = \varphi(\zeta)$ for $\zeta \in [-\tau, 0]$. Therefore, the equation can be restructured as follows:

$$v(\zeta) = \varphi(0) + \int_{-\tau}^{\zeta-\tau} \Omega_1 \frac{(\zeta - \tau - \lambda_1)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\lambda_1) d\lambda_1.$$

Now, let us consider a general interval such that $\zeta \in [m\tau, (m+1)\tau]$. Then, the equation obtained from (1) can be expressed as follows:

$${}^C D_{m\tau+}^\alpha v(\zeta) = \Omega_{m+1} v(\zeta - \tau)$$

Now, if the Riemann-Liouville fractional integral is applied, then

$$v(\zeta) = v(m\tau^+) + \Omega_{m+1} \frac{1}{\Gamma(\alpha)} \int_{m\tau}^\zeta (\zeta - x)^{\alpha-1} v(x - \tau) dx.$$

If the transformation $\lambda_{m+1} = x - \tau$ is applied, then the solution function is isolated:

$$v(\zeta) = v(m\tau^+) \int_{(m-1)\tau}^{\zeta-\tau} \frac{(\zeta - \tau - \lambda_{m+1})^{\alpha-1}}{\Gamma(\alpha)} v(\lambda_{m+1}) d\lambda_{m+1}. \quad (4)$$

It is so clear that for small values of $m \in \mathbb{N}$, finding the solution is relatively easy. As $m \in \mathbb{N}$ becomes sufficiently

large, finding the solution is getting difficult, so it becomes necessary to obtain a general form of the solution to the linear piecewise fractional system (1). For this purpose, let us derive the solution to the linear piecewise fractional system for the case $m = 1$, which is noted below:

$$\begin{cases} {}^c D_{\tau+}^{\alpha} v(\varsigma) = \Omega_2 v(\varsigma - \tau), & \varsigma \in [\tau, 2\tau), \\ {}^c D_{0+}^{\alpha} v(\varsigma) = \Omega_1 v(\varsigma - \tau), & \varsigma \in [0, \tau), \\ v(\varsigma) = \varphi(\varsigma), & \varsigma \in [-\tau, 0). \end{cases} \quad (5)$$

Based on the equation (4), the explicit solution of the first equation of the system (5) is presented as follows:

$$v(\varsigma) = v(\tau) + \Omega_2 \int_0^{\varsigma-\tau} \frac{(\varsigma - \tau - \lambda_2)^{\alpha-1}}{\Gamma(\alpha)} v(\lambda_2) d\lambda_2. \quad (6)$$

The explicit solution of the second equation of the system (5) is given by

$$v(\varsigma) = \varphi(0) + \Omega_1 \int_{-\tau}^{\varsigma-\tau} \frac{(\varsigma - \tau - \lambda_1)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\lambda_1) d\lambda_1. \quad (7)$$

Now, let us take for $\varsigma = \tau$ and $\varsigma = \lambda_1$ in the equation (7), and substitute them into the equation (6). One obtains the explicit solution to the linear piecewise fractional system (5) as noted below

$$\begin{aligned} v(\varsigma) &= \varphi(0) + \int_{\tau}^0 \Omega_1 \frac{(-\lambda_1)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\lambda_1) d\lambda_1 \\ &\quad + \int_0^{\varsigma-\tau} \Omega_2 \frac{(\varsigma - \tau - \lambda_2)^{\alpha-1}}{\Gamma(\alpha)} \left[\varphi(0) \right. \\ &\quad \left. + \int_{-\tau}^{\lambda_2-\tau} \Omega_1 \frac{(\lambda_2 - \tau - \lambda_1)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\lambda_1) d\lambda_1 \right] d\lambda_2 \\ &= \varphi(0) \int_{-\tau}^0 \Omega_1 \frac{(-\lambda_1)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\lambda_1) d\lambda_1 \\ &\quad + \int_0^{\varsigma-\tau} \Omega_2 \frac{(\varsigma - \tau - \lambda_2)^{\alpha-1}}{\Gamma(\alpha)} \varphi(0) d\lambda_2 \\ &\quad + \int_0^{\varsigma-\tau} \Omega_2 \frac{(\varsigma - \tau - \lambda_2)^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad \int_{-\tau}^{\lambda_2-\tau} \Omega_1 \frac{(\lambda_2 - \tau - \lambda_1)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\lambda_1) d\lambda_1 d\lambda_2. \end{aligned} \quad (8)$$

As can be seen in this solution, the expression is already quite cumbersome even for $m = 1$, and it is clear that it will be come much more complex form $m > 1$.

Intuitive Interpretation of the Generating Function

Now, we define the (fractional) generating function, which is produced by the initial function, to represent the explicit solution to the linear piecewise fractional system by

$$D_m(\varsigma, \varphi) = \begin{cases} \varphi(\varsigma), & m = 0, \\ \int_{(m-2)\tau}^{\varsigma-\tau} \Omega_m \frac{(\varsigma - \tau - \lambda_m)^{\alpha-1}}{\Gamma(\alpha)} \left(\sum_{i=0}^{m-2} D_i(i\tau, \varphi) + D_{m-1}(\lambda_m, \varphi) \right) d\lambda_m, & m \in \mathbb{N}_1. \end{cases} \quad (9)$$

The generating function $D_m(\varsigma, \varphi)$ provides a recursive mechanism to construct the solution interval by interval using the method of steps. Instead of solving the entire fractional delay system at once, the solution on each subinterval is built from the previously known history.

For $n = 1$, the system reduces to a single fractional equation on $[0, \tau)$, driven solely by the history function $\varphi(\varsigma - \tau)$. In this case, the generating function coincides with a fractional convolution between Ω_1 and the initial history.

For $n = 2$, the solution on $[\tau, 2\tau)$ depends both on the accumulated contribution at $\varsigma = \tau$ and on the solution constructed on $[0, \tau)$. The corresponding generating function D_2 therefore combines these two effects through a recursive integral representation.

The case $n = 2$ corresponds to the solution of system (5). Using the classical approach, this solution is expressed by means of equation (8), making use of the intermediate steps given in equations (6) and (7).

Below, the case $n = 2$, which represents the interval $[\tau, 2\tau)$ is reformulated according to the numbering of system (5) as the case $m = 1$, that is, the solution corresponding to the interval $[\tau, 2\tau)$. This solution is then derived step by step as an application of the generating function method.

Consequently, the generating function approach is explicitly illustrated through the cases $n = 1$ and $n = 2$ by considering $m = 1$. Moreover, it is shown that the obtained expression coincides with the solution of system (5) given in equation (8). This establishes the scientific validity of the proposed approach, demonstrating that the solution of equation (5) can indeed be represented by means of the generating function framework.

This pattern continues inductively: for general n , the generating function encodes the cumulative memory of all previous subintervals, allowing the explicit representation given in Theorem 2.

For $m = 1$, the explicit solution presented in (8) is rewritten in terms of the generating function (9):

$$v(\varsigma) = \sum_{i=0}^1 D_i(i\tau, \varphi) + D_2(\varsigma, \varphi) = D_0(0, \varphi) + D_1(\tau, \varphi) + D_2(\varsigma, \varphi), \quad \varsigma \in [\tau, 2\tau), \quad (10)$$

because, from the definition of the generating function,

$$D_0(\varsigma, \varphi) = \varphi(\varsigma) \Rightarrow D_0(0, \varphi) = \varphi(0), \quad (11)$$

$$\begin{aligned} D_1(\varsigma, \varphi) &= \int_{-\tau}^{\varsigma-\tau} \Omega_2 \frac{(\varsigma - \tau - \lambda_1)^{\alpha-1}}{\Gamma(\alpha)} \left(\sum_{i=0}^{-1} D_i(i\tau, \varphi) + D_0(\lambda_1, \varphi) \right) d\lambda_1, \\ &= \int_{-\tau}^{\varsigma-\tau} \Omega_1 \frac{(\varsigma - \tau - \lambda_1)^{\alpha-1}}{\Gamma(\alpha)} D_0(\lambda_1, \varphi) d\lambda_1 \\ &= \int_{-\tau}^{\varsigma-\tau} \Omega_1 \frac{(\varsigma - \tau - \lambda_1)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\lambda_1) d\lambda_1, \end{aligned}$$

$$D_1(\tau, \varphi) = \int_{-\tau}^0 \Omega_1 \frac{(-\lambda_1)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\lambda_1) d\lambda_1, \quad (12)$$

