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Research Article

Exploration of Ulam-Hyers stability for a system of fractional integro-coupled differential equations with integral boundary conditions

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ABSTRACT

This study examines existence, uniqueness, and Ulam-Hyers stability for solutions of non-linear coupled fractional integro-differential equations with integral boundary conditions. Fractional systems incorporating memory and hereditary effects serve as effective models for complex processes in science and engineering applications. This study proves solution existence and uniqueness by applying the Banach fixed-point theorem within carefully constructed function spaces, then extends our analysis to investigate Ulam-Hyers stability, a framework that reveals how solutions behave when initial conditions contain small errors. Our stability analysis demonstrates that minor perturbations in starting data translate to bounded solution variations, keeping the system stable within predictable limits, which we verify through a computational example showing how controlled initial changes produce correspondingly controlled solution deviations. These results advance stability theory for coupled fractional integro-differential systems, particularly where memory effects influence system behavior, mathematical models that appear frequently in applications ranging from viscoelastic materials to population dynamics, where past states influence current evolution. By establishing rigorous stability bounds, our work provides a theoretical foundation for implementing these models in real-world scenarios where measurement uncertainties and modeling approximations are inevitable.

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INTRODUCTION

Fractional calculus has proven remarkably useful across mathematics and its applications. What makes it fascinating is how we can generalize traditional calculus beyond integer orders, taking derivatives of order 0.5 or 1.7, for instance. This flexibility reveals new mathematical structures that better model phenomena where history matters, opening doors to more accurate descriptions of complex real-world systems. And it began when L'Hospital and Leibniz wrote a letter to each other in 1695. At present, there are a lot of theories and numerical findings on this topic in the scientific literature. The flow of groundwater in a stream can be described through a fractional differential equation

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that accounts for the non-integer order derivatives. The fractional derivative models the anomalous behavior of flow and transport in porous media. It may also be used in several fields, such as the theory of control, aerodynamics, communication and image filtering, and biology [1-4]. Manuals by Kilbas et al. [5], Miller and Ross [6], and Halfer were used to clarify the fundamental concepts of fractional calculus. Multiple writers have researched the existence and uniqueness of approaches to fractional order differential systems in finite and infinite-dimensional domains. Ahmad et al. discovered results involving non-linear fractional integro-differential equations utilizing integral boundary conditions [7-10].

The results obtained in this research demonstrate practical utility across multiple domains where complex differential systems with integral boundary conditions govern physical phenomena. Hydrogeological researchers have found fractional integro-differential equations particularly effective for modeling subsurface water flow dynamics [10], while chemical engineers apply these systems to understand reaction-diffusion mechanisms [11,12]. When studying population dynamics or circulatory systems, we need mathematical frameworks that can reliably predict how these systems evolve over time, making solution stability absolutely essential for any practical model [13]. Our Ulam-Hyers stability analysis tackles this challenge by establishing clear bounds on how solutions change when initial parameters vary slightly. This becomes especially important in hydrology and biomedical engineering, where professionals rely on model predictions to make decisions that can have serious consequences for public health or environmental protection. The stability criteria we've developed offer mathematical confidence that these models will behave consistently under real-world conditions, which is exactly what practitioners need when applying these tools to solve actual problems in environmental science and engineering.

Integral boundary conditions play a crucial role in hydrogeological modeling as they capture the dynamic interactions between aquifer systems and surrounding environments. These conditions can represent various physical processes, including groundwater recharge rates or discharge flows between aquifers and surface water bodies such as rivers and lakes. Additional details regarding integral boundary conditions can be found in [13-15]. To illustrate the practical significance of integral boundary conditions, consider the following real-world example

$$-\varpi'' = f(\phi)g(\phi,\varpi),\ \varpi(a)=0,\ \eta\varpi'(b)=\varpi(\beta)$$

where $\phi \in (a, b)$, $\beta \in (0,1]$ and η are positive constants, and a, b are constants. This represents a thermoregulator model. The problem admits well-defined solutions for the one-dimensional heat equation describing a heated bar equipped with a control system that modulates heat input based on temperature measurements from a sensor positioned at β.

This formulation extends to more general cases as the heat equation, incorporating nonlinear gradient terms and time-varying source components. The heated bar now features a controller positioned at β that regulates heat output according to temperature data collected from multiple sensors distributed along the bar [16-20]. This problem, therefore, may be stated as follows

$$\varpi'' = g(\phi,\varpi,\varpi'),\varpi(a)=0,\ \varpi'(b)=\int_a^b \varpi(s)dh(s)(\beta).$$

Ulam's type stability is a mathematical notion that describes how solutions to differential equations behave when tiny perturbations are introduced into the beginning conditions or equations themselves. It quantifies how stable or responsive a system is to disturbances. In 1940, Ulam [21] was the first to introduce the Ulam kind of stability. Hyers [22] subsequently refined this concept. Ulam and Hyers investigated stability properties across various integer-order differential equations, leading to what is now termed Hyers-Ulam stability for Cauchy functional equations. Their work produced stronger results concerning polynomials, isometric mappings, and convex functions [23-34]. Numerous researchers have since extended and generalized Hyers' framework for integer-order differential equations. The literature contains extensive investigations into additional stability concepts of Ulam type, documented in [35-44] and related works [45-51].

Motivated by the previously mentioned work related to integro-systems equipped with fractional order derivatives, we aim to extend the findings of [1, 51]. Integro-differential equations are important because they combine the properties of both differential and integral equations, allowing them to model a wide range of complex systems where current states depend on both rates of change and accumulated effects. Despite their significance, few studies have focused on investigating Ulam-type stability for integrodifferential equations involving fractional derivatives. Despite extensive work on fractional differential equations, Ulam's stability for coupled systems with integral boundary conditions remains unexplored. The proposed system is as follows

$$\begin{cases} {}^cD^m x(t) = k\big(t, y(t), \chi^y(t)\big), & m \in (1,2], t \in \aleph \\ {}^cD^n y(t) = l\big(t, x(t), \chi^x(t)\big), & n \in (1,2], t \in \aleph \\ \theta x(0) + \vartheta x'(0) = \int_0^1 b_1\big(x(s)\big)ds, \theta x(1) + \vartheta x'(1) = \int_0^1 b_2\big(x(s)\big)ds \\ \bar\theta y(0) + \bar\vartheta y'(0) = \int_0^1 \bar b_1\big(y(s)\big)ds, \bar\theta y(1) + \bar\vartheta y'(1) = \int_0^1 \bar b_2\big(y(s)\big)ds, \end{cases} \quad (1.1)$$

where ${}^cD$ is the Caputo fractional derivative, m, n represents the order of the derivative, $\aleph = [0,1]$ and $k, l\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for $\chi^y, \chi^x\colon [0,1] \times [0,1] \to [0,\infty)$. And

$$\chi^y(t) = \int_0^t \hat\gamma_1(t,s)y(s)ds \quad \text{and} \quad \chi^x(t) = \int_0^t \hat\gamma_2(t,s)x(s)ds.$$

Here $b_1, b_2, \bar b_1, \bar b_2\colon \mathbb{R} \to \mathbb{R}$ and $\theta, \bar\theta > 0, \vartheta, \bar\vartheta \geq 0$ are real numbers.

