Sigma J Eng Nat Sci, Vol. 44, No. 3, pp. 1915–1924, June, 2026

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2026.2072

Research Article

Accelerating cosmological universe with bilinear varying deceleration parameter in Lyra's geometry

R. SANTHIKUMAR[1,*], M. KRISHNA[2], K. Sobhan BABU[3]

[1]*Aditya Institute of Technology and Management, Tekkali, Srikakulam Dist, Andhra Pradesh, 532201, India*
[2]*Raghu Engineering College, Visakhapatnam, Andhra Pradesh, 531162, India*
[3]*Jawaharlal Nehru Technological University Kakinada University College of Engineering, Narasaraopeta, Andhra Pradesh, 522601, India*

ARTICLE INFO

Article history
Received: 29 October 2024
Revised: 30 January 2025
Accepted: 19 February 2025

Keywords:
Accelerating Expansion; Bilinear Varying Deceleration Parameter; Bulk Viscosity; Cosmic String

ABSTRACT

This paper investigates an anisotropic cosmological model Bianchi Type-VI_0 in the frame work of Lyra's geometry with barotropic matter, dark energy and bulk viscous effects using bilinearly varying deceleration parameter. The physical and kinematical properties will be studied.

Cite this article as: Santhikumar R, Krishna M, Babu KS. Accelerating cosmological universe with bilinear varying deceleration parameter in Lyra's geometry. Sigma J Eng Nat Sci 2026;44(3):1915–1924.

INTRODUCTION

By the motivation of several extensions of Riemannian geometry study. One of the most influential attempts was introduced by Hermann Weyl in 1918 [1], who proposed a generalised geometric framework to achieve a unified description of gravitational and electromagnetic interactions within one spacetime manifold. Although Weyl's theory represented a remarkable step toward unified field theory, it was affected by the non-integrability property of length transport, which led to serious physical objections and limited its acceptance.

In order to overcome this difficulty, A modified geometrical structure of Riemannian geometry was proposed by Günther Lyra in 1951 [2] by incorporating a gauge function into a structureless manifold. Lyra's modification successfully removed the non-integrability problem associated with Weyl's geometry and naturally introduced a displacement vector field into the gravitational framework. Motivated by this geometrical formulation, D. K. Sen [3], and Sen and Dunn [4], established a scalar–tensor theory of gravity founded on Lyra geometry. Further developments by W. D. Halford [5,6] demonstrated that the displacement vector field ϕ_i remains constant in Lyra geometry.

To understand the study of early stage of universe , we must study the high energy physical processes like symmetric phase breaking and trasition from one stage to another stage to produced topological relics in the form ofdomain walls, cosmic strings and monopoles. Out of these, the cosmic strings are more stable one-dimensional massive structures, and hence they are more important for the formation of large-scale cosmic structure, including galaxy clusters

***Corresponding author.**
*E-mail address: skrmahanthi@gmail.com

This paper was recommended for publication in revised form by Editor-in-Chief Ahmet Selim Dalkilic

Published by Yıldız Technical University Press, İstanbul, Turkey

and galaxies. By this motivation, String-based cosmological models have been widely analysed in classical general relativity and extended theories of gravity. Important contributions in this direction were made by researchers such as Patricio Letelier, Krori, Reddy, and Tripathy [7–13], who studied different classes of string-dominated cosmological models and analyzed their influence on the evolution and dynamics of the universe. Several investigations carried out by Bali [14], Rao [15], Reddy [16], Naidu [17], and Vijayasanthi [18] examined the combined effects of bulk viscosity, cosmic strings, and modified gravitational theories on the large-scale evolution of the universe.

The evolution of the universe is commonly described using the Hubble parameter H and the deceleration parameter q, both of which are determined by the behaviour of the cosmic scale factor $a(t)$. The observational discovery of the accelerated expansion of the universe through type Ia supernovae by Saul Perlmutter, Adam Riess, and their collaborators [19–21] generated significant interest in cosmological models with variable deceleration parameters. Earlier, M. S. Berman [22] introduced a constant deceleration parameter model that provided exact cosmological solutions, although it could not successfully explain the transition from a decelerating phase to an accelerating phase of the universe. To overcome this drawback, Özgür Akarsu and Cengiz Dereli [23], together with Misra [24–27], proposed cosmological models based on time-dependent deceleration parameters. In particular, the linearly varying deceleration parameter (LVDP) and bilinearly varying deceleration parameter (BVDP) models offer more realistic descriptions of cosmic evolution by explaining various stages of expansion, including the smooth transition from early-time deceleration to the presently observed accelerated expansion.

Modern observational evidence obtained from type Ia supernovae, CMB radiation, baryon acoustic oscillations [28–36]. Several researchers, including John D. Barrow [37], Santhikumar [38], and Krishna [39], studied Lyra geometry in different alternative theories of gravitation.

Motivated by these developments, the present work investigates a Bianchi type-VI_0 Lyra geometrical cosmological model by considering cosmic strings, bulk viscous fluid, with BVDP. The study aims to examine the influence of anisotropy, viscosity, and variable expansion dynamics on the evolution of the universe and to analyze the physical and geometrical behaviour of the resulting cosmological model in the light of modern observational cosmology.

Metric and Field Equations

Consider a Bianchi type-VI_0 anisotropic spacetime with spatial homogeneity is

$$ds^2 = -dt^2 + A^2dx^2 + B^2e^{2x}dy + C^2e^{-2x}dz^2 \quad (1)$$

Using the standard gauge in Lyra manifold, the modified form of Einstein's field equations can be written as

$$R_i^j - \frac{1}{2}g_i^j R + \left(\frac{3}{2}\phi_i\phi^j - \frac{3}{4}\phi_k\phi^k g_i^k\right) = -T_{ij} \quad (2)$$

Where the displacement vector $\phi_i = (0,0,0,\beta(_t))$

Here T_{ij} represents the energy–momentum tensor for a bulk viscous fluid coupled with a one-dimensional string distribution

$$T_{ij} = (\rho + \overline{p})u_iu_j + \overline{p}g_{ij} - \lambda x_ix_j \quad (3)$$

In this expression, λ represents the string tension, whereas ρ stands for the proper density of the system

The total pressure is

$$\overline{p} = p - 3\xi H \quad (4)$$

In this context, p is the proper pressure, $3\xi H$ corresponds to the bulk viscous pressure, $\xi(t)$ denotes the bulk viscosity coefficient, and H represents the Hubble expansion parameter.

