Sigma J Eng Nat Sci, Vol. 43, No. 1, pp. 234−240, February, 2025

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2025.00017

Research Article

Continuous dependence of solutions to a fourth order evolution equation

Sema BAYRAKTAR[1], Emine ÇELİK[1,*], Şevket GÜR[1]

[1]*Department of Mathematics, Sakarya University, Sakarya, 54050, Türkiye*

ARTICLE INFO

Article history
Received: 01 August 2023
Revised: 22 October 2023
Accepted: 10 March 2024

Keywords:
Continuous Dependence; Energy Estimates; Fourth-Order Parabolic Equations, 2020 Mathematics Subject Classification: 35G20, 35G31, 35Q35, 35K25, 35K55

ABSTRACT

We consider an initial-boundary value problem for a fourth-order nonlinear parabolic equation with constant coefficients. Our primary focus lies in establishing *a priori* estimates for the solution to this equation, with a particular emphasis on its continuous dependence on both the initial data and parameters. Using energy estimates, we establish the continuous dependency for both the solution and its gradient concerning the fourth-order nonlinear parabolic equation.

Cite this article as: Bayraktar S, Çelik E, Gür Ş. Continuous dependence of solutions to a fourth order evolution equation. Sigma J Eng Nat Sci 2025;43(1):234−240.

INTRODUCTION

Fourth-order nonlinear parabolic equations have been used in various scientific fields, such as material science and fluid dynamics.

For instance, Cahn-Hilliard equations, introduced by John W. Cahn and John E. Hilliard in 1958 [1], have been used to study phase separation in binary alloys. These equations are employed for studying the evolution of a homogeneous mixture of two components over time to form distinct phases. The basic idea behind the Cahn-Hilliard equations is to observe how thermodynamic forces when introduced to a system with a flow structure, lead to the formation of distinct phases. It can be described as a mass balance law with a phase flux $\mathcal{J}$ and a mobility function $M(\phi)$ that characterizes the rate at which the system can change, as follows:

$$\phi_t + \nabla \cdot \mathcal{J} = 0 \quad \text{with} \quad \mathcal{J} = -M(\phi)\nabla\left(\frac{\delta E(\phi)}{\delta \phi}\right), \tag{1}$$

where ϕ represents concentration, the term $\frac{\delta E(\phi)}{\delta \phi}$ represents the chemical potential, which drives the phase separation.

The Cahn-Hilliard equation [1] is a specific example of a fourth-order parabolic partial differential equation that describes the evolution of the concentration over space and time. In particular, Calderon and Kwembe [2] worked with Cahn-Hilliard equations and used these equations for image analysis. Elliott, Songmu, and Garcke, in papers [3], [4], also have worked on the existence and stability of solutions to Cahn-Hilliard equations.

Thin film equations represent another example of fourth-order nonlinear parabolic equations, and they have been studied to analyze the motion of a very thin layer of

***Corresponding author.**
*E-mail address: eminecelik@sakarya.edu.tr

This paper was recommended for publication in revised form by Editor-in-Chief Ahmet Selim Dalkilic

Published by Yıldız Technical University Press, İstanbul, Turkey

viscous incompressible fluids along an inclined plane, such as coating, draining of foams, and the movement of contact lenses.

A common form of the thin film equation is the fourth-order degenerate parabolic equation and can be described as follows (see [5,6]):

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(M(h)\frac{\partial^3 h}{\partial x^3} + f(h, h_x, h_{xx})\right) = 0, \tag{2}$$

where $h= h(x,t)$ represents the height of the thin film, and M is the mobility function that characterizes how fast the film evolves. One of the studies related to thin film equations is the work by Xu and Zhou [7] and [8]. Their paper has examined the existence and stability of solutions to thin film equations.

On the other hand, in the differential equation

$$\frac{du}{dt} = -c\nabla^2(\nabla^2 u) + \nabla \cdot J_{ne} + \eta,$$

of the motion for the evolution of the surface profile $u(x,t)$, Zangwill [9] used non-equilibrium current J_{ne} in a power series involving the surface slope ∇u and various powers and derivatives thereof to model epitaxial roughening in several different cases, also look at [10]. Here, the Gaussian random variable $\eta(x,t)$ describes the fluctuations in the average deposition flux. Moreover, Ortiz et al. [11] proposed a continuum model for epitaxial thin film growth, which accounts for nucleation and the transition to island growth and for the subsequent roughening and coarsening of the surface profile. They modify and present the model in [9] from different aspects.