We organize this work as follows: Section 2 presents essential definitions, notation, and preliminary results needed for our main analysis. In Section 3, we establish the existence and uniqueness of solutions for system (1.1). Section 4 develops our main stability results, focusing on Ulam-type stability criteria. Finally, Section 5 demonstrates these theoretical findings through a computational example that illustrates the practical implications of our stability bounds.

PRELIMINARIES

We start with preliminary results using a few helpful concepts, notations, and lemmas, which will be utilized in the subsequent parts.

Definition 1.[5] The Caputo derivative of a given function $y \in ((0,1),\mathbb{R})$ for a fractional order $m \in \mathbb{R}^+$ is presented as

$$ {}^{c}D^{m}y(t) = \frac{1}{\Gamma(\varsigma - m)}\int_0^t (t-s)^{\varsigma - m - 1} y^{(\varsigma)}(s)ds, \varsigma = [m]+1, $$

where $[m]$ denote the integer part of m and $\Gamma(\cdot)$ is the gamma function.

Definition 2. [5] The Riemann- Liouville fractional derivative of a function $y(t)$ of order m is presented as

$$ \mathcal{J}^{m}y(t) = \frac{1}{\Gamma(\varsigma - m)}\left(\frac{d}{dt}\right)^{\varsigma}\int_0^t \frac{y(s)}{(t-s)^{m-\varsigma+1}}ds, \varsigma = [m]+1, $$

provided the integral exists.

Lemma 1.[5] For any real number $m > 0$, the differential equation

$$ {}^{c}D^{m}y(t) = 0 $$

has a solution which is presented by

$$ y(t) = d_0 + d_1 t + d_2 t^2 + \cdots \ldots + d_{\varsigma-1}t^{\varsigma-1}, \quad d_\iota \in \mathbb{R}, \iota = 0,1,2,\ldots,\varsigma - 1, $$

where $\varsigma = [m] + 1$.

Lemma 2. [5] The solution for any real number $m > 0$, of the differential equation

$$ {}^{c}D^{m}y(t) = u(t) $$

will be presented as

$$ \mathcal{J}^{m}[{}^{c}D^{m}y(t)] = \mathcal{J}^{m}y(t) + d_0 + d_1 t + d_2 t^2 + \cdots \ldots + d_{\varsigma-1}t^{\varsigma-1}, \quad d_\iota \in \mathbb{R}, \iota = 0,1,2,\ldots,\varsigma - 1, $$

where $\varsigma = [m] + 1$.

Lemma 3. [9] For any function $u, \gamma_1, \gamma_2 \in C([0,1],\mathbb{R})$, the following boundary value problem has a unique solution of

$$ \begin{cases} {}^{c}D^{m}y(t) = u(t), t \in [0,1] \\ \theta y(0) + \vartheta y'(0) = \int_0^1 \gamma_1(s)ds, \theta y(1) + \vartheta y'(1) = \int_0^1 \gamma_2(s)ds \end{cases} \quad (2.1) $$

is presented as

$$ y(t) = \int_0^1 G_m(t,s)u(s)ds + \frac{1}{\theta^2}\left[(\theta(1-t)+\vartheta)\int_0^1 \gamma_1(s)ds + (\vartheta + \theta t)\int_0^1 \gamma_2(s)ds\right], \quad (2.2) $$

where $G_m(t,s)$ is the Green's function presented by

$$ G_m(t,s) = \begin{cases} \frac{\theta(t-s)^{m-1} + (\vartheta - \theta t)(1-s)^{m-1}}{\theta\Gamma(m)} + \frac{\vartheta(\vartheta - \theta t)(1-s)^{m-2}}{\theta^2\Gamma(m-1)}, & s \le t \\ \frac{(\vartheta - \theta t)(1-s)^{m-1}}{\theta\Gamma(m)} + \frac{\vartheta(\vartheta - \theta t)(1-s)^{m-2}}{\theta^2\Gamma(m-1)}, & t \le s \end{cases} \quad (2.3) $$

Proof: To prove lemma (3) for some real constants s_1, $s_2 \in \mathbb{R}$, we have

$$ y(t) = \mathcal{J}^{m}u(t) - s_1 - s_2 t = \int_0^t \frac{(t-s)^{m-1}}{\Gamma(m)}u(s)ds - s_1 - s_2 t $$

Given the relation ${}^{c}D^{m}\mathcal{J}^{m}y(t) = y(t)$ and $\mathcal{J}^{m}\mathcal{J}^{n} = \mathcal{J}^{m+n}y(t)$ for $m,n > 0, y \in L(0,1)$, we obtain

$$ y'(t) = \int_0^t \frac{(t-s)^{m-2}}{\Gamma(m-1)}u(s)ds - s_2 $$

Applying the boundary conditions for (2.1), we get the following

$$ s_1 = \frac{1}{\theta^2}\left[\vartheta\int_0^1 \gamma_2(s)ds - (\vartheta + \theta)\int_0^1 \gamma_1(s)ds\right] - \frac{\vartheta}{\theta\Gamma(m)}\int_0^1 (1-s)^{m-1}u(s)ds - \frac{\vartheta^2}{\theta^2\Gamma(m-1)}\int_0^1 (1-s)^{m-2}u(s)ds $$

$$ s_2 = \frac{1}{\theta}\left[\int_0^1 \gamma_1(s)ds - \int_0^1 \gamma_2(s)ds\right] + \frac{1}{\Gamma(m)}\int_0^1 (1-s)^{m-1}u(s)ds + \frac{\vartheta}{\theta\Gamma(m-1)}\int_0^1 (1-s)^{m-2}u(s)ds $$

The unique solution of (2.1) is therefore follows as

$$ \begin{aligned} y(t) = &\int_0^t \left[\frac{\theta(t-s)^{m-1} + (\vartheta - \theta t)(1-s)^{m-1}}{\theta\Gamma(m)} + \frac{\vartheta(\vartheta - \theta t)(1-s)^{m-2}}{\theta^2\Gamma(n-1)}\right]u(s)ds \\ &+ \int_t^1 \left[\frac{(\vartheta - \theta t)(1-s)^{m-1}}{\theta\Gamma(m)} + \frac{\vartheta(\vartheta - \theta t)(1-s)^{m-2}}{\theta^2\Gamma(n-1)}\right]u(s)ds \\ &+ \frac{1}{\theta^2}\left[(\theta(1-t)+\vartheta)\int_0^1 \gamma_1(s)ds + (\vartheta + \theta t)\int_0^1 \gamma_2(s)ds\right] \\ = &\int_0^1 G_m(t,s)u(s)ds + \frac{1}{\theta^2}\left[(\theta(1-t)+\vartheta)\int_0^1 \gamma_1(s)ds + (\vartheta + \theta t)\int_0^1 \gamma_2(s)ds\right] \end{aligned} $$

where $G_m(t,s)$ is given by (2.3). This is the required proof.