$u^i = \delta_\phi^i$ is a four-velocity vector and x^i space-like vector representing the strings' anisotropic direction.

$$g_{ij}x^ix_j = 1, \quad g_{ij}u^iu_j = -1 \text{ and } u^ix_i = 0 \quad (5)$$

The energy density of particles attached to the one-dimensional cosmic string is expressed as

$$\rho_\rho = \rho - \lambda \quad (6)$$

For the metric (1), from the field equations (2) –(5) are

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + \frac{1}{A^2} + \frac{3}{4}\beta^2 = -(\overline{p} - \lambda) \quad (7)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{1}{A^2} + \frac{3}{4}\beta^2 = -(\overline{p} - \lambda) \quad (8)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{1}{A^2} + \frac{3}{4}\beta^2 = -(\overline{p} - \lambda) \quad (9)$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{1}{A^2} - \frac{3}{4}\beta^2 = \rho \quad (10)$$

$$\frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0 \quad (11)$$

By conservative condition

$$T_{i;j}^j = 0 \quad (12)$$

Which leads to

$$\dot{\rho} + (\overline{p} + \rho)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 0 \quad (13)$$

Concentration on the LHS of equation (2) leads to

$$\left(R_i^j - \frac{1}{2}g_i^jR\right) + \left(\frac{3}{2}\phi_i\phi^j\right)_{;j} - \frac{3}{4}\left(\phi_k\phi^k g_i^j\right)_{;j} = 0 \quad (14)$$

which leads to

$$\frac{3}{2}\beta\dot{\beta} + \frac{3}{2}\beta^2\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 0 \quad (15)$$

Integrating equation (11), we have that,

$$B = lC \tag{16}$$

take $l = 1$

$$\text{Hence, } B = C \tag{17}$$

The average physical parameters are as followes

$$a = (AB^2)^{\frac{1}{3}} \tag{18}$$

$$V = (a)^3 = AB^2 \tag{19}$$

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \tag{20}$$

$$\theta = 3H = 3\frac{\dot{a}}{a} \tag{21}$$

$$\sigma = \frac{1}{2}\left[\left(\frac{\dot{A}}{A}\right)^2 + 2\left(\frac{\dot{B}}{B}\right)^2\right] - \frac{1}{6}\theta^2 \tag{22}$$

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -\left(\frac{\dot{H}+H^2}{H^2}\right) = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 \tag{23}$$

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 \tag{24}$$

Where $i = 1, 2, 3$

Solution of Field Equations

By Eq.(23) , we get

$$\frac{1}{H} = \int[1 + q(t)]dt + k_1 \tag{25}$$

$$H = \frac{1}{\int[1+q(t)]dt + k_1} \tag{26}$$

Since deceleration parameter (DP) models play an vital role to stidy the universe's expansion history, the Bilinear Varying Deceleration Parameter (BVDP) model is a more general approach than LVDP) and CDP**.** The advantages of BVDP over LVDP and CDP are that it can describe multiple evolutionary phases, including deceleration, acceleration, and potential future deceleration. Predicts an earlier transition to acceleration compared to LVDP.

we consider BVDP "q" by Mishra et al. (2016) is

$$q(t) = \frac{\alpha(1-t)}{1+t} \; for \; \alpha \geq 0 \tag{27}$$

By equation (24), we get

$$\frac{1}{H} = \int\left[1 + \frac{\alpha(1-t)}{1+t}\right]dt + k_1 = (1-\alpha)t + 2\alpha \, ln(1+t) + k_1$$
$$H = \frac{1}{(1-\alpha)t + 2\alpha \, ln(1+t) + k_1} \tag{28}$$

Since, as $t \to 0$, $H \to \infty$ for physical visibility, consider $k_1 = 0$,

During the early inflationary epoch, the expansion rate of the universe is extremely high

$$H = \frac{\dot{a}}{a} = \frac{1}{(1+\alpha)t} + \frac{\alpha}{(1+\alpha)^2} + \frac{-2\alpha+\alpha^2}{3(1+\alpha)}t + \frac{3\alpha-2\alpha^2+\alpha^3}{6(1+\alpha)^4}t^2 + \frac{-18\alpha+11\alpha^2-14\alpha^3+2\alpha^4}{45(1+\alpha)^5}t^3 + O(t^4) \tag{29}$$

Integrating on both sides,

$$a(t) = a_0 \, t^{\left(\frac{1}{1+\alpha}\right)} . e^{G(t)} \tag{30}$$

where

$$G(t) = \left\{\left[\frac{\alpha}{(1+\alpha)^2}\right]t + \left[\frac{-2\alpha+\alpha^2}{6(1+\alpha)}\right]t^2 + \left[\frac{3\alpha-2\alpha^2+\alpha^3}{18(1+\alpha)^4}\right]t^3 + \left[\frac{-18\alpha+11\alpha^2-14\alpha^3+2\alpha^4}{180(1+\alpha)^5}\right]t^4\right\} + O(t^5) \tag{31}$$