$$\begin{aligned}
D_2(\varsigma, \varphi) &= \int_0^{\varsigma-\tau} \Omega_2 \frac{(\varsigma-\tau-\lambda_2)^{\alpha-1}}{\Gamma(\alpha)} \left(\sum_{i=0}^0 D_i(i\tau, \varphi) + D_1(\lambda_2, \varphi) \right) d\lambda_2 \\
&= \int_0^{\varsigma-\tau} \Omega_2 \frac{(\varsigma-\tau-\lambda_2)^{\alpha-1}}{\Gamma(\alpha)} \left(D_0(0, \varphi) \right. \\
&\quad \left. + \int_{-\tau}^{\lambda_2-\tau} \Omega_1 \frac{(\lambda_2-\tau-\lambda_1)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\lambda_1) d\lambda_1 \right) d\lambda_2 \\
&= \int_0^{\varsigma-\tau} \Omega_2 \frac{(\varsigma-\tau-\lambda_2)^{\alpha-1}}{\Gamma(\alpha)} \varphi(0) d\lambda_2 \\
&\quad + \int_0^{\varsigma-\tau} \Omega_2 \frac{(\varsigma-\tau-\lambda_2)^{\alpha-1}}{\Gamma(\alpha)} \int_{-\tau}^{\lambda_2-\tau} \Omega_1 \frac{(\lambda_2-\tau-\lambda_1)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\lambda_1) d\lambda_1 d\lambda_2.
\end{aligned} \quad (13)$$

This, in fact, coincides with the solution given in (8) by combining (11), (12), and (13), as we claimed.

Remark 1. The fractional generating function reduces to the iterative function introduced in [29] when the alpha parameter is equal to one.

Explicit Solution Via Generating Functions

In this subsection, we derive an explicit interval-wise representation of the solution to the linear system (2) by employing a generating function approach.

Theorem 2. An explicit solution to the following linear piecewise fractional system

$$\begin{cases} {}^c D_{(m-1)\tau}^\alpha v(\varsigma) = \Omega_m v(\varsigma - \tau), & \varsigma \in [(m-1)\tau, m\tau), \\ {}^c D_{(m-2)\tau}^\alpha v(\varsigma) = \Omega_{m-1} v(\varsigma - \tau), & \varsigma \in [(m-2)\tau, (m-1)\tau), \\ \vdots \\ {}^c D_{\tau}^\alpha v(\varsigma) = \Omega_2 v(\varsigma - \tau), & \varsigma \in [\tau, 2\tau), \\ {}^c D_0^\alpha v(\varsigma) = \Omega_1 v(\varsigma - \tau), & \varsigma \in [0, \tau), \\ v(\varsigma) = \varphi(\varsigma), & \varsigma \in [-\tau, 0), \end{cases} \quad (14)$$

has the following form with the help of the generating function

$$v(\varsigma) = \sum_{i=0}^{m-1} D_i(i\tau, \varphi) + D_m(\varsigma, \varphi). \quad (15)$$

Proof. To show that it holds for any $m \in \mathbb{N}$, we will use the method of mathematical induction on m . For $m = 1$, considering the solutions (8) and (10), we can conclude that the statement holds. Now, assume that our claim holds for $m = n$. We will now prove that it also holds for $m = n + 1$. That is, the solution given in (15) for $k = m$ to the system (14) must hold:

$$v(\varsigma) = \sum_{i=0}^{m-1} D_i(i\tau, \varphi) + D_m(\varsigma, \varphi). \quad (16)$$

When $k = m + 1$, the solution becomes:

$$v(\varsigma) = \sum_{i=0}^m D_i(i\tau, \varphi) + D_{m+1}(\varsigma, \varphi). \quad (17)$$

We need to show that it holds. When $\varsigma \in [m\tau, (m+1)\tau)$, our equation becomes

$${}^c D_{m\tau}^\alpha v(\varsigma) = \Omega_{m+1} v(\varsigma - \tau),$$

whose explicit solution followed from the equation (4) is given by:

$$v(\varsigma) = v(m\tau^+) + \int_{(m-1)\tau}^{\varsigma-\tau} \Omega_{m+1} \frac{(\varsigma-\tau-\lambda_{m+1})^{\alpha-1}}{\Gamma(\alpha)} v(\lambda_{m+1}) d\lambda_{m+1}. \quad (18)$$

Now, if we take $\varsigma = m\tau$ in the equation (16), then one gets

$$\begin{aligned} v(m\tau) &= \sum_{i=0}^{m-1} D_i(i\tau, \varphi) + D_m(m\tau, \varphi) \\ &= \sum_{i=0}^m D_i(i\tau, \varphi). \end{aligned}$$

If we take $\varsigma = \lambda_{m+1}$ in the equation (16), then one has

$$v(\lambda_{m+1}) = \sum_{i=0}^{m-1} D_i(i\tau, \varphi) + D_m(\lambda_{m+1}, \varphi).$$

If these equations are substituted into the equation (18), then the following is obtained:

$$v(\varsigma) = \sum_{i=0}^m D_i(i\tau, \varphi) + \int_{(m-1)\tau}^{\varsigma-\tau} \Omega_{m+1} \frac{(\varsigma-\tau-\lambda_{m+1})^{\alpha-1}}{\Gamma(\alpha)} \left(\sum_{i=0}^{m-1} D_i(i\tau, \varphi) + D_m(\lambda_{m+1}, \varphi) \right) d\lambda_{m+1}.$$

Based on the definition of the generating function, one can observe that:

$$D_{m+1}(\varsigma, \varphi) = \int_{(m-1)\tau}^{\varsigma-\tau} \Omega_{m+1} \frac{(\varsigma-\tau-\lambda_{m+1})^{\alpha-1}}{\Gamma(\alpha)} \left(\sum_{i=0}^{m-1} D_i(i\tau, \varphi) + D_m(\lambda_{m+1}, \varphi) \right) d\lambda_{m+1}.$$

By rearranging this system for $\varsigma \in [m\tau, (m+1)\tau)$, the corresponding explicit solution to the corresponding system is found as stated below

$$v(\varsigma) = \sum_{i=0}^m D_i(i\tau, \varphi) + D_{m+1}(\varsigma, \varphi),$$

which gives the desired solution. This concludes the induction proof.

Alternative Proof. Because of ${}^c D_{\alpha}^\beta c = 0$ for any constant $c \in \mathbb{R}$, one has

$$\begin{aligned} {}^c D_{(m-1)\tau}^\alpha \left(\sum_{i=0}^{m-1} D_i(i\tau, \varphi) + D_m(\varsigma, \varphi) \right) &= {}^c D_{(m-1)\tau}^\alpha D_m(\varsigma, \varphi) \\ &= {}^c D_{(m-1)\tau}^\alpha \int_{(m-2)\tau}^{\varsigma-\tau} \Omega_m \frac{(\varsigma-\tau-\lambda_m)^{\alpha-1}}{\Gamma(\alpha)} \left(\sum_{i=0}^{m-2} D_i(i\tau, \varphi) + D_{m-1}(\lambda_m, \varphi) \right) d\lambda_m \\ &= {}^c D_{(m-1)\tau}^\alpha \int_{(m-1)\tau}^{\varsigma-\tau} \Omega_m \frac{(\varsigma-\lambda_m)^{\alpha-1}}{\Gamma(\alpha)} \left(\sum_{i=0}^{m-2} D_i(i\tau, \varphi) + D_{m-1}(\lambda_m - \tau, \varphi) \right) d\lambda_m, \end{aligned}$$

applying the substitution $\theta = \lambda_m + \tau$, one has

$$\begin{aligned} {}^c D_{(m-1)\tau}^\alpha \left(\sum_{i=0}^{m-1} D_i(i\tau, \varphi) + D_m(\varsigma, \varphi) \right) &= {}^c D_{(m-1)\tau}^\alpha I_{(m-1)\tau}^\alpha \Omega_m \left(\sum_{i=0}^{m-2} D_i(i\tau, \varphi) \right. \\ &\quad \left. + D_{m-1}(\varsigma - \tau, \varphi) \right) \\ &= \Omega_m \left(\sum_{i=0}^{m-2} D_i(i\tau, \varphi) + D_{m-1}(\varsigma - \tau, \varphi) \right) \\ &= \Omega_m v(\varsigma - \tau). \end{aligned}$$

Remark 3. In the special case $\alpha = 1$, the Caputo fractional derivative reduces to the classical first-order derivative. Consequently, the linear piecewise fractional delay system (14) becomes a classical piecewise linear state-delay system. In this setting, the explicit solution representation (15) is consistent with the structure of solutions studied for piecewise linear state-delay systems in the literature, such as [29]. Hence, the present result recovers known classical models when $\alpha = 1$, while extending them to the fractional-order case $0 < \alpha < 1$ and providing explicit interval-wise analytical solution representations together with nite-time stability bounds.