Lemma 4. The space $\mathcal{H} = y(t) \mid y \in C(\aleph)$ is a Banach space under the defined norm $\|y\|_{\mathcal{H}} = \max_{t\in\mathcal{N}}|y(t)|$. Similarly, the norm on product space is defined as $\|(y,\bar{y})\|_{\mathcal{H}\times\mathcal{H}} = \|y\|_{\mathcal{H}} + \|\bar{y}\|_{\mathcal{H}}$.Obviously$(\mathcal{H}\times\mathcal{H}, \|(.,.)\|_{\mathcal{H}\times\mathcal{H}})$ is a Banach space. Moreover the cone $\mathcal{U} \subset \mathcal{H}\times\mathcal{H}$ is defined as

$$\mathcal{U} = (y, \bar{y}) \in \mathcal{H} \times \mathcal{H} \mid y(t) \geq 0, \bar{y}(t) \geq 0.$$

Here system (1.1) is transformed into a fixed point problem. Let $\mathcal{S}: \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ be the operator defined as

$$\mathcal{S}(x,y)(t) = \begin{pmatrix} \int_0^1 G_m(t,s)k\big(s, y(s), \chi^y(s)\big)ds + \frac{1}{\theta^2}\left[(\theta(1-t)+\vartheta)\int_0^1 b_1\big(x(s)\big)ds\right] \\ +(\vartheta + \theta t)\int_0^1 b_2\big(x(s)\big)ds \\ \int_0^1 G_n(t,s)l\big(s, x(s), \chi^x(s)\big)ds + \frac{1}{\theta^2}\left[(\bar{\theta}(1-t)+\bar{\vartheta})\int_0^1 \bar{b}_1\big(y(s)\big)ds\right] \\ +(\bar{\vartheta} + \bar{\theta} t)\int_0^1 \bar{b}_2\big(y(s)\big)ds \end{pmatrix}. \quad (2.4)$$

It follows that the fixed point of the operator coincides with the solution of the proposed coupled system (1.1).

To prove the existence and uniqueness of solutions to system (1.1), we require the subsequent conditions

(B1) For $t \in \aleph, \exists\, \mu_1, \mu_2, \mu_3 \in C(\aleph, \mathbb{R}^+)$ such that

$$\left|k\big(t, y(t), \chi^y(t)\big)\right| \leq \mu_1(t) + \mu_2(t)|y(t)| + \mu_3(t)|\chi^y(t)|, \quad \forall y(t) \in C(\aleph, \mathbb{R})$$

with

$$\mu_1^* = \sup_{t\in\aleph}\mu_1(t),\ \mu_2^* = \sup_{t\in\aleph}\mu_2(t),\ \mu_3^* = \sup_{t\in\aleph}\mu_3(t).$$

Similarly, For $t \in \aleph, \exists \nu_1, \nu_2, \nu_3 \in C(\aleph, \mathbb{R}^+)$ such that

$$\left|l\big(t, x(t), \chi^x(t)\big)\right| \leq \nu_1(v) + \nu_2(t)|x(t)| + \nu_3(t)|\chi^x(t)|, \quad \forall x(t) \in C(\aleph, \mathbb{R})$$

with

$$\nu_1^* = \sup_{t\in\aleph}\nu_1(t),\ \nu_2^* = \sup_{t\in\aleph}\nu_2(t),\ \nu_3^* = \sup_{t\in\aleph}\nu_3(t).$$

(B2) For $t \in \aleph, \exists\, L_{b_1}, L_{b_2}$ such that

$$\left|b_1\big(x(t)\big)\right| \leq L_{b_1}|x(t)| \text{ and } \left|b_2\big(x(t)\big)\right| \leq L_{b_2}|x(t)|, \quad \forall x(t) \in C(\aleph, \mathbb{R}).$$

Similarly, For $t \in \aleph, \exists \quad L_{\bar{b}_1}, L_{\bar{b}_2}$ such that

$$\left|\bar{b}_1\big(y(t)\big)\right| \leq L_{b_1}|y(t)| \text{ and } \left|\bar{b}_2\big(y(t)\big)\right| \leq L_{\bar{b}_2}|y(t)|, \quad \forall y(t) \in C(\aleph, \mathbb{R}).$$

(B3) For $t \in \aleph,\ \exists \quad L_{\hat{\omega}_1}, L_{\hat{\omega}_2}$ such that

$$\int_0^1 |\hat{\gamma}_1(t,s)|ds \leq L_{\hat{\omega}_1} \text{ and } \int_0^1 |\hat{\gamma}_2(t,s)|ds \leq L_{\hat{\omega}_2}.$$

(B4)

$$\mathcal{M}_1 = \int_0^1 |G_m(t,s)|\mu_1(s) < \infty, \mathcal{N}_1 = \int_0^1 |G_m(t,s)|\mu_2(s)ds + L_{\hat{\omega}_1}\int_0^1 |G_m(t,s)|\mu_3(s)ds + \frac{(\theta+\vartheta)\big(L_{b_1}+L_{b_2}\big)}{\theta^2} < \frac{1}{2}$$

$$\mathcal{M}_2 = \int_0^1 |G_n(t,s)|\nu_1(s) < \infty, \mathcal{N}_2 = \int_0^1 |G_n(t,s)|\nu_2(s)ds + L_{\hat{\omega}_2}\int_0^1 |G_n(t,s)|\nu_3(s)ds + \frac{(\bar{\theta}+\bar{\vartheta})\big(L_{\bar{b}_1}+L_{\bar{b}_2}\big)}{\bar{\theta}^2} < \frac{1}{2}$$

(B5) For all $\bar{y} \in C(\aleph, \mathbb{R})$ and for each $t \in \aleph$ there exist positive real constants $\mathfrak{R}_k, \bar{\mathfrak{R}}_k$, such that

$$\left|k\big(t, y(t), \chi^y(t)\big) - k\big(t, \bar{y}(t), \bar{\chi}^y(t)\big)\right| \leq \mathfrak{R}_k|y - \bar{y}| + \bar{\mathfrak{R}}_k|\chi^y - \bar{\chi}^y|$$

Similarly, $\forall x, \bar{x} \in C(\aleph, \mathbb{R})$ and for each $t \in \aleph$ there exist positive constants $\mathfrak{R}_l, \bar{\mathfrak{R}}_l$, such that

$$\left|l\big(t, x(t), \chi^x(v)\big) - k\big(t, \bar{x}(t), \bar{\chi}^x(t)\big)\right| \leq \mathfrak{R}_l|x - \bar{x}| + \bar{\mathfrak{R}}_l|\chi^x - \bar{\chi}^x|$$

(B6) For all $\bar{x} \in C(\aleph, \mathbb{R})$ and for each $t \in \aleph$ there exist positive real constants $\mathfrak{R}_{b_1}, \mathfrak{R}_{b_2}$, such that

$$|b_1(x) - b_1(\bar{x})| \leq \mathfrak{R}_{b_1}|x - \bar{x}| \text{ and } |b_2(x) - b_2(\bar{x})| \leq \mathfrak{R}_{b_2}|x - \bar{x}|$$