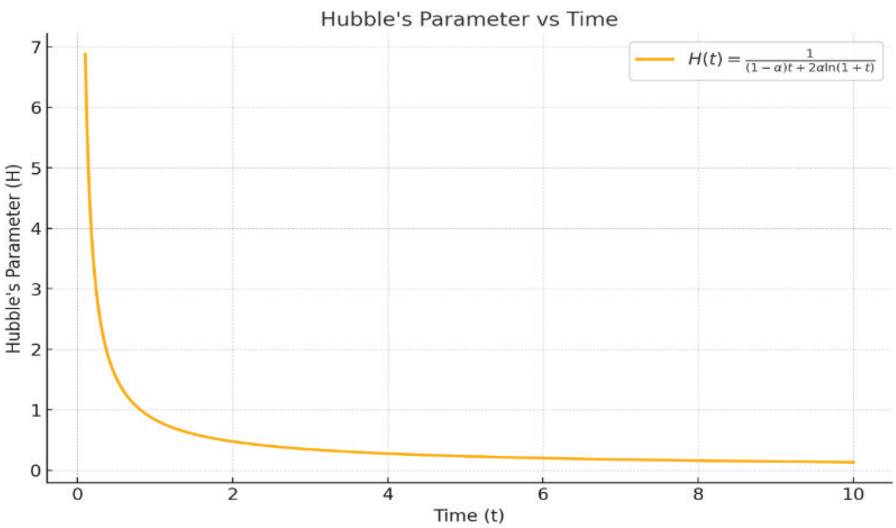

Figure 1. Hubble's parameter V_s cosmictime t.

Where a_0 is the constant of integration.

The behaviour from Figure 1 of Hubble's parameter $H = \frac{1}{(1-\alpha)t+2\alpha\, ln(1+t)}$ over time. This indicates rapid expansion during the early universe; the rate of decrease slows, reflecting a transition phase in the universe's evolution during the middle stage, and then stabilises and approaches zero, showing slower expansion in the late universe.

$$A = \left[a_0 . t^{\left(\frac{1}{1+\alpha}\right)} . e^{G(t)}\right]^{\frac{3}{2n+1}} \tag{32}$$

$$B = C = A^n = \left[a_0 . t^{\left(\frac{1}{1+\alpha}\right)} . e^{G(t)}\right]^{\frac{3n}{2n+1}} \tag{33}$$

Using $a_0 = 1$the metric equation (1), we can write it as

$$\begin{aligned} ds^2 = -dt^2 + \left[a_0 . t^{\left(\frac{1}{1+\alpha}\right)} . e^{G(t)}\right]^{\frac{6}{2n+1}} dx \\ + \left[a_0 . t^{\left(\frac{1}{1+\alpha}\right)} . e^{G(t)}\right]^{\frac{6n}{2n+1}} (e^{2x} dy \; + \; e^{-2x}\, dz) \end{aligned} \tag{34}$$

Physical Properties of the Model

Equation (34) describes the Bianchi type-VI_0 cosmological model in Lyra geometry in the presence of cosmic strings and bulk viscosity with BVDP.

We obtain the average physical and kunematical parameters are

$$V = a^3 = a_0{}^3 t^{\frac{3}{1+\alpha}} . e^{3G(t)} \tag{35}$$

$$\theta = 3H = \frac{3}{(1-\alpha)t+2\alpha\, ln(1+t)} \tag{36}$$

$$\sigma^2 = \frac{3}{2}\left[\frac{3(1+2n^2)}{(2n+1)^2} - 1\right]\left[\frac{1}{(1-\alpha)t+2\alpha\, ln(1+t)}\right]^2 \tag{37}$$

Clearly

$$\frac{\sigma^2}{\theta^2} = \frac{1}{3}\frac{(n-1)^2}{(2n+1)^2} \neq 0 \tag{38}$$

Since

$$\begin{aligned} H &= \frac{\dot a}{a} = \frac{1}{3}\left(\frac{\dot A}{A} + \frac{\dot B}{B} + \frac{\dot C}{C}\right) = \frac{1}{3}\left(\frac{\dot A}{A} + 2\frac{\dot B}{B}\right) \\ H &= \frac{1}{3}\left(\frac{\dot A}{A} + 2n\frac{\dot A}{A}\right) = \left(\frac{2n+1}{3}\right)\frac{\dot A}{A} \end{aligned} \tag{39}$$

$$\begin{aligned} H_1 &= \frac{\dot A}{A} = \frac{3H}{2n+1} = \frac{3}{2n+1}\left[\frac{1}{(1-\alpha)t+2\alpha\, ln(1+t)}\right], \\ H_2 = H_3 &= \frac{\dot B}{B} = \frac{\dot C}{C} = n\frac{\dot A}{A} = \frac{3n}{2n+1}\left[\frac{1}{(1-\alpha)t+2\alpha\, ln(1+t)}\right] \end{aligned} \tag{40}$$

$$A_m = \frac{2(n-1)^2}{(2n+1)^2} \tag{41}$$

Integrating equation (15), we get

$$\begin{aligned} \beta &= \beta_0 a^{-1} \\ \beta &= \beta_0 \left[a_0 t^{\frac{1}{1+\alpha}} . e^{G(t)}\right]^{-1} \end{aligned} \tag{42}$$

Using equation (10), we get

$$\rho = \frac{3(n^2+2n)}{2n+1}\left(\frac{\dot a}{a}\right) - \frac{1}{a^{\frac{6}{2n+1}}} - \frac{3}{4}(\beta_0 a^{-1})^2 \tag{43}$$

$$\begin{aligned} \rho = \frac{3n(n+2)}{2n+1}\left[\frac{1}{(1-\alpha)t+2\alpha\, ln(1+t)}\right] - \left[a_0 . t^{\left(\frac{1}{1+\alpha}\right)} . G(t)\right]^{-\frac{6}{2n+1}} \\ -\frac{3}{4}(\beta_0{}^2)\left[a_0 . t^{\left(\frac{1}{1+\alpha}\right)} . G(t)\right]^{-2} \end{aligned} \tag{44}$$