One of the recent works on thin equations is by King [12]. He characterized nonnegative solutions of generalized thin equations and used moving boundary conditions. Specifically, King et al. [13] used a fourth-order parabolic equation to model epitaxial nanoscale thin film growth, which had another major interest among researchers in material science recently. They worked on the existence, uniqueness, and regularity of solutions to the equation in a suitable function space.

The model was derived by the following fourth-order nonlinear evolution equation in [13]:

$$u_t + \gamma\Delta u + \alpha\Delta^2 u - \beta\nabla \cdot (|\nabla u|^2 \nabla u) = g, \tag{3}$$

where α, β, γ are positive constants and $g(x,t)$ denotes the deposition flux. Physical interpretations of special derivatives of the height $u(x,t)$ of a film in epitaxial growth in equation (3) are as following:

- $\gamma\Delta u$ is diffusion due to evaporation–condensation [14, 15],
- $\alpha\Delta^2 u$ is capillarity-driven surface diffusion [15,16],
- $\beta\nabla \cdot(|\nabla u|^2\nabla u)$ is hopping of atoms [10].

Liu [17] then studied the regularity of solutions for the fourth-order parabolic equation. He worked on the problem in King's paper for a one-dimensional case.

The model was worked considering different versions in [18] and [19]. For example, Polat [18] established a finite time blow-up result for a thin-film equation including a diffusion term, a fourth-order term, and a nonlocal source term under the periodic boundary conditions. Bertsch et al. [20] also examined how a thin layer of liquid moves over a dry surface due to capillary forces, especially when the liquid only partially wets the surface. The behavior of the liquid film was described by a class of fourth-order degenerate equations. They proved the existence of the weak solutions to this equation in their paper.

There are studies on the model with variable exponents looking at the behavior of solutions asymptotically. For example, Zhang and Zhou [21] established the existence, uniqueness, and long-time behavior of weak solutions for the initial-boundary value problem of a fourth-order degenerate parabolic equation with the variable exponent of nonlinearity. Antontsev and Shmarev [22] also worked on the finite time blow-up phenomenon of the solutions to the nonlinear parabolic problem of fourth order with variable exponents of nonlinearity and their equation includes coefficients depending on both x and t. These coefficients are assumed to be bounded from below and above. On the other hand, Shangerganesh et al. [23] used difference and variations methods to study the existence and uniqueness of weak solutions to fourth-order parabolic equations with variable exponents.

Other studies consider coefficients depend on time. Philippin and Piro [24] work on the finite time blow-up phenomenon of the solutions to the nonlinear parabolic problem of the fourth order with time-dependent coefficients.

Recent studies on models with constant exponents vary. For instance, Zhang et al. [25] studied a fourth-order parabolic equation modeling epitaxial thin film growth, and Han [26] in his paper focused on the solutions to an initial boundary value problem for a fourth-order parabolic equation with a general nonlinearity. They obtained the decay estimate of weak solutions and provided upper and lower estimates for the blow-up time. Additionally, Jansen et al. [27] analyzed the long-time behavior of positive weak solutions to quasilinear doubly degenerate parabolic problems. These problems are of fourth order and involve models for power law fluids and Ellis-fluids. You can also refer to the work on a double degenerate fourth-order parabolic equation with a nonlinear second-order diffusion by Liang et al. [28]. They studied the existence of weak solutions for the boundary degeneracy problem. Some related numerical studies on fourth-order parabolic equations can be found in references such as [29-34] and will not be discussed further in this paper. There is a vast body of work on the existence and uniqueness of solutions to various specific types of fourth-order nonlinear parabolic equations; see [35-37]. Many of these works employ numerical methods and regularization techniques to establish existence and uniqueness. In contrast, we focus on establishing continuous dependence and a priori estimates using energy estimates.