Similarly, For all $y, \bar{y} \in C(\aleph, \mathbb{R})$ and for each $t \in \aleph$ there exist positive real constants $\mathfrak{R}_{\bar{b}_1}, \mathfrak{R}_{\bar{b}_2}$, such that

$$\left|\bar{b}_1(y) - \bar{b}_1(\bar{y})\right| \leq \mathfrak{R}_{\bar{b}_1}|y - \bar{y}| \text{ and } \left|\bar{b}_2(y) - \bar{b}_2(\bar{y})\right| \leq \mathfrak{R}_{\bar{b}_2}|y - \bar{y}|$$

(B7) Let

(i). $\sigma_1 = \mathcal{K}_{(n\bar{\theta}\bar{\vartheta})}\mathfrak{R}_l + \mathfrak{R}_{\theta\vartheta}$, where $\mathfrak{R}_{\theta\vartheta} = \frac{(\theta+\vartheta)(\mathfrak{R}_{b_1}+\mathfrak{R}_{b_2})}{\theta^2}$

and

$$\begin{aligned}\mathcal{K}_{(n\bar{\theta}\bar{\vartheta})} &= \max_{t\in[0,1]}\left|\int_0^1 G_n(t,s)ds\right| \\ &= \left|\int_0^t \left[\frac{\bar{\theta}(t-s)^{(n-1)} + (\bar{\bar{\vartheta}} - \bar{\theta})(1-s)^{n-1}}{\theta\Gamma(n)} + \frac{\bar{\vartheta}(\bar{\vartheta} - \bar{\theta}t)(1-s)^{n-2}}{\theta^2\Gamma(n-1)}\right]ds\right| \\ &\quad + \left|\int_t^1 \left[\frac{(\bar{\vartheta} - \bar{\theta}t)(1-s)^{(n-1)}}{\theta\Gamma(n)} + \frac{\bar{\theta}(\bar{\theta} - \bar{\theta}t)(1-s)^{n-2}}{\theta^2\Gamma(n)}\right]ds\right| \\ &= \frac{1}{\Gamma(n+1)} + \frac{2(\bar{\theta}+\bar{\vartheta})}{\Gamma(n+1)} + \frac{2(\vartheta^2 + \bar{\theta}\bar{\vartheta})}{\theta^2\Gamma(n)}\end{aligned}$$

(ii). $\sigma_2 = \mathcal{K}_{(m\theta\vartheta)}\mathfrak{R}_k + \mathfrak{R}_{\bar{\theta}\bar{\vartheta}}$, where $\mathfrak{R}_{\bar{\theta}\bar{\vartheta}} = \frac{(\bar{\theta}+\bar{\vartheta})\left(\mathfrak{R}_{\bar{b}_1}+\mathfrak{R}_{\bar{b}_2}\right)}{\theta^2}$

and

$$\begin{aligned}\mathcal{K}_{(m\theta\vartheta)} &= \max_{t\in[0,1]}\left|\int_0^1 G_m(t,s)ds\right| \\ &= \frac{1}{\Gamma(m+1)} + \frac{2(\theta+\vartheta)}{\Gamma(m+1)} + \frac{2(\vartheta^2 + \theta\vartheta)}{\theta^2\Gamma(m)}\end{aligned}$$

EXISTENCE AND UNIQUENESS ANALYSIS

Theorem 3.1. If all of the prerequisites ($B1$) - ($B7$) are satisfied and $\sigma = \max(\sigma_1, \sigma_2) < 1$, so the suggested fractional order paired framework (1.1) has a unique solution.

Proof: For a positive number

$$\rho = \max\left(\frac{2\mathcal{M}_1}{1-2\mathcal{N}_1}, \frac{2\mathcal{M}_2}{1-2\mathcal{N}_2}\right),$$

we define a set

$$\mathcal{V} = \{(x,y) \in \mathcal{H} \times \mathcal{H} \colon \| (x,y) \|_{\mathcal{H}\times\mathcal{H}} \leq \rho\}.$$

First, to establish that $\mathcal{S}$ maps $\mathcal{V}$ into itself, we have

$$\begin{aligned}
|\mathcal{S}_m(n,m)(t)| \leq & \int_0^1 |G_m(t,s)| \left| k\big(s, y(s), \chi^y(s)\big) ds \right| \\
& + \frac{1}{\alpha^2}\left[|(\theta(1-t)+\vartheta)| \int_0^1 \left|b_1\big(x(s)\big)\right| ds + |(\vartheta + \theta t)| \int_0^1 \left|b_2\big(x(s)\big)\right| ds \right] \\
& \leq \int_0^1 |G_m(t,s)| \mu_1(s) ds + \int_0^1 |G_m(t,s)| [\mu_2(s)|y(s)|] ds + \int_0^1 |G_m(t,s)| [\mu_3(s)|\chi^y(s)|] ds \\
& + \frac{1}{\theta^2}\left[(\theta+\vartheta) \int_0^1 \left|b_1\big(x(s)\big)\right| ds + (\theta+\vartheta) \int_0^1 \left|b_2\big(x(s)\big)\right| ds \right] \\
& \leq \int_0^1 |G_m(t,s)| \mu_1(s) ds + \rho\left[\int_0^1 |G_m(t,s)| \mu_2(s) ds + L_{\hat{\omega}_1} \int_0^1 |G_m(t,s)| \mu_3(s) ds \right.
\end{aligned}$$

Upon applying the maximization operation to both sides of the aforementioned inequality over the set $\mathcal{T}$, we derive the following expression:

$$\|\mathcal{S}_m(n,m)\|_{\mathcal{H}} \leq \frac{\rho}{2}.$$

Similarly,

$$\|\mathcal{S}_n(m,n)\|_{\mathcal{H}} \leq \frac{\rho}{2}.$$

Hence, we can conclude that,

$$\| \mathcal{S}(m,n) \|_{\mathcal{H}\times\mathcal{H}} \leq \rho. \tag{3.2}$$

The aforementioned inequality serves as confirmation that the mapping $\mathcal{S}$ effectively maps the set $\mathcal{V}$ into itself. Subsequently, to demonstrate that $\mathcal{S}$ qualifies as the contraction operator under the condition where $t \in \aleph$, we proceed with the following analysis.

$$\begin{aligned}
|\mathcal{S}_m(n,m)(t) - \mathcal{S}_m(\bar{y},\bar{x})(t)| \leq & \int_0^1 |G_m(t,s)| \left| k\big(s, y(s), \chi^y(s)\big) - k\big(s, \bar{y}(s), \bar{\chi}^y(s)\big) \right| ds \\
& + \frac{1}{\theta^2}\left[|(\theta(1-t)+\vartheta)| \int_0^1 \left| b_1\big(x(s)\big) - b_1\big(\bar{x}(s)\big) \right| ds \right. \\
& \leq \mathcal{K}_{(m\theta\vartheta)} \Re_k |y(t) - \bar{y}(t)| \\
& + \frac{(\theta+\vartheta)\big(\Re_{b_1} + \Re_{b_2}\big)}{\theta^2} |x(t) - \bar{x}(t)|.
\end{aligned}$$