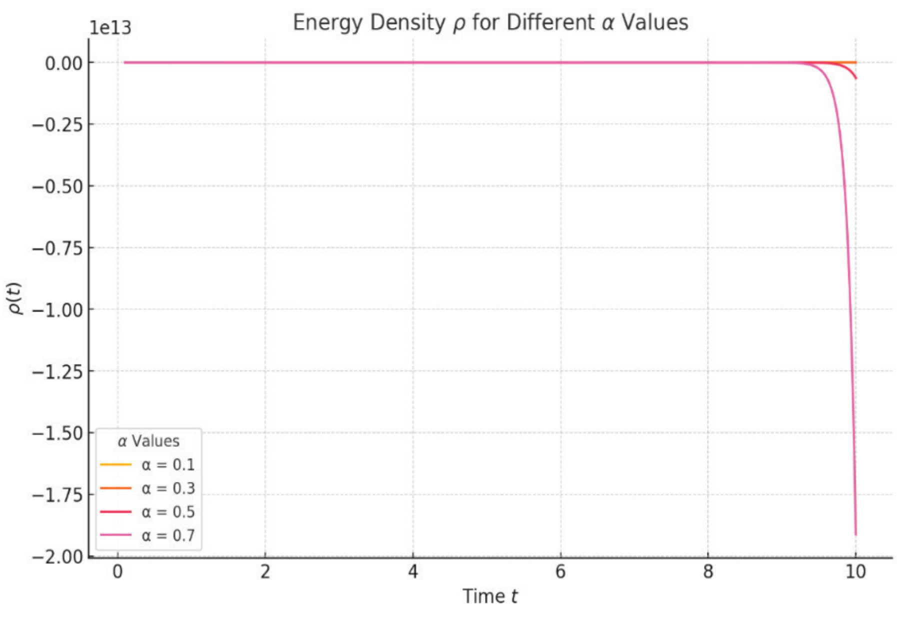

Figure 2. Energy density $\rho(t)$ V_s cosmictime t.

Figure 2 shows that the behaviour indicates high energy density in the early stages of evolution. In the middle stage, the impact of exponential terms $F(t)$ begins to moderate the growth, leading to a smoother transition. At late times, the energy density $\rho(t)$ stabilises or asymptotically approaches smaller values as exponential decay dominates. Smaller α leads to steeper growth at early times and slower decay at late times. More prominent moderates the energy density, leading to a more stable behaviour.

As we observed the behaviour , the energy density increases sharply as $t \to 0$, showing a modelis has singularity. Since the first term, $\frac{3n(n+2)}{2n+1}\left[\frac{1}{(1-\alpha)t+2\alpha\, ln(1+t)}\right]$, dominates because the denominator $(1-\alpha)t+2\alpha\, ln(1+t)$. It becomes small for small t, leading to a rapid increase in energy density and to singular behaviour. Possible sharp changes or divergences might occur at specific intermediate times, depending on the balance among polynomial, logarithmic, and exponential terms. The contributions from the $\left[a_0 t^{\frac{1}{1+\alpha}} F(t)\right]^{-\frac{6}{2n+1}}$ terms are minimal because the factors F(t) and $t^{\frac{1}{1+\alpha}}$ are small for $t \to 0$. The factor $F(t)$, being a series expansion, grows gradually, and its impact on the energy density becomes noticeable. At Late Times ($t \to \infty$), Energy density asymptotically approaches a constant or nearly constant value governed by the last exponential terms. Stabilisation reflects the model's convergence to a steady state in the late universe. Because the terms involving $\left[a_0 t^{\frac{1}{1+\alpha}} F(t)\right]^{-\frac{6}{2n+1}}$ dominate because the exponential decay and polynomial components become stable as $F(t)$ reaches an asymptotic behaviour. The first term diminishes as $\left(\frac{1}{(1-\alpha)t+2\alpha\, ln(1+t)}\right)$ to zero.

Clearly, it follows that in the initial stages of the universe, ρ grows rapidly due to the dominance of the first term, reflecting instability or singular behaviour. The model transitions to a phase of stabilisation, in which competing terms determine the energy density's behaviour, leading to a smoother evolution during the intermediate stage. And, Exponential and polynomial terms dominate, causing ρ to stabilise, consistent with the late-time behaviour of an expanding universe.

Since $\rho = r\lambda$ we have the string tension density is

$$\lambda = \frac{\rho}{r} = \frac{1}{r}\left[\frac{3n(n+2)}{2n+1}\left[\frac{1}{(1-\alpha)t+2\alpha\, ln(1+t)}\right] - \left[a_0 . t^{\left(\frac{1}{1+\alpha}\right)} . F(t)\right]^{-\frac{6}{2n+1}} - \frac{3}{4}\left(\beta_0{}^2\right)\left[a_0 . t^{\left(\frac{1}{1+\alpha}\right)} . F(t)\right]^{-2}\right] \tag{45}$$

Where r is an arbitrary constant.

The graph (Figure 3) illustrates the behaviour of $\lambda(t)$ for different r values. The curves demonstrate how $\lambda(t)$ varies with time t, showing distinct behaviours for different r values. We observed that for smaller values of r (e.g., r = 0.5), $\lambda(t)$ shows steeper variations, indicating more significant changes in early times. As r increases (e.g., r = 5), the variations in $\lambda(t)$ become smoother, and the overall magnitude decreases. At later times ($t \to \infty$), all curves exhibit stabilization trends depending on the dominant terms in the expression.