Remark 4. For the special case of a constant initial history $\varphi(\varsigma) \equiv \varphi_0$, the solution admits a closed piecewise representation. Define the shorthand coefficients

$$P_{j,r} := \prod_{q=0}^{r-1} \Omega_{j-q} \quad (r \geq 1), \quad S_j := \varphi_0 + \sum_{r=1}^j P_{j,r} \frac{\tau^{\alpha r}}{\Gamma(\alpha r + 1)}, \quad S_0 := \varphi_0.$$

Then, for $m = 6$ the solution is given explicitly by

$$v(\varsigma) = \begin{cases} \varphi_0, & \varsigma \in [-\tau, 0), \\ S_0 + \Omega_1 \frac{\varsigma^\alpha}{\Gamma(\alpha + 1)}, & \varsigma \in [0, \tau), \\ S_1 + \Omega_2 \frac{(\varsigma - \tau)^\alpha}{\Gamma(\alpha + 1)} + \Omega_2 \Omega_1 \frac{(\varsigma - \tau)^{2\alpha}}{\Gamma(2\alpha + 1)}, & \varsigma \in [\tau, 2\tau), \\ S_2 + \Omega_3 \frac{(\varsigma - 2\tau)^\alpha}{\Gamma(\alpha + 1)} + \Omega_3 \Omega_2 \frac{(\varsigma - 2\tau)^{2\alpha}}{\Gamma(2\alpha + 1)} \\ + \Omega_3 \Omega_2 \Omega_1 \frac{(\varsigma - 2\tau)^{3\alpha}}{\Gamma(3\alpha + 1)}, & \varsigma \in [2\tau, 3\tau), \\ S_3 + \Omega_4 \frac{(\varsigma - 3\tau)^\alpha}{\Gamma(\alpha + 1)} + \Omega_4 \Omega_3 \frac{(\varsigma - 3\tau)^{2\alpha}}{\Gamma(2\alpha + 1)} \\ + \Omega_4 \Omega_3 \Omega_2 \frac{(\varsigma - 3\tau)^{3\alpha}}{\Gamma(3\alpha + 1)} + \Omega_4 \Omega_3 \Omega_2 \Omega_1 \frac{(\varsigma - 3\tau)^{4\alpha}}{\Gamma(4\alpha + 1)}, & \varsigma \in [3\tau, 4\tau), \\ S_4 + \Omega_5 \frac{(\varsigma - 4\tau)^\alpha}{\Gamma(\alpha + 1)} + \Omega_5 \Omega_4 \frac{(\varsigma - 4\tau)^{2\alpha}}{\Gamma(2\alpha + 1)} \\ + \Omega_5 \Omega_4 \Omega_3 \frac{(\varsigma - 4\tau)^{3\alpha}}{\Gamma(3\alpha + 1)} + \Omega_5 \Omega_4 \Omega_3 \Omega_2 \frac{(\varsigma - 4\tau)^{4\alpha}}{\Gamma(4\alpha + 1)} \\ + \Omega_5 \Omega_4 \Omega_3 \Omega_2 \Omega_1 \frac{(\varsigma - 4\tau)^{5\alpha}}{\Gamma(5\alpha + 1)}, & \varsigma \in [4\tau, 5\tau), \\ S_5 + \Omega_6 \frac{(\varsigma - 5\tau)^\alpha}{\Gamma(\alpha + 1)} + \Omega_6 \Omega_5 \frac{(\varsigma - 5\tau)^{2\alpha}}{\Gamma(2\alpha + 1)} \\ + \Omega_6 \Omega_5 \Omega_4 \frac{(\varsigma - 5\tau)^{3\alpha}}{\Gamma(3\alpha + 1)} + \Omega_6 \Omega_5 \Omega_4 \Omega_3 \frac{(\varsigma - 5\tau)^{4\alpha}}{\Gamma(4\alpha + 1)} \\ + \Omega_6 \Omega_5 \Omega_4 \Omega_3 \Omega_2 \frac{(\varsigma - 5\tau)^{5\alpha}}{\Gamma(5\alpha + 1)} \\ + \Omega_6 \Omega_5 \Omega_4 \Omega_3 \Omega_2 \Omega_1 \frac{(\varsigma - 5\tau)^{6\alpha}}{\Gamma(6\alpha + 1)}, & \varsigma \in [5\tau, 6\tau). \end{cases}$$

Here, the constants S_j accumulate the contributions from the preceding intervals, explicitly,

$$\begin{aligned} S_1 &= \varphi_0 + \Omega_1 \frac{\tau^\alpha}{\Gamma(\alpha + 1)}, \\ S_2 &= S_1 + \Omega_2 \frac{\tau^\alpha}{\Gamma(\alpha + 1)} + \Omega_2 \Omega_1 \frac{\tau^{2\alpha}}{\Gamma(2\alpha + 1)}, \\ S_3 &= S_2 + \Omega_3 \frac{\tau^\alpha}{\Gamma(\alpha + 1)} + \Omega_3 \Omega_2 \frac{\tau^{2\alpha}}{\Gamma(2\alpha + 1)}, \\ S_4 &= S_3 + \Omega_4 \frac{\tau^\alpha}{\Gamma(\alpha + 1)} + \Omega_4 \Omega_3 \frac{\tau^{2\alpha}}{\Gamma(2\alpha + 1)} \\ &\quad + \Omega_4 \Omega_3 \Omega_2 \frac{\tau^{3\alpha}}{\Gamma(3\alpha + 1)} + \Omega_4 \Omega_3 \Omega_2 \Omega_1 \frac{\tau^{4\alpha}}{\Gamma(4\alpha + 1)}, \\ S_5 &= S_4 + \Omega_5 \frac{\tau^\alpha}{\Gamma(\alpha + 1)} + \Omega_5 \Omega_4 \frac{\tau^{2\alpha}}{\Gamma(2\alpha + 1)} \\ &\quad + \Omega_5 \Omega_4 \Omega_3 \frac{\tau^{3\alpha}}{\Gamma(3\alpha + 1)} + \Omega_5 \Omega_4 \Omega_3 \Omega_2 \frac{\tau^{4\alpha}}{\Gamma(4\alpha + 1)} \\ &\quad + \Omega_5 \Omega_4 \Omega_3 \Omega_2 \Omega_1 \frac{\tau^{5\alpha}}{\Gamma(5\alpha + 1)}. \end{aligned}$$

This piecewise expression shows that at each node $\varsigma = k\tau$ the left and right limits coincide, and hence the solution $v(\varsigma)$ is continuous across all subintervals.

The Continuity of the Explicit Solution

Now, the continuity of the explicit solution we have constructed is debated in the following theorem.

Theorem 5. The analytical solution expressed in (15) to the linear piecewise fractional system (14) is continuous on $[-\tau, \infty)$.

Proof. Before proceeding to the junction-point analysis, we briefly justify the continuity of the generating functions $D_m(\varsigma, \varphi)$ and the convergence properties of the recursive integral representations involved. We proceed by induction on m . For $m = 0$, one has $D_0(\varsigma, \varphi) = \varphi(\varsigma)$, which is continuous on $[-\tau, 0]$ by assumption. Assume that, for some $m \geq 1$, the functions $D_i(\cdot, \varphi)$, $i = 0, 1, \dots, m - 1$, are continuous and bounded on their respective domains.

By definition, $D_m(\varsigma, \varphi)$ is given by a fractional integral whose integrand consists of the bounded matrix Ω_m , the weakly singular kernel $(\varsigma - \tau - \lambda)^{\alpha-1}/\Gamma(\alpha)$ with $0 < \alpha < 1$, and the continuous bounded function $\sum_{i=0}^{m-2} D_i(i\tau, \varphi) + D_{m-1}(\lambda, \varphi)$. Since the kernel is integrable on any finite interval and the remaining factors are bounded, standard properties of fractional integrals imply that $D_m(\cdot, \varphi)$ is continuous on $[(m-1)\tau, m\tau]$. This completes the induction step and establishes continuity of all D_m .