When we operate the maximum on both sides of the above-mentioned inequality over $\aleph$, we obtain

$$\|\mathcal{S}_m(y,x) - \mathcal{S}_m(\bar{y},\bar{x})\|_{\mathcal{H}} \leq \mathcal{K}_{(m\theta\vartheta)} \Re_k \| y - \bar{y} \|_{\mathcal{H}} + \frac{(\theta+\vartheta)\big(\Re_{b_1} + \Re_{b_2}\big)}{\theta^2} \| x - \bar{x} \|_{\mathcal{H}}. \tag{3.4}$$

Similarly, the following can be obtained

$$\|\mathcal{S}_n(x,y) - \mathcal{S}_n(\bar{x},\bar{y})\|_{\mathcal{H}} \leq \mathcal{K}_{(n\bar{\theta}\bar{\vartheta})} \Re_l \| x - \bar{x} \|_{\mathcal{H}} + \frac{(\bar{\theta}+\bar{\vartheta})\big(\Re_{\bar{b}_1} + \Re_{\bar{b}_2}\big)}{\bar{\theta}^2} \| y - \bar{y} \|_{\mathcal{H}}. \tag{3.5}$$

From the above two equations, we obtain

$$\| \mathcal{S}(x,y) - \mathcal{S}(\bar{x},\bar{y}) \|_{\mathcal{H}\times\mathcal{H}} \leq \sigma \| (x,y) - (\bar{x},\bar{y}) \|_{\mathcal{H}\times\mathcal{H}}. \tag{3.6}$$

Henceforth, it is evident that the operator $\mathcal{S}$ exhibits strict contraction properties. Employing Banach's fixed-point method, specifically designed for establishing a unique fixed point, becomes instrumental in determining the sole solution to the proposed coupled system (1.1). Consequently, the fulfillment of our primary objective is achieved.

ULAM'S STABILITY ANALYSIS

In this section, we endeavor to investigate the Ulam-type stability characteristics inherent in the envisaged coupled system denoted as (1.1).

For some real number $\varepsilon = \max(\varepsilon_m, \varepsilon_n) > 0$. We assume the inequality given as follows

$$\begin{cases} \left| \ {}^cD^m x(t) - k\big(t, y(t), \chi^y(t)\big) \right| \leq \varepsilon_m, t \in \aleph \\ \left| \ {}^cD^n y(t) - l\big(t, x(t), \chi^x(t)\big) \right| \leq \varepsilon_n, \ t \in \aleph. \end{cases} \tag{4.1}$$

The following definitions are inspired by Rus [19].

Definition 3: The Ulam-Hyers stability of the coupled system denoted by (1.1) is established under the condition that there exists a positive constant $\mathcal{K}_{(mn\theta v\bar{\theta}\bar{v})}$ such that, for any solution $(x, y) \in \mathcal{H} \times \mathcal{H}$ of the inequality (4.1), there exists a unique solution $(\zeta, \eta) \in \mathcal{H} \times \mathcal{H}$. This unique solution satisfies the inequality:

$$|(x,y)(t) - (\zeta,\eta)(t)| \leq \mathcal{K}_{(mn\theta\vartheta\bar{\vartheta}\bar{\vartheta})}\varepsilon, \quad t \in \aleph$$

Definition 4: The generalized Ulam's-Hyers stability of the coupled system denoted by (1.1) is posited under the condition that there exists a function $\varpi(\varepsilon) \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\varpi(0) = 0$. This function is required to satisfy the condition that, for any solution $(x, y) \in \mathcal{H} \times \mathcal{H}$ of the inequality (4.1), there exists a unique solution $(\zeta, \eta) \in \mathcal{H} \times \mathcal{H}$ for the system. Additionally, the unique solution must adhere to the inequality:

$$|(x,y)(t) - (\zeta,\eta)(t)| \leq \varpi(\varepsilon), \quad t \in \aleph$$

Remark 1. Consider the pair $(x, y) \in \mathcal{H} \times \mathcal{H}$ to be a solution of inequality (4.1) if there exist functions ϕ and ψ belonging to the space $C(\aleph, \mathbb{R})$, contingent upon the variables x and y respectively, such that

(R1). $|\phi(t)| \leq \varepsilon_m$ and $|\psi(t)| \leq \varepsilon_n, \ t \in \aleph$;

(R2). and

$$\begin{cases} {}^cD^m x(t) = k\big(t, y(t), \chi^y(t)\big) + \phi(t), t \in \aleph \\ {}^cD^n y(t) = l\big(t, x(t), \chi^x(t)\big) + \psi(t), t \in \aleph. \end{cases} \tag{4.2}$$

Lemma 5. Let $(x, y) \in \mathcal{H} \times \mathcal{H}$ be deemed as a solution to inequality (4.1). Subsequently, the ensuing inequality is guaranteed to be satisfied:

$$\begin{cases} \left| x(t) - \int_0^1 G_m(t,s)k(s,y(s),\chi^y(s))ds - \frac{1}{\theta^2}\left[(\theta(1-t)+\vartheta)\int_0^1 b_1(x(s))ds \right.\right. \\ \left.\left. + (\vartheta+\theta t)\int_0^1 b_2(x(s))ds\right] \right| \leq \mathcal{K}_{m\theta\vartheta}\varepsilon_m, t \in \aleph \\ \left| y(t) - \int_0^1 G_n(t,s)l(s,x(s),\chi^x(s))ds - \frac{1}{\bar\theta^2}\left[(\bar\theta(1-t)+\bar\vartheta)\int_0^1 \bar b_1(x(s))ds \right.\right. \\ \left.\left. +(\bar\vartheta+\bar\theta t)\int_0^1 \bar b_2(x(s))ds\right] \right| \leq \mathcal{K}_{n\bar\theta\bar\vartheta}\varepsilon_n, t \in \aleph. \end{cases}$$

Proof. By Remark 1 (R2), we have

$$\begin{cases} {}^cD^m x(t) = k(t,y(t),\chi^y(t)) + \phi(t), \quad t \in \aleph \\ {}^cD^n y(t) = l(t,x(t),\chi^x(t)) + \psi(t), \quad t \in \aleph \\ \theta x(0) + \vartheta x'(0) = \int_0^1 b_1(x(s))ds, \theta x(1) + \vartheta x'(1) = \int_0^1 b_2(x(s))ds \\ \bar\theta y(0) + \bar\vartheta y'(0) = \int_0^1 \bar b_1(y(s))ds, \bar\theta y(1) + \bar\vartheta y'(1) = \int_0^1 \bar b_2(y(s))ds. \end{cases} \tag{4.3}$$