The energy density of the particles coupled with the string is defined as

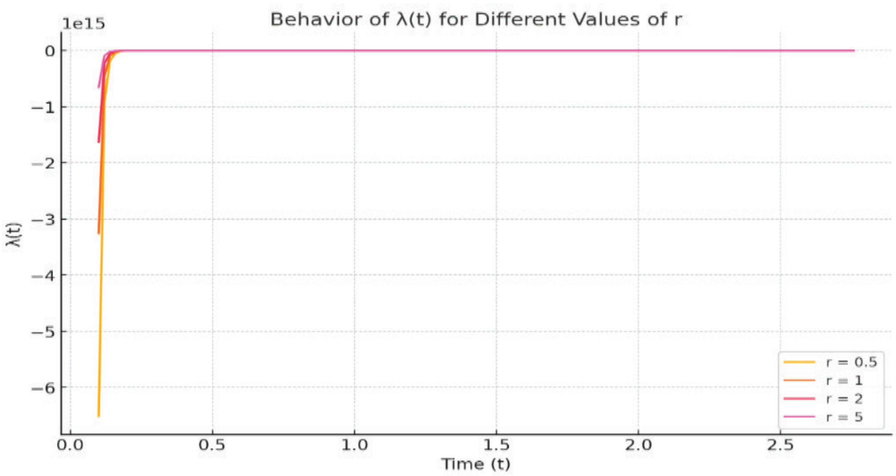

Figure 3. String tension density $\lambda(t)$ $V_{s.}$ cosmictime t.

$$\rho_\rho = \left(\frac{r-1}{r}\right)\rho = \left(\frac{r-1}{r}\right)\left\{\frac{3n(n+2)}{2n+1}\left[\frac{1}{(1-\alpha)t+2\alpha\, ln(1+t)}\right] - \left[a_0.t^{\left(\frac{1}{1+\alpha}\right)}.F(t)\right]^{-\frac{6}{2n+1}} - \frac{3}{4}\left(\beta_0{}^2\right)\left[a_0.t^{\left(\frac{1}{1+\alpha}\right)}.F(t)\right]^{-2}\right\} \tag{46}$$

Figure 4 illustrates the behaviour of ρ_p for varying values of r. As r changes, the energy density shows distinct trends, emphasising the impact of r on the model's evolution with respect to time. This visualisation helps to understand how ρ_ρ behaves across different r values, with early-time rapid changes and late-time stabilisation or decay.

Pressure

$$p = \omega\rho = \omega\left[\frac{3n(n+2)}{2n+1}\left[\frac{1}{(1-\alpha)t+2\alpha\, ln(1+t)}\right] - \left[a_0.t^{\left(\frac{1}{1+\alpha}\right)}.G(t)\right]^{-\frac{6}{2n+1}} - \frac{3}{4}\left(\beta_0{}^2\right)\left[a_0.t^{\left(\frac{1}{1+\alpha}\right)}.G(t)\right]^{-2}\right] \tag{47}$$

The graph (Figure 5) gives the pressure $p(t)$ as a function of time with respect to different values of ω. Each curve represents a specific ω value, and the variations highlight how ω influences the pressure evolution: if $\omega < 0$, Pressure values tend to be negative, representing a contracting or repulsive phase. For Zero ω $(\omega = 0)$, it represents a neutral scenario in which the pressure behaviour stabilises without additional scaling effects. For Positive $\omega(\omega > 0)$, which leads to positive pressure values associated with an accelerating universe or an expanding phase.

The coefficient of Bulk viscosity is

Since

$$p - 3\xi H = \varepsilon\rho \Rightarrow \omega\rho - 3\xi H = \varepsilon\rho \Rightarrow (\omega-\varepsilon)\rho = 3\xi H \Rightarrow \xi = \frac{\rho(\omega-\varepsilon)}{3H}$$

$$\xi = \frac{\omega-\varepsilon}{3}\left[\frac{\frac{3n(n+2)}{2n+1}\left[\frac{1}{(1-\alpha)t+2\alpha\, ln(1+t)}\right] - \left[a_0 t^{\frac{1}{1+\alpha}}.G(t)\right]^{-\frac{6}{2n+1}} - \frac{3}{4}\left(\beta_0{}^2\right)\left[a_0 t^{\frac{1}{1+\alpha}}.G(t)\right]^{-2}}{\frac{1}{(1-\alpha)t+2\alpha\, ln(1+t)}}\right]$$

$$\xi = \frac{\omega-\varepsilon}{3}[(1-\alpha)t + 2\alpha\, ln(1+t)]\left[\frac{3n(n+2)}{2n+1}\left[\frac{1}{(1-\alpha)t+2\alpha\, ln(1+t)}\right] - \left[a_0.t^{\left(\frac{1}{1+\alpha}\right)}.G(t)\right]^{-\frac{6}{2n+1}} - \frac{3}{4}\left(\beta_0{}^2\right)\left[a_0.t^{\left(\frac{1}{1+\alpha}\right)}.G(t)\right]^{-2}\right] \tag{48}$$

Figure 6 illustrates the behaviour of $\xi(t)$ as a function of cosmic time t for different values of ω. At the early stage of the formation of the universe, the values of $\xi(t)$ vary significantly. For smaller values of ω, the initial value of $\xi(t)$ is relatively minor compared to higher ω. At the middle stage formation of the universe, as time progresses, the rate of change of $\xi(t)$ reduces, indicating stabilisation. For Late Times $(t \to \infty)$, for all ω, $\xi(t)$ stabilises or approaches asymptotic values, as the exponential decay dominates the behaviour. The impact of ω is higher: higher ω results in a larger magnitude of $\xi(t)$ across all times, reflecting the proportional dependence of $\xi(t)$ on ω. This analysis highlights the role of the parameter ω in determining the temporal evolution of $\xi(t)$, with smooth stabilisation observed at later times.

Clearly $\rho = r\lambda$, $\rho_\rho = \left(1-\frac{1}{r}\right)\rho, \overline{p} = p - 3\xi H = \varepsilon\rho$, indicates that $\lambda \propto \rho, \rho_\rho \propto \rho,\ \overline{p} \propto p,\ p \propto \rho, \xi \propto \rho$. Therefore, the behaviour of λ, ρ_ρ, $\bar{p}$, p, ξ are the same as the behaviour of energy density ρ. i.e., we concluded as at early times, ρ, λ, ρ_ρ, $\bar{p}$, p, ξ exhibits rapid growth due to the

Figure 4. ρ_ρ V_s. cosmictime t.