Moreover, the integrands defining D_m are dominated by an integrable majorant independent of ς on compact subintervals. Hence, the family of recursive integrals converges uniformly on compact sets. As a consequence, limit operations with respect to ς can be interchanged with the integral sign, which justifies the limit arguments used below.

It is evident that $\varsigma \in [(m-1)\tau, m\tau]$, $m \in \mathbb{N}$, is continuous at every interior point. At the critical point $\varsigma = m\tau$, it suffices to show that the right-hand limit, the left-hand limit, and the function's value coincide. For $\varsigma = m\tau$,

$$v(\zeta) = \sum_{i=0}^m D_i(i\tau, \varphi) + D_{m+1}(\zeta, \varphi).$$

Evaluating at $\zeta = m\tau$ yields

$$v(m\tau) = \sum_{i=0}^m D_i(i\tau, \varphi) + D_{m+1}(m\tau, \varphi).$$

Since the integration limits of $D_{m+1}(m\tau, \varphi)$ coincide, one has $D_{m+1}(m\tau, \varphi) = 0$, and therefore

$$v(m\tau) = \sum_{i=0}^m D_i(i\tau, \varphi).$$

The right-hand limit is

$$\begin{aligned} \lim_{\zeta \rightarrow m\tau^+} v(\zeta) &= \sum_{i=0}^m D_i(i\tau, \varphi) + \lim_{\zeta \rightarrow m\tau^+} D_{m+1}(\zeta, \varphi) \\ &= \sum_{i=0}^m D_i(i\tau, \varphi), \end{aligned}$$

where the limit of D_{m+1} follows from the continuity established above. For the left-hand limit, one uses

$$v(\zeta) = \sum_{i=0}^{m-1} D_i(i\tau, \varphi) + D_m(\zeta, \varphi),$$

which gives

$$\begin{aligned} \lim_{\zeta \rightarrow m\tau^-} v(\zeta) &= \sum_{i=0}^{m-1} D_i(i\tau, \varphi) + D_m(m\tau, \varphi) \\ &= \sum_{i=0}^m D_i(i\tau, \varphi). \end{aligned}$$

Since the left-hand limit, right-hand limit, and the function value coincide at $\zeta = m\tau$, the solution is continuous at each junction point. Therefore, the explicit solution expressed in (15) is continuous on $[-\tau, \infty)$.

Finite-Time Stability

In this section, we first present the formal definition of finite-time stability. To support the main result, two auxiliary lemmas are established, followed by the proof of the principal theorem on the finite-time stability of the system under consideration.

Definition 6. [33] The system described by the homogeneous state equation (1), and subject to the initial condition

$$v(\zeta) = \varphi(\zeta), \quad -\tau \leq \zeta \leq 0,$$

is said to be finite-time stable with respect to the parameters $\{\delta, \varepsilon, \zeta_0, J\}$, under the condition $\delta < \varepsilon$ if and only if

$$\|\varphi\|_\infty < \delta,$$

implies

$$\|v(\zeta)\| < \varepsilon, \quad \forall \zeta \in J,$$

where ζ_0 denotes the initial observation time of the system, and J is the observation interval given by $J = [\zeta_0, \zeta_0 + T] \subset \mathbb{R}$.

Lemma 7. Let $\tau \geq 0$, $\alpha, M > 0$ be given. Define

$$S_k := \frac{M^{k+1}\tau^{(k+1)\alpha}}{\Gamma((k+1)\alpha+1)}, \quad T_j = \sum_{i=0}^j S_i, \quad k, j \geq 0.$$

Then,

$$S_0 + \sum_{i=1}^{m-2} 2^{i-1}T_i + 2^{m-2}T_{m-1} \leq 2^{m-1}T_m, \quad \forall m \geq 2.$$

Proof. Based on the given definition, one has without any calculations

$$\begin{aligned} S_0 + \sum_{i=1}^{m-2} 2^{i-1}T_i + 2^{m-2}T_{m-1} &\leq T_1 \\ &+ \sum_{i=1}^{m-2} 2^{i-1}T_i + 2^{m-2}T_{m-1}, \quad \forall m \geq 2. \end{aligned}$$

Because of $S_k \geq 0$, T_j is monotonically increasing and so $T_i \leq T_m$ for all $i \leq m$. Thus

$$\begin{aligned} T_1 + \sum_{i=1}^{m-2} 2^{i-1}T_i + 2^{m-2}T_{m-1} \\ &\leq T_m + \sum_{i=1}^{m-2} 2^{i-1}T_i + 2^{m-2}T_{m-1} \\ &\leq \left(1 + \sum_{i=1}^{m-2} 2^{i-1} + 2^{m-2}\right)T_m \\ &= (1 + 2^{m-2} - 1 + 2^{m-2})T_m \\ &= 2^{m-1}T_m, \end{aligned}$$

which is, indeed, we want to show.

Lemma 8. Let $D_m(\zeta, \varphi)$ be given in the equation (9), that is,

$$\begin{aligned} D_0(\zeta, \varphi) &= \varphi(\zeta), \\ D_m(\zeta, \varphi) &= \int_{(m-2)\tau}^{\zeta-\tau} \Omega_m \frac{(\zeta - \tau - \lambda_{m+1})^{\alpha-1}}{\Gamma(\alpha)} \left(\sum_{i=0}^{m-2} D_i(i\tau, \varphi) + D_{m-1}(\lambda, \varphi) \right) d\lambda, \end{aligned}$$

for $m \in \mathbb{N}_1$ and $\zeta \in [(m-1)\tau, m\tau]$. Then,

$$\sup_{\zeta \in [(m-1)\tau, m\tau]} \|D_m(\zeta, \varphi)\| \leq 2^{m-1}\delta \sum_{k=0}^m \frac{M^k \tau^{k\alpha}}{\Gamma(k\alpha+1)}. \quad (19)$$

Proof. We proceed by the mathematical induction on m . For the base case $m = 1$,

$$D_1(\varsigma, \varphi) = \int_{-\tau}^{\varsigma-\tau} \Omega_1 \frac{(\varsigma - \tau - \lambda)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\lambda) d\lambda, \quad \varsigma \in [0, \tau].$$

Using $\|\varphi\|_{\infty} \leq \delta$ and the change $u = \varsigma - \tau - \lambda \in [0, \varsigma]$,

$$\|D_1(\varsigma, \varphi)\| \leq \frac{\|\Omega_1\|}{\Gamma(\alpha)} \delta \int_0^{\varsigma} u^{\alpha-1} du = \frac{\|\Omega_1\|}{\Gamma(\alpha-1)} \varsigma^{\alpha} \delta \leq \frac{M\tau^{\alpha}}{\Gamma(\alpha-1)} \delta.$$

Taking the supremum in $\varsigma \in [0, \tau]$ yields

$$\sup \|D_1\| \leq 2^{1-1} \delta \sum_{k=0}^1 \frac{M^k \tau^{k\alpha}}{\Gamma(k\alpha+1)},$$

which is (19) with $m = 1$ (since $2^0 = 1$ and the sum has only $k = 0$ and $k = 1$). Consider the induction step, that is, assume that for each $1 \leq r \leq m-1$,

$$B_r := \sup_{\varsigma \in [(m-1)\tau, m\tau]} \|D_1\| \leq 2^{r-1} \delta \sum_{k=0}^r \frac{M^k \tau^{k\alpha}}{\Gamma(k\alpha+1)}, \quad (20)$$

for fixed $\varsigma \in [(m-1)\tau, m\tau]$, taking norms in the recursive formula and using linearity,

$$\begin{aligned} \|D_m(\varsigma, \varphi)\| &\leq \|\Omega_m\| \int_{(m-2)\tau}^{\varsigma-\tau} \frac{(\varsigma - \tau - \lambda)^{\alpha-1}}{\Gamma(\alpha)} \left(\sum_{i=0}^{m-2} \|D_i(i\tau, \varphi)\| + \|D_{m-1}(\lambda, \varphi)\| \right) d\lambda \\ &\leq \|\Omega_m\| \left[\frac{(\varsigma - (m-1)\tau)^{\alpha}}{\Gamma(\alpha+1)} \right] \left(\delta + \sum_{i=1}^{m-2} B_i + B_{m-1} \right), \end{aligned}$$

we used $\int_{(m-2)\tau}^{\varsigma-\tau} \frac{(\varsigma - \tau - \lambda)^{\alpha-1}}{\Gamma(\alpha)} d\lambda = \frac{(\varsigma - (m-1)\tau)^{\alpha}}{\Gamma(\alpha+1)}$ and $D_0(0, \varphi) = \varphi(0)$. Since $\varsigma \leq m\tau$, we have $(\varsigma - (m-1)\tau)^{\alpha} \leq \tau^{\alpha}$, and thus

$$\|D_m(\varsigma, \varphi)\| \leq \frac{M\tau^{\alpha}}{\Gamma(\alpha+1)} \left(\delta + \sum_{i=1}^{m-2} B_i + B_{m-1} \right). \quad (21)$$