By applying Lemma 3, the solution of (4.3) is given as follows:

$$\begin{cases} x(t) = \int_0^1 G_m(t,s)k(s,y(s),\chi^y(s))ds + \int_0^1 G_m(t,s)\phi(s)ds + \frac{1}{\theta^2}[(\theta(1-t)+\vartheta) \\ \int_0^1 b_1(x(s))ds + (\vartheta+\theta t)\int_0^1 b_2(x(s))\Big] ds, t \in \aleph \\ y(t) = \int_0^1 G_n(t,s)l(s,x(s),\chi^x(s))ds + \int_0^1 G_n(t,s)\psi(s)ds + \frac{1}{\bar\theta^2}[(\bar\theta(1-t)+\bar\vartheta) \\ \int_0^1 \bar b_1(x(s))ds + (\bar\vartheta+\bar\theta t)\int_0^1 \bar b_2(x(s))\Big] ds, t \in \aleph. \end{cases} \tag{4.4}$$

Considering the first equation of system (4.4), we have

$$\left| x(t) - \int_0^1 G_m(t,s)k(s,y(s),\chi^y(s))ds - \frac{1}{\theta^2}\left[(\theta(1-t)+\vartheta)\int_0^1 b_1(x(s))ds + (\vartheta+\theta t)\int_0^1 b_2(x(s))\right] ds \right| \leq \left|\int_0^1 G_\theta(t,s)\phi(s)ds\right| \leq \int_0^1 |G_\theta(t,s)\phi(s)|ds.$$

By Remark 1 (R1) and using the condition of (B6), we get

$$\left| x(t) - \int_0^1 G_m(t,s)k(s,y(s),\chi^y(s))ds - \frac{1}{\theta^2}\left[(\theta(1-t)+\vartheta)\int_0^1 b_1(x(s))ds + (\vartheta+\theta t)\int_0^1 b_2(x(s))\right] ds \right| \leq \mathcal{K}_{(m\theta\vartheta)}\varepsilon_m. \tag{4.5}$$

By applying the same process for the second equation of system (4.4), we get

$$\left| y(t) - \int_0^1 G_n(t,s)l(s,x(s),\chi^x(s))ds - \frac{1}{\bar\theta^2}\left[(\bar\theta(1-t)+\bar\vartheta)\int_0^1 \bar b_1(x(s))ds + (\bar\vartheta+\bar\theta t)\int_0^1 \bar b_2(x(s))\right] ds \right| \leq \mathcal{K}_{(n\bar\theta\bar\vartheta)}\varepsilon_n. \tag{4.6}$$

Theorem 2. Assuming that conditions (B5) to (B7) are satisfied, the fractional order coupled system denoted by equation (1.1) can be deemed Ulam's-Hyers stable. This assertion extends to the broader context of being generalized Ulam ‘s-Hyers stable.

$$(1-\Re_{\theta\vartheta})(1-\Re_{\bar\theta\bar\vartheta}) - \Re_{m\theta\vartheta}\Re_{n\bar\theta\bar\vartheta} \neq 0$$

Proof: Let $(x, y) \in \mathcal{H} \times \mathcal{H}$ be the solution of the system (12) and $(\zeta, \eta) \in \mathcal{H} \times \mathcal{H}$ be the unique solution to the following considered system:

$$\begin{cases} {}^cD^m\zeta(t) = k(t,\eta(t),\chi^\eta(t)), t \in \aleph \\ {}^cD^n\eta(t) = l\left(t,\zeta(t),\chi^\zeta(t)\right), t \in \aleph \\ \theta\zeta(0) + \vartheta\zeta'(0) = \int_0^1 b_1(\zeta(s))ds, \theta\zeta(1) + \vartheta\zeta'(1) = \int_0^1 b_2(\zeta(s))ds \\ \bar\theta\eta(0) + \bar\vartheta\eta'(0) = \int_0^1 \overline{b_1}(\eta(s))ds, \bar\theta\eta(1) + \bar\vartheta\eta'(1) = \int_0^1 \overline{b_2}(\eta(s))ds. \end{cases}$$

Using Lemma 3, the solution of the above system is

$$\begin{cases} \zeta(t) = \int_0^1 G_m(t,s)k(s,\eta(s),\chi^\eta(s))ds + \frac{1}{\theta^2}\left[(\theta(1-t)+\vartheta)\int_0^1 b_1(\zeta(s))ds \right. \\ \eta(t) = \int_0^1 G_n(t,s)l\left(s,\zeta(s),\chi^\zeta(s)\right)ds + \frac{1}{\bar\theta^2}\left[(\bar\theta(1-t)+\bar\vartheta)\int_0^1 \bar b_1(\eta(s))ds \right. \\ \left. +(\bar\vartheta+\bar\theta t)\int_0^1 \bar b_2(\eta(s))\right] ds, \quad t \in \aleph. \end{cases}$$

We have

$$\begin{aligned} |x(t)-\zeta(t)| &= \left| x(t) - \int_0^1 \right| G_m(t,s)k(s,\eta(s),\chi^\eta(s))ds - \frac{1}{\theta^2}\left[(\theta(1-t)+\vartheta)\int_0^1 b_1(\zeta(s))ds \right. \\ &\quad \left. +(\vartheta+\theta t)\int_0^1 b_2(\zeta(s))ds\right] | \\ &\leq \left| x(t) - \int_0^1 \right| G_m(t,s)k(s,y(s),\chi^y(s))ds - \frac{1}{\theta^2}\left[(\theta(1-t)+\vartheta)\int_0^1 b_1(x(s))ds \right. \\ &\quad \left. +(\vartheta+\theta t)\int_0^1 b_2(x(s))ds\right] | \\ &\quad + \left|\int_0^1 \right| G_m(t,s)k(s,y(s),\chi^y(s))ds - \int_0^1 |G_m(t,s)k(s,\eta(s),\chi^\eta(s))ds| \\ &\quad + \left| \frac{1}{\theta^2}\left[(\theta(1-t)+\vartheta)\int_0^1 b_1(x(s))ds + (\vartheta+\theta t)\int_0^1 b_2(x(s))ds\right] \right. \\ &\quad \left. - \frac{1}{\theta^2}\left[(\theta(1-t)+\vartheta)\int_0^1 b_1(\zeta(s))ds + (\vartheta+\theta t)\int_0^1 b_2(\zeta(s))ds\right] \right\| \\ &\leq \mathcal{K}_{m\theta\vartheta}\varepsilon_m + \Re_{m\theta\vartheta}|y(t)-\eta(t)| + \Re_{\theta\vartheta}|x(t)-\zeta(t)|, \end{aligned}$$

where $\Re_{m\theta\vartheta} = \mathcal{K}_{m\theta\vartheta}\Re_k$.

Hence, we get

$$(1-K_{\theta\vartheta}) \parallel x-\zeta \parallel_{\mathcal{H}} \leq \mathcal{K}_{(m\theta\vartheta)}\varepsilon_m + \Re_{m\theta\vartheta} \parallel y-\eta \parallel_{\mathcal{H}}. \tag{4.9}$$

Similarly, we have

$$(1-\Re_{\bar\theta\bar\vartheta}) \parallel y-\eta \parallel_{\mathcal{H}} \leq \mathcal{K}_{(n\bar\theta\bar\vartheta}\varepsilon_n + \Re_{n\bar\theta\bar\vartheta} \parallel x-\zeta \parallel_{\mathcal{H}}, \tag{4.10}$$

where $\Re_{n\bar\theta\bar\vartheta} = \mathcal{K}_{n\bar\theta\bar\vartheta}\Re_l$.