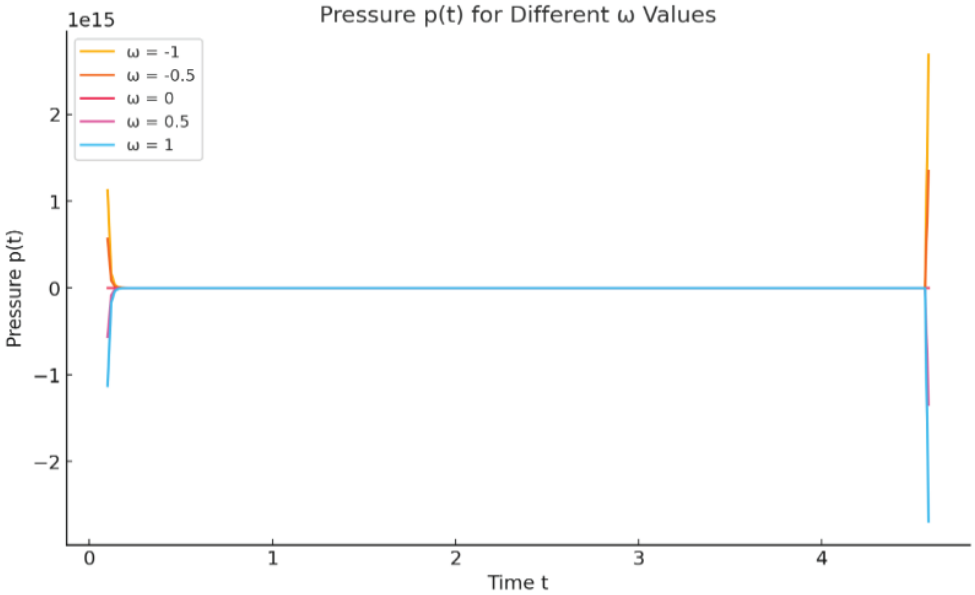

Figure 5. Pressure $p(t)$ $V_{s.}$ cosmic time.

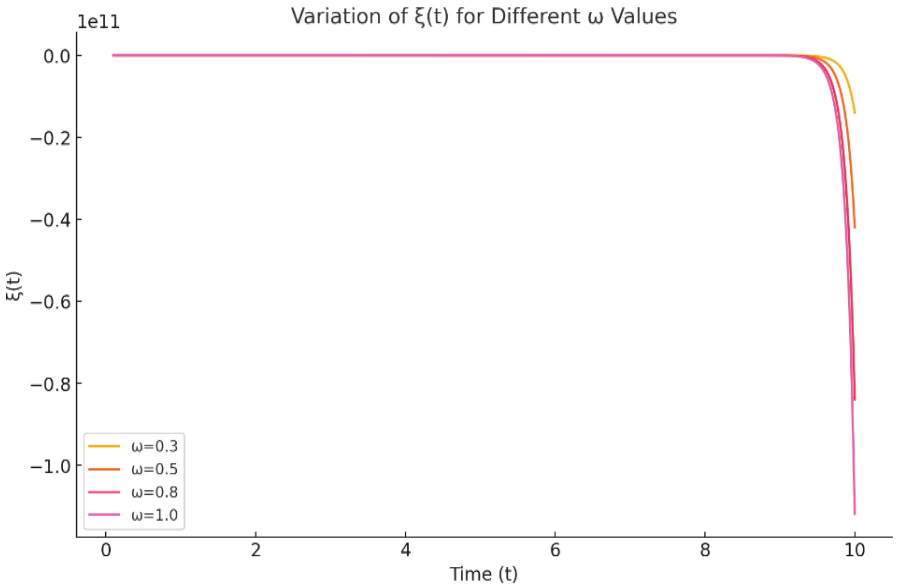

Figure 6. The coefficient of bulk viscosity $\xi(t)$ V_s. cosmictime t.

dominance of the first term, reflecting instability or singular behaviour at the initial stages of the universe. At middle of the formation of universe the model transitions to a phase of stabilisation. At Late Times**,** Exponential and polynomial terms dominate, causing ρ, λ, ρ_p, $\bar{p}$, p, ξ to stabilise, consistent with the late-time behaviour of an expanding universe.

The Behavior of the Displacement Vector $\beta(t)$

From equation (15), we have a displacement vector.

$$\beta(t) = \beta_0 a^{-3} = \beta_0 \left[a_0 t^{\frac{1}{1+\alpha}}. e^{G(t)}\right]^{-3} \tag{49}$$

From the graph (figure 7), the behaviour of the displacement vector can be explained as follows: Early Time ($t \to 0$), the term. $t^{\frac{1}{1+\alpha}}$ dominates the behaviour of the scale factor a, causing $\beta(t)$ to show rapid growth or decay depending on the parameter values. $F(t)$ has minimal contribution at small t, making the power-law term the primary driver.

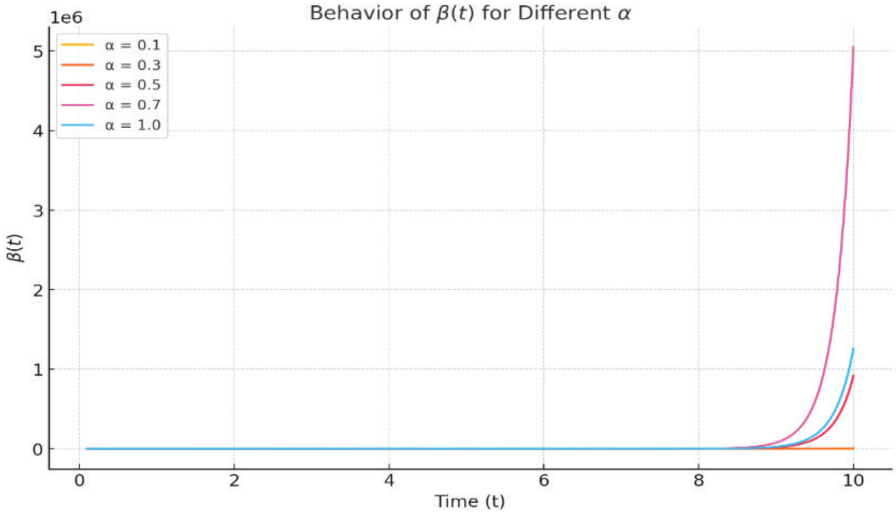

Figure 7. Displacement vector $\beta(t)$ $V_{s.}$ time t.