We study on the following initial-boundary value problem of fourth-order nonlinear parabolic equation for $u(x,t)$:

$$u_t - \alpha\Delta u + \beta\Delta^2 u + \gamma u|u|^{p-1} = 0, \quad x \in \Omega,\ t > 0 \tag{4}$$

$$u = \Delta u = 0, \quad x \in \partial\Omega \tag{5}$$

$$u(x,0) = f(x), \quad x \in \Omega, \tag{6}$$

where α, β and γ are positive constants, Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and $p > 1$.

In this current work, we consider coefficients as positive constants. We study the continuous dependence of solutions of the homogeneous Dirichlet problem for the parabolic equation (4) above.

We employ the notation

$$\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p \quad \text{and} \quad \|\cdot\|_{L^2(\Omega)} = \|\cdot\|,$$

where

$$\| f \|_{L^p(\Omega)} = \left(\int_\Omega |f|^p\, dx\right)^{\frac{1}{p}}.$$

Let $\mathring{W}^{1,p}(\Omega)$ be the space of functions in $W^{1,p}(\Omega)$ with vanishing traces on the boundary. For any $1 \le p \le n$, we denote its Sobolev conjugate exponent by p^*, i.e., $p^* = \frac{np}{n-p}$.

We recall Sobolev-Poincaré's inequality.

$$\| f \|_{L^{p^*}(\Omega)} \le C_p \| \nabla f \|_{L^p(\Omega)} \quad \text{for } f \in \mathring{W}^{1,p}(U), \tag{7}$$

where the positive constant C_p depends on p, n and the domain Ω.

Throughout this paper, we assume that

$$p \le \frac{n}{n-2}. \tag{8}$$

This paper is organized as follows. In section 2, we derived *a priori* estimates on ∇u and Δu that will be used in our subsequent analysis. Section 3 proves the continuous dependence of the solutions on the coefficient α while sections 4 and 5 work on the case of β and γ-dependency, respectively.

A Priori Estimates

Lemma 1. *Let $u(x,t)$ be solution of (4), (5) and (6). Then one has*

$$\| \nabla u \|^2 \le A_1, \qquad \| \Delta u \|^2 \le A_2, \tag{9}$$

where A_1, $A_2 > 0$ depends on the parameters of the fourth order parabolic equation (4) and the initial value f.

Proof. Multiplying (4) by u and integrating over Ω, using integration by parts, we have

$$\frac{1}{2}\frac{d}{dt} \| u \|^2 + \alpha \| \nabla u \|^2 + \beta \| \Delta u \|^2 + \gamma \| u \|_{p+1}^{p+1} = 0. \tag{10}$$

Now multiplying (4) by u_t and integrating over Ω, we obtain

$$\frac{d}{dt}\left[\frac{\alpha}{2} \| \nabla u \|^2 + \frac{\beta}{2} \| \Delta u \|^2 + \frac{\gamma}{p+1} \| u \|_{p+1}^{p+1}\right] + \| u_t \|^2 = 0. \tag{11}$$

Adding (10) and (11), we have

$$\frac{d}{dt}E(t) + \| u_t \|^2 + \alpha \| \nabla u \|^2 + \beta \| \Delta u \|^2 + \gamma \| u \|_{p+1}^{p+1} = 0, \tag{12}$$

where $E(t) = \frac{1}{2}\| u \|^2 + \frac{\alpha}{2}\| \nabla u \|^2 + \frac{\beta}{2}\| \Delta u \|^2 + \frac{\gamma}{p+1}\| u \|_{p+1}^{p+1}$. Integrating (12) in t, we can derive that

$$E(t) \le E(0). \tag{13}$$

Thus

$$\| \nabla u \|^2 \le \frac{2}{\alpha}E(0) \quad \text{and} \quad \| \Delta u \|^2 \le \frac{2}{\beta}E(0) \tag{14}$$

which, by defining $A_1 = \frac{2}{\alpha}E(0)$ and $A_2 = \frac{2}{\beta}E(0)$, implies (9).