Applying the induction hypothesis (20) to each B_i gives

$$\begin{aligned} \delta + \sum_{i=1}^{m-2} B_i + B_{m-1} &\leq \delta + \sum_{i=1}^{m-2} 2^{i-1} \delta \sum_{k=0}^i \frac{M^k \tau^{k\alpha}}{\Gamma(k\alpha+1)} \\ &\quad + 2^{m-2} \delta \sum_{k=0}^{m-1} \frac{M^k \tau^{k\alpha}}{\Gamma(k\alpha+1)}. \end{aligned}$$

By multiplying by $M\tau^{\alpha}/\Gamma(\alpha+1)$ and using the Beta-identity bound $\frac{M\tau^{\alpha}}{\Gamma(\alpha+1)} \cdot \frac{M^k \tau^{k\alpha}}{\Gamma(k\alpha+1)} \leq \frac{M^{k+1} \tau^{(k+1)\alpha}}{\Gamma((k+1)\alpha+1)}$, and using Lemma 7, we obtain

$$\sup_{\varsigma \in [(m-1)\tau, m\tau]} \|D_m(\varsigma, \varphi)\| \leq 2^{m-1} \delta \sum_{k=0}^m \frac{M^k \tau^{k\alpha}}{\Gamma(k\alpha+1)}.$$

This is exactly (19) and completes the mathematical induction on m .

Theorem 9. The considered system (1) is finite-time stable if

$$\sum_{i=1}^m 2^{i-1} \sum_{k=0}^i \frac{M^k \tau^{k\alpha}}{\Gamma(k\alpha+1)} < \frac{\varepsilon}{\delta}. \quad (22)$$

Proof. From Theorem 2, the explicit continuous solution of the homogeneous piecewise fractional system (1) is expressed by

$$v(\varsigma) = \sum_{i=0}^{m-1} D_i(i\tau, \varphi) + D_m(\varsigma, \varphi),$$

Taking the norm of both sides, we obtain

$$\|v(\varsigma)\| \leq \sum_{i=0}^{m-1} \|D_i(i\tau, \varphi)\| + \|D_m(\varsigma, \varphi)\|.$$

Based on the definition of finite-time stability, let $\|\varphi\|_{\infty}$ hold. Now, let's take the maximum value for $\varsigma \in [(m-1)\tau, m\tau]$ on the right-hand side together with employing Lemma 8

$$\begin{aligned} \|v(\varsigma)\| &\leq \sum_{i=0}^m \sup_{\varsigma \in [(m-1)\tau, m\tau]} \|D_m(\varsigma, \varphi)\| \\ &\leq \delta \sum_{i=1}^m 2^{i-1} \sum_{k=0}^i \frac{M^k \tau^{k\alpha}}{\Gamma(k\alpha+1)}. \end{aligned}$$

Hence, employing the basic condition of this theorem, relation (23) yields:

$$\|v(\varsigma)\| < \varepsilon, \quad \forall \varsigma \in J.$$

The desired is readily obtained. Consequently, the considered system (1) is finite-time stable under the established inequality.

Remark 10. Although the sufficient condition (22) is explicit, the double sum may be come computationally heavy for large m . In practice, it can be evaluated efficiently by precomputing the coefficients $a_k := \frac{(M\tau^{\alpha})^k}{\Gamma(k\alpha+1)}$ and updating the partial sums recursively. Specifically, one may compute $T_i := \sum_{k=0}^i a_k$ incrementally via $T_i = T_{i-1} + a_i$, and then accumulate $S_m := \sum_{i=1}^m 2^{i-1} T_i$ in $O(m)$ time. Moreover, for large k , $\Gamma(k\alpha+1)$ grows superlinearly, so a_k decays rapidly once k is moderate; hence, truncating the inner sum at a finite index (based on a prescribed tolerance) yields an accurate and stable numerical check of (22). Finally, (22) explicitly reveals parameter sensitivity: the left-hand side increases with τ and M , and typically decreases as α increases (because larger α produces larger $\Gamma(k\alpha+1)$, reducing the terms). Therefore, smaller delays τ and/or larger fractional orders α enlarge the admissible stability margin ε/δ , which can guide design and tuning of the piecewise coefficients.

Remark 11. Although the finite-time stability condition obtained in this study is explicit and directly verifiable, it may exhibit a certain degree of conservatism. This conservatism mainly originates from the use of uniform norm bounds and the exponential-type growth factors introduced in the recursive estimation procedure. Such bounding techniques are commonly adopted in the finite-time stability analysis of fractional delay systems to ensure analytical tractability and broad applicability. While sharper bounds may be achievable for particular system structures or specific coefficient configurations, the proposed criterion provides a reliable and computable sufficient condition guaranteeing finite-time stability over the prescribed time interval.

Examples

In this section, several illustrative examples are provided to demonstrate the applicability of the theoretical results. The following examples include explicit numerical simulations and trajectory plots in order to visually verify the continuity at switching points and to support the finite-time stability results derived theoretically.

Example 12. Consider the case $m = 6$, $\alpha = 1/2$, $\tau = 1$, constant history $\varphi(\varsigma) \equiv 1$, and coefficient sequence $(\Omega_1, \dots, \Omega_6) = (0.8, 0.5, -0.3, 0.2, -0.1, 0.05)$. Using the closed representation in Remark 1, the accumulated constants are

$$\begin{aligned} S_0 &= 1, \\ S_1 &= 1 + 0.8 \frac{1}{\Gamma(1.5)} \approx 1.9027, \\ S_2 &= S_1 + 0.5 \frac{1}{\Gamma(1.5)} + 0.4 \frac{1}{\Gamma(2)} \approx 2.8669, \\ S_3 &= S_2 - 0.3 \frac{1}{\Gamma(1.5)} - 0.15 \frac{1}{\Gamma(2)} \\ &\quad - 0.12 \frac{1}{\Gamma(2.5)} \approx 2.2881, \\ S_4 &= S_3 + 0.2 \frac{1}{\Gamma(1.5)} - 0.06 \frac{1}{\Gamma(2)} \\ &\quad - 0.03 \frac{1}{\Gamma(2.5)} - 0.024 \frac{1}{\Gamma(3)} \approx 2.4192, \\ S_5 &= S_4 - 0.1 \frac{1}{\Gamma(1.5)} - 0.02 \frac{1}{\Gamma(2)} \\ &\quad + 0.006 \frac{1}{\Gamma(2.5)} + 0.003 \frac{1}{\Gamma(3)} \\ &\quad + 0.0024 \frac{1}{\Gamma(3.5)} \approx 2.3131. \end{aligned}$$

Hence, the solution $v(\varsigma)$ is given piecewise as in (23). Before presenting the graphical representation, we emphasize that the piecewise expression (23) has been constructed so that the left- and right-hand limits coincide at each junction point $\varsigma = k\tau$, $k = 0, 1, \dots, 5$, in accordance with the continuity result established in Theorem 5.