At the intermediate time**,** as t increases, the exponential term $e^{F(t)}$ becomes significant. $F(t)$, with its higher-order polynomial corrections, contributes to a gradual growth/decay rate adjustment. At late time ($t \to \infty$), the scale factor $a(t)$ grows significantly, causing $\beta(t)$ to decay towards zero asymptotically. The exponential term $e^{F(t)}$ stabilises, leading to smoother behaviour of $\beta(t)$ at larger t. This behaviour demonstrates how $\beta(t)$ transitions through various phases of cosmic evolution, controlled by the power-law and exponential contributions from $a(t)$.

CONCLUSION

In this paper, the field equations for the Bianchi type-IV_0 cosmological model within the framework of Lyra geometry are obtained by considering a bilinearly varying deceleration parameter in the presence of cosmic strings and bulk viscous fluid. The analysis reveals that the BVDP model provides more realistic, physically acceptable results than the LVDP and constant deceleration parameter (CDP) models. In particular, the BVDP model predicts an earlier transition from decelerated to accelerated expansion with a comparatively smaller deceleration parameter, making it better suited to explaining the universe's current accelerated phase. Hence, the BVDP approach appears to be an effective framework for investigating several cosmological phenomena while remaining compatible with current observational evidence.

From the above analysis, the spatial volume becomes zero at the primordial stage of the universe and gradually increases with cosmic time, signifying the late-time cosmic expansion. The physical parameters such H, ρ, σ, θ, and p attain very large values at $t = 0$, while all these quantities decrease and approach zero when $t \to \infty$. Since A_m remains constant throughout the evolution, the general dynamics of the universe do not change over time. Moreover, the constancy of the ratio σ^2 / θ^2 indicates that the model does have anisotropy. The parameter β, being inversely related to the deceleration parameter, has a large value at the initial stage and gradually diminishes during the late accelerated phase of the universe. Therefore, the obtained cosmological model represents an expanding universe without an initial singularity.

The behaviour of the string tension density λ, particle density ρ_p, effective pressure p^{-}, isotropic pressure p, and bulk viscous coefficient ξ strongly depends on the evolution of the energy density ρ. During the early universe, these quantities increase rapidly due to the dominance of the leading terms, marking an unstable, highly energetic phase of cosmic evolution. At the intermediate stage, the competing contributions from different terms lead to smoother, more stable behaviour of the physical parameters. At late times, the exponential and polynomial contributions dominate the dynamics, causing ρ, λ, ρ_p, p^{-}, p, and ξ to stabilise gradually, which is consistent with the observed behaviour of the present expanding universe.

The evolution of the displacement field $\beta(t)$ Further supports the dynamical Nature of the model. In the early epoch ($t \to 0$), the power-law term $t^{\frac{1}{1+\alpha}}$ governs the behaviour of the scale factor, resulting in a rapid variation of $\beta(t)$. In this region, the contribution of the exponential term remains negligible. However, at later times ($t \to \infty$), the scale factor increases significantly, forcing $\beta(t)$to decay asymptotically towards zero. Simultaneously, the exponential contribution stabilises the overall behaviour, leading to a smooth cosmic evolution. This transition illustrates how the displacement field evolves through different cosmological phases under the combined influence of power-law and exponential terms.

Overall, the study of anisotropic models contributes significantly to our understanding of cosmic evolution and large-scale structure formation, while providing potential observational tests for future research.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

STATEMENT ON THE USE OF ARTIFICIAL INTELLIGENCE

Artificial intelligence was not used in the preparation of the article.

REFERENCES

[1] Weyl H. Gravitation und Elektrizität. Sitzungsber Preuss Akad Wiss Berlin Math Phys 1918;1918:465–480.

[2] Lyra G. Über eine Modifikation der Riemannschen Geometrie. Math Z 1951;54:52–64. [CrossRef]

[3] Sen DK. A static cosmological model. Z Phys 1957;149:311–323. [CrossRef]

[4] Sen DK, Dunn KA. A scalar-tensor theory of gravitation in a modified Riemannian manifold. J Math Phys 1971;12:578–586. [CrossRef]

[5] Halford WD. Cosmological theory based on Lyra's geometry. Aust J Phys 1970;23:863–869. [CrossRef]

[6] Halford WD. Scalar-tensor theory of gravitation in a Lyra manifold. J Math Phys 1972;13:1699–1703. [CrossRef]

[7] Letelier PS. String cosmologies. Phys Rev D 1983;28:2414–2419. [CrossRef]

[8] Krori KD, Chaudhury T, Mahanta CR, Mazumdar A. Some exact solutions in string cosmology. Gen Relativ Gravit 1990;22:123–130. [CrossRef]

[9] Reddy DRK. A string cosmological model in Brans-Dicke theory of gravitation. Astrophys Space Sci 2003;286:365–371. [CrossRef]

[10] Reddy DRK. A string cosmological model in a scalar-tensor theory of gravitation. Astrophys Space Sci 2003;286:359–363. [CrossRef]

[11] Tripathy SK, Sahu SK, Routray TR. String cloud cosmologies for Bianchi type-III models with electromagnetic field. Astrophys Space Sci 2008;315:105–110. [CrossRef]