From (4.9) and (4.10), it can be written as

$$\begin{cases} (1-\Re_{\theta\vartheta}) \parallel x-\zeta \parallel_{\mathcal{H}} - \Re_{m\theta\vartheta} \parallel y-\eta \parallel_{\mathcal{H}} \leq \mathcal{K}_{(m\theta\vartheta)}\varepsilon_m \\ (1-\Re_{\bar\theta\bar\vartheta}) \parallel y-\eta \parallel_{\mathcal{H}} - \Re_{n\bar\theta\bar\vartheta} \parallel x-\zeta \parallel_{\mathcal{H}} \leq \mathcal{K}_{(n\bar\theta\bar\vartheta)}\varepsilon_n. \end{cases} \tag{4.11}$$

The matrix representation of the above equation is as follows

$$\begin{pmatrix} (1-\mathfrak{R}_{\theta\vartheta}) & -\mathfrak{R}_{m\theta\vartheta} \\ -\mathfrak{R}_{n\bar{\theta}\bar{\vartheta}} & (1-\mathfrak{R}_{\bar{\vartheta}\bar{\vartheta}}) \end{pmatrix} \begin{pmatrix} \| x-\zeta \|_{\mathcal{H}} \\ \| y-\eta \|_{\mathcal{H}} \end{pmatrix} \leq \begin{pmatrix} \mathcal{K}_{(m\theta\vartheta)}\varepsilon_m \\ \mathcal{K}_{(n\bar{\theta}\bar{\vartheta})}\varepsilon_n \end{pmatrix}.$$

Simplifying the above inequality yields

$$\begin{pmatrix} \| x-\zeta \|_{\mathcal{H}} \\ \| y-\eta \|_{\mathcal{H}} \end{pmatrix} \leq \begin{pmatrix} \frac{(1-\mathfrak{R}_{\bar{\theta}\bar{\vartheta}})}{\Delta} & \frac{\mathfrak{R}_{m\theta\vartheta}}{\Delta} \\ \frac{\mathfrak{R}_{n\bar{\theta}\bar{\vartheta}}}{\Delta} & \frac{(1-\mathfrak{R}_{\theta\vartheta})}{\Delta} \end{pmatrix} \begin{pmatrix} \mathcal{K}_{(m\theta\vartheta)}\varepsilon_m \\ \mathcal{K}_{(n\bar{\theta}\bar{\vartheta})}\varepsilon_n \end{pmatrix}.$$

where $\Delta = (1-\mathfrak{R}_{\theta\vartheta})(1-\mathfrak{R}_{\bar{\theta}\bar{\vartheta}}) - \mathfrak{R}_{m\theta\vartheta}\mathfrak{R}_{n\bar{\theta}\bar{\vartheta}} \neq 0$. Further simplification gives

$$\| x-\zeta \|_{\mathcal{H}} \leq \frac{(1-\mathfrak{R}_{\bar{\theta}\bar{\vartheta}})\mathcal{K}_{(m\theta\vartheta)}\varepsilon_m}{\Delta} + \frac{\mathfrak{R}_{m\theta\vartheta}\mathcal{K}_{(n\bar{\theta}\bar{\vartheta})}\varepsilon_n}{\Delta} \tag{4.12}$$

$$\| y-\eta \|_{\mathcal{H}} \leq \frac{\mathfrak{R}_{n\bar{\vartheta}\bar{\vartheta}}\mathcal{K}_{(m\theta\vartheta)}\varepsilon_m}{\Delta} + \frac{(1-\mathfrak{R}_{\theta\vartheta})\mathcal{K}_{(n\bar{\theta}\bar{\vartheta})}\varepsilon_n}{\Delta}, \tag{4.13}$$

From equations (4.12) and (4.13), we have

$$\| x-\zeta \|_{\mathcal{H}} + \| y-\eta \|_{\mathcal{H}} \leq \frac{(1-\mathfrak{R}_{\bar{\theta}\bar{\vartheta}})\mathcal{K}_{(m\theta\vartheta)}\varepsilon_m}{\Delta} + \frac{\mathfrak{R}_{m\theta\vartheta}\mathcal{K}_{(n\bar{\theta}\bar{\vartheta})}\varepsilon_n}{\Delta} + \frac{\mathfrak{R}_{n\bar{\theta}\bar{\vartheta}}\mathfrak{R}_{(m\theta\vartheta)}\varepsilon_m}{\Delta} + \frac{(1-\mathfrak{R}_{\theta\vartheta})\mathcal{K}_{(n\bar{\vartheta}\bar{\vartheta})}\varepsilon_n}{\Delta}$$

Therefore, we have

$$\| (x,y)-(\zeta,\eta) \|_{\mathcal{H}\times\mathcal{H}} \leq \mathcal{K}_{(mn\theta\vartheta\bar{\theta}\bar{v})}\varepsilon, \tag{4.14}$$

where $\varepsilon = \max(\varepsilon_m, \varepsilon_n)$ and

$$\mathcal{K}_{(mn\theta\vartheta\bar{\theta}\bar{\vartheta})} = \frac{(1-\mathfrak{R}_{\bar{\theta}\bar{\vartheta}}\mathcal{K}_{(m\theta\vartheta)}}{\Delta} + \frac{\mathfrak{R}_{m\theta\vartheta}\mathcal{K}_{(n\bar{\theta}\bar{\vartheta})}}{\Delta} + \frac{\mathfrak{R}_{n\bar{\vartheta}}\mathfrak{R}_{(m\theta\vartheta)}}{\Delta} + \frac{(1-\mathfrak{R}_{\theta\vartheta})\mathcal{K}_{(n\bar{\theta}\bar{\vartheta})}}{\Delta}$$

Therefore, based on the preceding inequality, it can be deduced that the coupled system denoted as (1.1) exhibits Ulam ‘s-Hyers stability. Additionally, the aforementioned inequality can be expressed in the form:

$$\| (x,y)-(\zeta,\eta) \|_{\mathcal{H}\times\mathcal{H}} \leq \varpi(\varepsilon), \text{ where } \varpi(0) = 0.$$

Through this inequality, it becomes evident that the proposed coupled system (1.1) adheres to the principles of generalized Ulam ‘s-Hyers stability.