$$v(\varsigma) = \begin{cases} 1, & \varsigma \in [-1, 0), \\ S_0 + 0.8 \frac{\varsigma^{1/2}}{\Gamma(1.5)}, & \varsigma \in [0, 1), \\ S_1 + 0.5 \frac{(\varsigma-1)^{1/2}}{\Gamma(1.5)} + 0.4 \frac{(\varsigma-1)}{\Gamma(2)}, & \varsigma \in [1, 2), \\ S_2 - 0.3 \frac{(\varsigma-2)^{1/2}}{\Gamma(1.5)} - 0.15 \frac{(\varsigma-2)}{\Gamma(2)} \\ \quad - 0.12 \frac{(\varsigma-2)^{3/2}}{\Gamma(2.5)}, & \varsigma \in [2, 3), \\ S_3 + 0.2 \frac{(\varsigma-3)^{1/2}}{\Gamma(1.5)} - 0.06 \frac{(\varsigma-3)}{\Gamma(2)} \\ \quad - 0.03 \frac{(\varsigma-3)^{3/2}}{\Gamma(2.5)} - 0.024 \frac{(\varsigma-3)^2}{\Gamma(3)}, & \varsigma \in [3, 4), \\ S_4 - 0.1 \frac{(\varsigma-4)^{1/2}}{\Gamma(1.5)} - 0.02 \frac{(\varsigma-4)}{\Gamma(2)} \\ \quad + 0.006 \frac{(\varsigma-4)^{3/2}}{\Gamma(2.5)} + 0.003 \frac{(\varsigma-4)^2}{\Gamma(3)} \\ \quad + 0.0024 \frac{(\varsigma-4)^{5/2}}{\Gamma(3.5)}, & \varsigma \in [4, 5), \\ S_5 + 0.05 \frac{(\varsigma-5)^{1/2}}{\Gamma(1.5)} - 0.005 \frac{(\varsigma-5)}{\Gamma(2)} \\ \quad - 0.001 \frac{(\varsigma-5)^{3/2}}{\Gamma(2.5)} + 0.0003 \frac{(\varsigma-5)^2}{\Gamma(3)} \\ \quad + 0.00015 \frac{(\varsigma-5)^{5/2}}{\Gamma(3.5)} + 0.00012 \frac{(\varsigma-5)^3}{\Gamma(4)}, & \varsigma \in [5, 6). \end{cases}$$

This explicit expression, whose graph is presented in Figure 2, confirms the validity of the generating-function construction and shows that the solution remains continuous across each junction $\varsigma = k\tau$. Figure 2 visually confirms that the explicit solution constructed in (23) is continuous at each junctionpoint $\varsigma = k\tau$, $k = 0, 1, \dots, 5$, despite the piecewise definition. In particular, no jump discontinuities or cusp-like behaviors are observed at the switching instants, which is fully consistent with the analytical continuity proof provided in Section 3.

We now verify the finite-time stability condition for this example. Here, $\alpha = 1/2$, $\tau = 1$, $m = 6$, and $M := \max_{1 \leq k \leq 6} |\Omega_k| = 0.8$. For the constant history $\varphi(\varsigma) \equiv 1$, we have

$\|\varphi\|_\infty = \delta = 1$. Evaluating the stability bound in Theorem 9, we obtain

$$\sum_{i=1}^6 2^{i-1} \sum_{k=0}^i \frac{M^k \tau^{k\alpha}}{\Gamma(k\alpha + 1)} = \sum_{i=1}^6 2^{i-1} \sum_{k=0}^i \frac{0.8^k}{\Gamma(k/2 + 1)} < C,$$

for some finite constant C . Consequently, by choosing $\varepsilon > C$, the inequality (22) is satisfied, and the system is finite-time stable on $J = [0, 6]$. This computation shows how the abstract finite-time stability condition of Theorem 9 can be explicitly evaluated in practice using the system parameters,

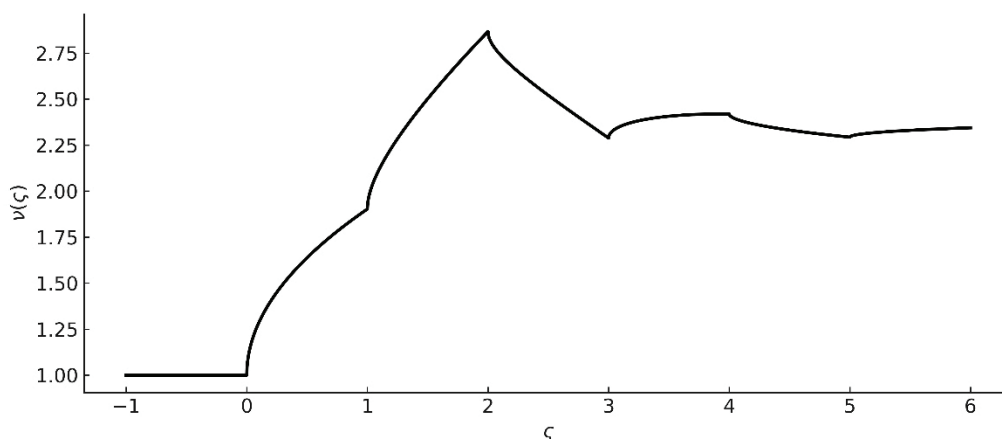


Figure 2. Graph of the explicit solution (23), illustrating the continuity of $v(\zeta)$ across the junction points $\zeta = k\tau$.

there by linking the analytical criterion to concrete numerical verification. Overall, this example demonstrates that the proposed generating function-based solution framework not only yields explicit closed-form expressions, but also allows a direct verification of continuity and finite-time stability through both analytical bounds and graphical inspection.

Example 13. Consider the following linear piecewise fractional system

$$\begin{cases} {}^c D_{2+}^{1/2} v(\zeta) = \Omega_3 v(\zeta - 1), & \zeta \in [2, 3), \\ {}^c D_{1+}^{1/2} v(\zeta) = \Omega_2 v(\zeta - 1), & \zeta \in [1, 2), \\ {}^c D_{0+}^{1/2} v(\zeta) = \Omega_1 v(\zeta - 1), & \zeta \in [0, 1), \\ v(\zeta) = \varphi(\zeta), & \zeta \in [-1, 0), \end{cases} \quad (24)$$

where

$$\varphi(\theta) = \begin{pmatrix} 1 + \theta \\ 2 - \theta \end{pmatrix}, \quad \theta \in [-1, 0), \quad \Omega_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\Omega_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & -0.3 \end{pmatrix}, \quad \Omega_3 = \begin{pmatrix} 0.2 & 0.1 \\ 0 & 0.1 \end{pmatrix}.$$

Based on the definition of the generating function $D_m(\zeta, \varphi)$ ($=: D_m(\zeta)$ for simplicity) in (9), for $m = 1$, $\zeta \in [0, 1)$, one obtains

$$D_1(\zeta) = \frac{\Omega_1}{\Gamma(1/2)} \int_{-1}^{\zeta-1} (\zeta - 1 - \lambda)^{-1/2} \varphi(\lambda) d\lambda = \begin{pmatrix} \frac{6\zeta^{1/2} - \frac{4}{3}\zeta^{3/2}}{\sqrt{\pi}} \\ 0 \end{pmatrix}.$$

Explicit computation of $D_1(1)$ (matrix-valued case).

For the reader's convenience, we detail the computation of $D_1(1)$. Setting $\zeta = 1$ in the definition of $D_1(\zeta)$ yields

$$D_1(\zeta) = \frac{\Omega_1}{\Gamma(1/2)} \int_{-1}^0 (-\lambda)^{-1/2} \varphi(\lambda) d\lambda.$$

Since

$$\varphi(\lambda) = \begin{pmatrix} 1 + \lambda \\ 2 - \lambda \end{pmatrix},$$

the integral can be evaluated componentwise as

$$\int_{-1}^0 (-\lambda)^{-1/2} \varphi(\lambda) d\lambda = \begin{pmatrix} \int_{-1}^0 (-\lambda)^{-1/2} (1 + \lambda) d\lambda \\ \int_{-1}^0 (-\lambda)^{-1/2} (2 - \lambda) d\lambda \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{14}{3} \end{pmatrix}.$$

Consequently, applying the matrix coefficient

$$\Omega_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

gives

$$D_1(1) = \frac{1}{\Gamma(1/2)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{4}{3} \\ \frac{14}{3} \end{pmatrix} = \frac{1}{\sqrt{\pi}} \begin{pmatrix} \frac{14}{3} \\ 0 \end{pmatrix}.$$

This computation illustrates explicitly how the matrix coefficient acts on the vector-valued fractional integral of the history function. Hence $D_1(1) = \left(\frac{14}{3\sqrt{\pi}}, 0\right)^T$ and

$$S_1 = \varphi(0^+) + D_1(1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{14}{3\sqrt{\pi}} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + \frac{14}{3\sqrt{\pi}} \\ 2 \end{pmatrix}.$$