[12] Tripathy SK, Nayak SK, Sahu SK, Routray TR. Massive string cloud cosmologies in Saez-Ballester theory of gravitation. Int J Theor Phys 2009;48:213–225. [CrossRef]

[13] Tripathy SK, Nayak SK, Sahu SK, Routray TR. Time-dependent viscous string cloud cosmological models. Astrophys Space Sci 2009;323:91–96. [CrossRef]

[14] Bali R, Pareek UK, Pradhan A. Bianchi type-I massive string magnetized barotropic perfect fluid cosmological model in general relativity. Chin Phys Lett 2007;24:2455–2458. [CrossRef]

[15] Rao VUM, Vijaya Santhi M, Vinutha T, Sree Devi Kumari G. LRS Bianchi type-I dark-energy cosmological model in Brans-Dicke theory of gravitation. Int J Theor Phys 2012;51:3303–3310. [CrossRef]

[16] Reddy DRK, Santhi Kumar R, Pradeep Kumar TV. Kaluza-Klein universe with cosmic strings and bulk viscosity in a scalar-tensor theory of gravitation. Int J Theor Phys 2013;52:1214–1220. [CrossRef]

[17] Naidu RL, Satyanarayana B, Reddy DRK. LRS Bianchi type-II universe with cosmic strings and bulk viscosity in a scalar-tensor theory of gravitation. Astrophys Space Sci 2012;338:351–354. [CrossRef]

[18] Vijaya Santhi M, Rao VUM, Aditya Y. Bianchi type-I bulk viscous string model in gravity. J Dyn Syst Geom Theor 2019;17:23–38. [CrossRef]

[19] Katore SD, Hatkar SP, Gore SV. Cosmology of string bulk viscosity in theory of gravitation. Int J Geom Methods Mod Phys 2018;15:1850116. [CrossRef]

[20] Perlmutter S, Aldering G, della Valle M, Deustua S, Ellis RS, Fabbro S, et al. Discovery of a supernova explosion at half the age of the universe. Nature 1998;391:51–54. [CrossRef]

[21] Riess AG, Filippenko AV, Challis P, Clocchiatti A, Diercks A, Garnavich PM, et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. Astron J 1998;116:1009–1038. [CrossRef]

[22] Riess AG, Nugent PE, Gilliland RL, Schmidt BP, Tonry J, Dickinson M, et al. The farthest known supernova: Support for an accelerating universe and a glimpse of the epoch of deceleration. Astrophys J 2001;560:49–71. [CrossRef]

[23] Berman MS, Gomide FM. Cosmological models with constant deceleration parameter. Gen Relativ Gravit 1988;20:191–198. [CrossRef]

[24] Akarsu Ö, Dereli T. Cosmological models with linearly varying deceleration parameter. Int J Theor Phys 2012;51:612–621. [CrossRef]

[25] Chawla C, Mishra RK, Pradhan A. String cosmological models from early deceleration to current acceleration phase with varying and. Eur Phys J Plus 2012;127:137. [CrossRef]
[26] Chawla C, Mishra RK. Bianchi type-I viscous fluid cosmological models with variable deceleration parameter. Rom J Phys 2013;58:75–85.
[27] Mishra RK, Pradhan A, Chawla C. Anisotropic viscous fluid cosmological models from deceleration to acceleration in string cosmology. Int J Theor Phys 2013;52:2546–2559. [CrossRef]
[28] Mishra RK, Chand A, Pradhan A. Dark-energy models in theory with variable deceleration parameter. Int J Theor Phys 2016;55:1241–1254. [CrossRef]
[29] Perlmutter S, Aldering G, Goldhaber G, Knop RA, Nugent P, Castro PG, et al. Measurements of and from 42 high-redshift supernovae. Astrophys J 1999;517:565–586. [CrossRef]
[30] Bennett CL, Halpern M, Hinshaw G, Jarosik N, Kogut A, Limon M, et al. First-year Wilkinson Microwave Anisotropy Probe observations: Preliminary maps and basic results. Astrophys J Suppl Ser 2003;148:1–27. [CrossRef]
[31] Spergel DN, Verde L, Peiris HV, Komatsu E, Nolta MR, Bennett CL, et al. First-year Wilkinson Microwave Anisotropy Probe observations: Determination of cosmological parameters. Astrophys J Suppl Ser 2003;148:175–194. [CrossRef]
[32] Oli S. Interacting and noninteracting two-fluid cosmological models in Bianchi type- spacetimes. Indian J Phys 2012;86:755–761. [CrossRef]
[33] Tegmark M, Strauss MA, Blanton MR, Abazajian K, Dodelson S, Sandvik H, et al. Cosmological parameters from SDSS and WMAP. Phys Rev D 2004;69:103501. [CrossRef]
[34] Chimento LP. Extended tachyon field, Chaplygin gas, and solvable -essence cosmologies. Phys Rev D 2004;69:123517. [CrossRef]
[35] Reddy DRK, Santhi Kumar R. Some anisotropic cosmological models in a modified theory of gravitation. Astrophys Space Sci 2013;344:253–257. [CrossRef]
[36] Land K, Magueijo J. Examination of evidence for a preferred axis in the cosmic radiation anisotropy. Phys Rev Lett 2005;95:071301. [CrossRef]
[37] Barrow JD. Helium formation in cosmologies with anisotropic curvature. Mon Not R Astron Soc 1984;211:221–227. [CrossRef]
[38] Santhi Kumar R, Satyannarayana B. Accelerating anisotropic cosmological model in theory of gravity. Indian J Phys 2017;91:1293–1296. [CrossRef]
[39] Krishna M, Koppala S, Rajamahanthi RS. Accelerating plane-symmetric cosmological model with bulk viscous fluid and cosmic strings in Lyra's geometry. Indian J Phys 2024;98:3733–3740. [CrossRef]