APPLICATION

Example. Here, we validate our findings related to existence, uniqueness, and stability by presenting a concrete example. Let the following fractional order connected coupled system as follows:

$$\begin{cases} {}^{c}D^{\frac{1}{2}}x(t) = \frac{1}{(t+)^2}\frac{|y(t)|}{1+|y(x)|} + \int_0^t \frac{e^{-(s-t)}}{49} y(s)ds, m \in (0,1], t \in \aleph \\ {}^{c}D^{\frac{1}{2}}y(t) = \frac{1}{100}[t\cos x(t) + x(v)\sin t] + \int_0^t \frac{7}{s^2+7^2} x(s)ds, n \in (0,1], t \in \aleph \\ x(0)+x'(0) = \int_0^1 \frac{|x(s)|}{11+|x(s)|}ds, x(1)+x'(1) = \int_0^1 \frac{|x(s)|}{13+|x(s)|}ds \\ y(0)+y'(0) = \int_0^1 \frac{1}{22}[\cos y(s) + \sin y(s)]ds \\ y(1)+y'(1) = \int_0^1 \frac{1}{26}[\cos y(s) + \sin y(s)]ds. \end{cases}$$

By comparing coupled system (5.1) with system (1.1), the following values are derived:

$$m = n = \frac{1}{2}, \theta = \vartheta = \bar{\theta} = \bar{\vartheta} = 1, \mathfrak{R}_{b_1} = \mathfrak{R}_{\bar{b}_1} = \frac{1}{11}$$

$$\text{and } \mathfrak{R}_{b_2} = \mathfrak{R}_{\bar{b}_2} = \frac{1}{13}.$$

Also,

$$k(t, y(t), \chi^y(t)) = \frac{1}{(t+7)^2}\frac{|y(t)|}{1+|y(t)|} + \int_0^t \frac{e^{-(s-t)}}{49} y(s)ds$$

and

$$l(t, x(t), \chi^x(t)) = \frac{1}{100}[t\cos x(t) + x(t)\sin t] + \int_0^t \frac{7}{s^2+7^2} x(s)ds,$$

whereas $|k(t,y,\chi^y) - k(t,\bar{y},\bar{\chi}^y)| \leq \left(\frac{1}{49}\right) \| y-\bar{y} \| + \|\chi^y - \bar{\chi}^y\|$ and $|l(t,x,\chi^x) - l(t,\bar{x},\bar{\chi}^x)| \leq \frac{1}{50} \| x-\bar{x} \| + \|\chi^x - \bar{\chi}^x\|$.

Therefore, (B4) is satisfied with $\mathfrak{R}_k, \bar{\mathfrak{R}}_k = \frac{1}{49}$ and $\mathfrak{R}_l, \bar{\mathfrak{R}}_l = \frac{1}{50}$. Further, we have

$$\begin{aligned} \sigma_1 &= \left(\frac{1}{\Gamma(n+1)} + \frac{2(\bar{\theta}+\bar{\vartheta})}{\Gamma(n+1)} + \frac{2(\bar{\theta}^2+\bar{\theta}\bar{\theta})}{\bar{\theta}^2\Gamma(n)}\right)\mathfrak{R}_l + \frac{(\theta+\vartheta)(\mathfrak{R}_{b_1}+\mathfrak{R}_{b_2})}{\theta^2} \\ &= \left(\frac{1}{\Gamma(n+1)} + \frac{4}{\Gamma(n+1)} + \frac{4}{\Gamma(n)}\right)\frac{1}{50} + 2\left(\frac{1}{11}+\frac{1}{13}\right) \\ &= \left(\frac{5}{\Gamma(n+1)} + \frac{4}{\Gamma(n)}\right)\frac{1}{50} + \left(\frac{48}{143}\right) = \left(\frac{5}{\Gamma\left(\frac{1}{2}+1\right)} + \frac{4}{\Gamma\left(\frac{1}{2}\right)}\right)\frac{1}{50} + \left(\frac{48}{143}\right) \\ &= \left(\frac{10}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi}}\right)\frac{1}{50} + \left(\frac{48}{143}\right) = \left(\frac{14}{50\sqrt{\pi}}\right) + \left(\frac{48}{143}\right) < 1. \end{aligned}$$

Similarly,

$$\begin{aligned} \sigma_2 &= \left(\frac{1}{\Gamma(m+1)} + \frac{2(\theta+\vartheta)}{\Gamma(m+1)} + \frac{2(\theta^2+\theta\vartheta)}{\theta^2\Gamma(m)}\right)\mathfrak{R}_k + \frac{(\bar{\theta}+\bar{\vartheta})(\mathfrak{R}_{\bar{b}_1}+\mathfrak{R}_{\bar{b}_2})}{\bar{\theta}^2} \\ &= \left(\frac{1}{\Gamma(m+1)} + \frac{4}{\Gamma(m+1)} + \frac{4}{\Gamma(m)}\right)\frac{1}{49} + 2\left(\frac{1}{11}+\frac{1}{13}\right) \\ &= \left(\frac{5}{\Gamma(m+1)} + \frac{4}{\Gamma(m)}\right)\frac{1}{49} + \left(\frac{48}{143}\right) = \left(\frac{5}{\Gamma\left(\frac{1}{2}+1\right)} + \frac{4}{\Gamma\left(\frac{1}{2}\right)}\right)\frac{1}{49} + \left(\frac{48}{143}\right) \\ &= \left(\frac{10}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi}}\right)\frac{1}{49} + \left(\frac{48}{143}\right) = \left(\frac{14}{49\sqrt{\pi}}\right) + \left(\frac{48}{143}\right) < 1. \end{aligned}$$

Consequently, it can be asserted that the coupled system denoted as (5.1) possesses a singular solution. Furthermore, the condition stipulated in Theorem 2, expressed as $(1-\mathfrak{R}_{\theta\vartheta})(1-\mathfrak{R}_{\bar{\theta}\vartheta}) - \mathfrak{R}_{m\theta\vartheta}\mathfrak{R}_{n\bar{\theta}\bar{\vartheta}} \neq 0$, is also satisfied.

Consequently, the coupled system (5.1) demonstrates both Ulam-Hyers stability and generalized Ulam-Hyers stability.

CONCLUSION

We investigate Ulam-Hyers and generalized Ulam-Hyers stability for fractional integro-differential coupled systems with integral boundary conditions, an area where existing research has focused mainly on simpler, isolated equations rather than the coupled systems that appear more frequently in practice. Our approach differs by analyzing how these coupled systems behave when subjected to small perturbations, using fixed-point theorems and carefully constructed function spaces to establish both solution existence and uniqueness.

What makes this work particularly valuable is its relevance to real applications in engineering, finance, and physics, where model stability directly impacts whether computational results can be trusted for decision-making. Our quantitative analysis shows that when initial conditions vary slightly, the resulting solution changes remain bounded and predictable, exactly the kind of stability guarantee practitioners need. Beyond these immediate results, the mathematical framework we've developed opens doors to studying more complex systems, including those with neutral delays, impulsive effects, or various inclusion properties that arise when modeling systems with both finite and infinite time delays. Rather than simply filling theoretical gaps, this work provides concrete tools that researchers can apply when analyzing stability in increasingly sophisticated differential systems.

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AUTHORSHIP CONTRIBUTIONS

Brijendra Kumar Chaurasiya: Investigation, Methodology, Validation, Writing-original draft, Writing-review and editing. Avadhesh Kumar: Conceptualization, Investigation, Methodology, Supervision, Validation, Writing-original draft, Writing-review and editing.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

STATEMENT ON THE USE OF ARTIFICIAL INTELLIGENCE

Artificial intelligence was not used in the preparation of the article.

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