For $m = 2$, $\zeta \in [1, 2)$, one gets

$$\begin{aligned} D_2(\zeta) &= \frac{\Omega_2}{\Gamma(1/2)} \int_0^{\zeta-1} (\zeta - 1 - \lambda)^{-1/2} (\varphi(0^+) + D_1(\lambda)) d\lambda \\ &= \begin{pmatrix} \frac{1}{\sqrt{\pi}} (\zeta - 1)^{1/2} + \frac{3}{2} (\zeta - 1) - \frac{1}{4} (\zeta - 1)^2 \\ -\frac{1.2}{\sqrt{\pi}} (\zeta - 1)^{1/2} \end{pmatrix}. \end{aligned}$$

In particular,

$$D_2(2) = \begin{pmatrix} \frac{1}{\sqrt{\pi}} + \frac{5}{4} \\ -\frac{1.2}{\sqrt{\pi}} \end{pmatrix},$$

$$S_2 = S_1 + D_2(2) = \begin{pmatrix} 1 + \frac{14}{3\sqrt{\pi}} + \frac{1}{\sqrt{\pi}} + \frac{5}{4} \\ 2 - \frac{1.2}{\sqrt{\pi}} \end{pmatrix} = \begin{pmatrix} \frac{14}{3\sqrt{\pi}} + \frac{1}{\sqrt{\pi}} + \frac{9}{4} \\ 2 - \frac{1.2}{\sqrt{\pi}} \end{pmatrix}.$$

$$v(\varsigma) = \begin{cases} \varphi(\varsigma) = \begin{pmatrix} 1 + \varsigma \\ 2 - \varsigma \end{pmatrix}, & \varsigma \in [-1, 0), \\ \varphi(0) + D_1(\varsigma) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 6\varsigma^{1/2} - \frac{4}{3}\varsigma^{3/2} \\ 0 \end{pmatrix}, & \varsigma \in [0, 1), \\ S_1 + D_2(\varsigma) = \begin{pmatrix} 1 + \frac{14}{3\sqrt{\pi}} \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{\pi}}(\varsigma - 1)^{1/2} + \frac{3}{2}(\varsigma - 1) \\ -\frac{1}{4}(\varsigma - 1)^2 - \frac{1.2}{\sqrt{\pi}}(\varsigma - 1)^{1/2} \end{pmatrix}, & \varsigma \in [1, 2), \\ S_2 + D_3(\varsigma), & \varsigma \in [2, 3), \end{cases}$$

For $m = 3$, $\varsigma \in [2, 3)$, one has

$$\begin{aligned} D_3(\varsigma) &= \frac{\Omega_3}{\Gamma(1/2)} \int_1^{\varsigma-1} (\varsigma - 1 - \lambda)^{-1/2} (\varphi(0^+) \\ &\quad + D_1(1) + D_2(\lambda)) d\lambda \\ &= \frac{\Omega_3}{\sqrt{\pi}} \left[2(\varsigma - 2)^{1/2} \begin{pmatrix} 1 + \frac{14}{3\sqrt{\pi}} \\ 2 \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} \frac{\sqrt{\pi}}{2}(\varsigma - 2) + 2(\varsigma - 2)^{3/2} \\ -\frac{4}{15}(\varsigma - 2)^{5/2} - 0.6\sqrt{\pi}(\varsigma - 2) \end{pmatrix} \right]. \end{aligned}$$

Based on Theorem 2, the explicit solution to the linear piecewise fractional system (24) is expressed by

with S_2 and $D_3(\varsigma)$ as explicitly given above. Note that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, $\Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4}$ have been employed above. The left and right limits match at $\varsigma = 0, 1, 2$, hence v is continuous on $[-1, 3)$. We emphasize that the piecewise representation above is constructed so that the accumulated terms S_1 and S_2 ensure the matching of left- and right-hand limits at the switching instants $\varsigma = 0, 1, 2$, in agreement with the continuity result established in Theorem 5.

Figure 3 depicts the two components of the solution $v(\varsigma) = (v_1(\varsigma), v_2(\varsigma))^T$ on $[-1, 3)$ and visually confirms the continuity at the junction points $\varsigma = 0, 1, 2$. In particular, Figure 3 shows that both components v_1 and v_2 evolve smoothly across the switching points, despite the presence of distinct system matrices on successive intervals. This confirms that the continuity property is preserved component wise, not merely in a normed sense.

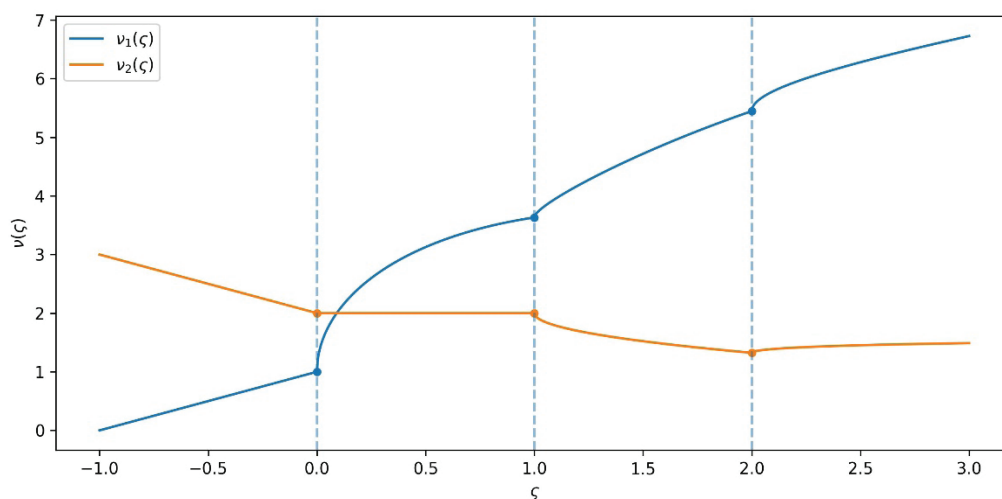


Figure 3. Componentwise solution of Example 13 illustrating the continuity of $v_1(\varsigma)$ and $v_2(\varsigma)$ across the junction points $\varsigma = 0, 1, 2$.

The infinity-norm trajectory shown in Figure 4 provides a quantitative visualization of the finite-time stability concept. By plotting the infinity norm of the solution together with the prescribed bound $\varepsilon = 0.80$, the figure illustrates how the analytical bound derived in Theorem 9 constrains the entire solution trajectory within the stability margin. This representation is consistent with the definition of finite-time stability and visually confirms the condition stated in Definition 6 and Theorem 9, showing that the solution remains within the prescribed bound for all $\varsigma \in [0, 3]$.

Overall, this example illustrates how the proposed framework simultaneously delivers explicit matrix-valued solutions, guarantees continuity at switching points, and enables direct numerical verification of finite-time stability, thereby bridging the gap between abstract theory and applied analysis.

Example 14. Consider the following homogeneous piecewise fractional system

$$\begin{cases} {}^C D_{2+}^\alpha v(\varsigma) = \Omega_3 v(\varsigma - 1), & \varsigma \in [2, 3], \\ {}^C D_{1+}^\alpha v(\varsigma) = \Omega_2 v(\varsigma - 1), & \varsigma \in [1, 2], \\ {}^C D_{0+}^\alpha v(\varsigma) = \Omega_1 v(\varsigma - 1), & \varsigma \in [0, 1], \\ v(\varsigma) = \varphi(\varsigma), & \varsigma \in [-1, 0). \end{cases} \quad (25)$$

Let $\delta = 0.05$, $\varepsilon = 0.80$ be given. Then, $M := \max_{1 \leq k \leq 3} |\Omega_k| = 0.30$ and $\varepsilon/\delta = 16$. In the light of the statement of Theorem 9, one has to calculate the following sum

$$S(3) := \sum_{i=1}^3 2^{i-1} \sum_{k=0}^i \frac{M^k \tau^{k\alpha}}{\Gamma(k\alpha + 1)} = \sum_{i=1}^3 2^{i-1} \sum_{k=0}^i \frac{(0.30)^k}{\Gamma(k/2 + 1)}.$$

For $i = 1$:

$$2^0 \left(1 + \frac{0.30}{\Gamma(1.5)} \right) = 1 \left(1 + \frac{0.30}{0.8862269} \right) = 1(1 + 0.3385) \approx 1.3385,$$

For $i = 2$:

$$\begin{aligned} 2^1 \left(1 + \frac{0.30}{\Gamma(1.5)} + \frac{0.30^2}{\Gamma(2)} \right) &= 2 \cdot \left(1 + \frac{0.30}{0.8862269} + 0.09 \right) \\ &= 2 \cdot (1 + 0.3385 + 0.09) \\ &= 2 \cdot 1.4285 \approx 2.8570, \end{aligned}$$

for $i = 3$:

$$\begin{aligned} 2^2 \left(1 + \frac{0.30}{\Gamma(1.5)} + \frac{0.30^2}{\Gamma(2)} + \frac{0.30^3}{\Gamma(2.5)} \right) &= 4 \cdot \left(1 + \frac{0.30}{0.8862269} + 0.09 + \frac{0.027}{1.3293404} \right) \\ &= 4 \cdot (1.4488) \approx 5.7953. \end{aligned}$$

Then,

$$S(3) \approx 1.3385 + 2.8570 + 5.7953 = 9.9908.$$

Since

$$S(3) \approx 9.9908 < \frac{\varepsilon}{\delta} = 16,$$

the hypothesis (22) holds. Therefore, by Theorem 9, the homogeneous piecewise fractional system (25) is finite-time stable on $J = [0, 3]$ for all histories with $\|\varphi\|_\infty < \delta = 0.05$, and its solution satisfies $\|v(\varsigma)\| < \varepsilon = 0.80$ for all $\varsigma \in J$. To complement the analytical verification above, we provide a numerical trajectory of the solution in order to visualize the finite-time stability property predicted by Theorem 9. Figure 5 illustrates the evolution of $v(\varsigma)$ over $\varsigma \in [0, 3]$ obtained by applying the Caputo fractional derivative of order $\alpha = 1/2$ with piecewise coefficients $\Omega_1 = 0.30$, $\Omega_2 = 0.20$, and $\Omega_3 = 0.25$. The initial history is assumed as

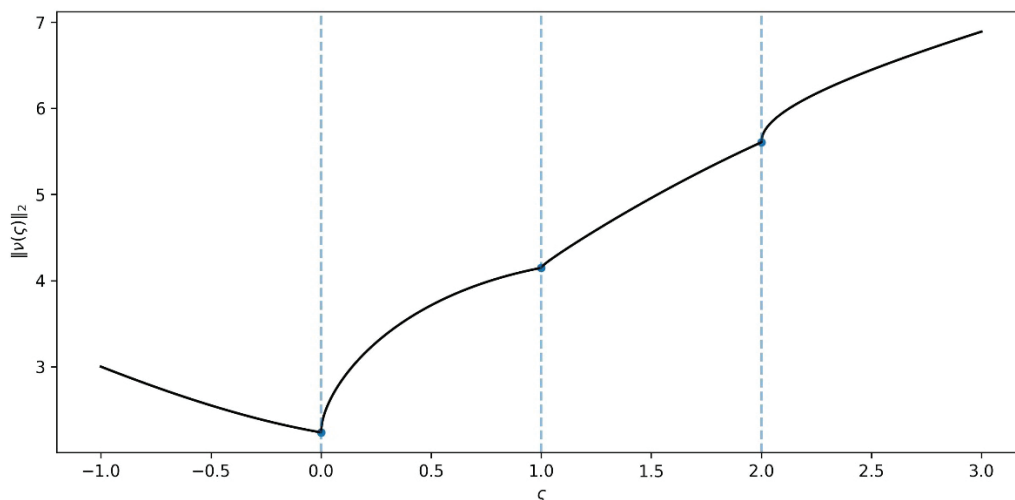


Figure 4. Infinity norm of the solution $\|v(\varsigma)\|_\infty$ for Example 13. The dashed horizontal line indicates the prescribed stability bound $\varepsilon = 0.80$, while the vertical dashed lines mark the junction points $\varsigma = 0, 1, 2$. The trajectory remains within the bound, confirming finite-time stability on $J = [0, 3]$.

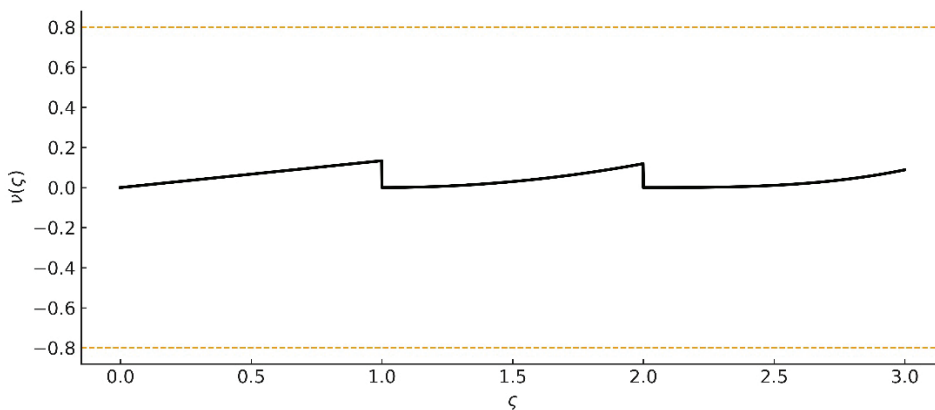


Figure 5. Solution profile illustrating finite-time stability for the system to the system 25.

$\varphi(\zeta) \equiv \delta = 0.05$ for $\zeta \in [-1, 0)$. The dashed horizontal lines correspond to the prescribed bound $|v(\zeta)| = \varepsilon = 0.80$. It is clearly observed that the entire trajectory of $v(\zeta)$ remains within the stability margin $|v(\zeta)| < \varepsilon$, confirming the finite-time stability condition $S(3) \approx 9.99 < \varepsilon/\delta = 16$ predicted by Theorem 9.

We emphasize that the above finite-time stability verification is not restricted to constant histories. Indeed, the stability condition in Theorem 9 depends only on the bound $\|\varphi\|_\infty < \delta$ and not on the specific functional form of $\varphi(\zeta)$. For instance, choosing a non-constant history $\varphi(\zeta) = \delta(1 + \zeta)$ for $\zeta \in [-1, 0)$, one still has $\|\varphi\|_\infty = \delta = 0.05$ because

$$\|\varphi\|_\infty = \sup_{\zeta \in [-1, 0)} |\delta(1 + \zeta)| = \delta \sup_{\zeta \in [-1, 0)} (1 + \zeta).$$

Since $1 + \zeta$ is continuous and monotonically increasing on $[-1, 0)$, its maximum is attained at $\zeta \rightarrow 0^-$, yielding

$$\sup_{\zeta \in [-1, 0)} (1 + \zeta) = 1.$$

Hence,

$$\|\varphi\|_\infty = \delta = 0.05.$$

Hence, the inequality $S(3) < \varepsilon/\delta$ remains unchanged and the homogeneous piecewise fractional system (25) is finite-time stable on $J = [0, 3]$. This shows that the numerical validation applies to a broader class of initial histories, including non-constant functions.

The numerical trajectory shown in Figure 5 remains strictly within the prescribed bound $|v(\zeta)| < \varepsilon$ for all $\zeta \in [0, 3]$, providing a direct visual confirmation of the finite-time stability result and demonstrating the practical sharpness of the analytical estimate.

Note that the figures presented in the examples are intended to complement the analytical results by providing a direct visual interpretation of the explicit solution construction and the finite-time stability analysis. In particular,

the solution trajectories illustrate the continuity of the piecewise-defined solutions at the switching points, while the plotted norms demonstrate that the solutions remain within the prescribed stability bounds over the finite time interval. These graphical representations therefore serve as numerical confirmations of the theoretical findings, bridging the gap between the abstract analytical conditions and their observable dynamical behavior.

CONCLUSION

We develop an explicit interval-wise solution for linear piecewise fractional systems with Caputo derivatives and a single delay via a generating-function approach. The solution is continuous across junctions, and a computable sum inequality yields a finite-time stability condition. A couple of examples are presented to validate our findings.

The originality of this work lies in the explicit construction of solutions for piecewise fractional delay systems together with a directly applicable finite-time stability criterion, extending existing results that are mainly based on implicit formulations or restrictive assumptions.

There are many directions for future work. For example, is our system controllable? Is it exponentially or asymptotically stable? One may also consider the nonhomogeneous type of the system under study.

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AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The authors declared no conflict of interest.

ETHICS

There are no ethical issues with the publication of this manuscript.

STATEMENT ON THE USE OF ARTIFICIAL INTELLIGENCE

The authors declared that no AI tools were used in the development of the scientific content of this paper; however, minor language and stylistic revisions were carried out.

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