Continuous Dependence On α

In this section, we will investigate how the solution of the system (4), (5) and (6) depends on the coefficient α of the term Δu. For this purpose, assume that u is the solution to the following problem:

$$u_t - \alpha_1\Delta u + \beta\Delta^2 u + \gamma u|u|^{p-1} = 0, \quad x \in \Omega,\ t > 0 \tag{15}$$

$$u = \Delta u = 0, \quad x \in \partial\Omega \tag{16}$$

$$u(x,0) = f(x), \quad x \in \Omega \tag{17}$$

and v be the solution for

$$v_t - \alpha_2\Delta v + \beta\Delta^2 v + \gamma v|v|^{p-1} = 0, \quad x \in \Omega,\ t > 0 \tag{18}$$

$$v = \Delta v = 0, \quad x \in \partial\Omega \tag{19}$$

$$v(x,0) = f(x), \quad x \in \Omega. \tag{20}$$

Define $w = u - v$, and $\bar{\alpha} = \alpha_1 - \alpha_2$. Then w is the solution of

$$w_t - \alpha_1\Delta w - \bar{\alpha}\Delta v + \beta\Delta^2 w + \gamma[u|u|^{p-1} - v|v|^{p-1}] = 0, \quad x \in \Omega,\ t > 0 \tag{21}$$

$$w = \Delta w = 0, \quad x \in \partial\Omega \tag{22}$$

$$w(x,0) = 0, \quad x \in \Omega. \tag{23}$$

Theorem 2. *Assume (8). One has*

$$\frac{\alpha_1}{2}\parallel \nabla w \parallel^2 + \frac{\beta}{2}\parallel \Delta w \parallel^2 \leq A_3 \bar{\alpha}^2 e^{M_3 t}, \tag{24}$$

where constants $M_3 > 0$ and $A_3 > 0$ depends on initial data (17), (20), and parameters of (21).

Proof. Multiplying equation (21) by w_t, integrating over Ω, and using the integration by parts, we find that

$$\frac{d}{dt}E_1 + \|w_t\|^2 - \bar{\alpha}(\Delta v, w_t) + \gamma\int_\Omega (u|u|^{p-1} - v|v|^{p-1}) w_t dx = 0, \tag{25}$$

where $E_1 = \frac{\alpha_1}{2}\parallel \nabla w \parallel^2 + \frac{\beta}{2}\parallel \Delta w \parallel^2$. Then

$$\frac{d}{dt}E_1 + \|w_t\|^2 \leq |\bar{\alpha}||(\Delta v, w_t)| + \gamma\left|\int_\Omega (u|u|^{p-1} - v|v|^{p-1}) w_t dx\right|. \tag{26}$$

Applying Cauchy-Schwarz and Cauchy's inequalities to the first term on the right-hand side of (26), we have the following:

$$|\alpha||(\Delta v, w_t)| \leq |\bar{\alpha}|\|\Delta v\|\|w_t\| \leq |\bar{\alpha}|^2\|\Delta v\|^2 + \frac{1}{4}\|w_t\|^2. \tag{27}$$

For the second term on the right-hand side of (26), by Mean Value Theorem and Generalized Hölder's inequality for powers $\frac{2n}{n-2}$, 2 and n, we obtain

$$\begin{aligned}&\gamma\left|\int_\Omega (u|u|^{p-1} - v|v|^{p-1}) w_t dx\right| \\ &\leq \gamma p\int_\Omega (|u|^{p-1} + |v|^{p-1})|w||w_t|dx \\ &\leq \gamma p\left(\|u\|_{(p-1)n}^{p-1} + \|v\|_{(p-1)n}^{p-1}\right)\|w\|_{\frac{2n}{n-2}}\|w_t\|.\end{aligned} \tag{28}$$

Now, under the assumption $p \leq \frac{n}{n-2}$, applying Hölder's inequality to the right-hand side of (27) yields

$$\begin{aligned}&\gamma\left|\int_\Omega (u|u|^{p-1} - v|v|^{p-1}) w_t dx\right| \\ &\leq C_1\gamma p\left(\|u\|_{\frac{2n}{n-2}}^{p-1} + \|v\|_{\frac{2n}{n-2}}^{p-1}\right)\|w\|_{\frac{2n}{n-2}}\|w_t\|,\end{aligned} \tag{29}$$

where $C_1 > 0$ is a constant depends on n, p and the space. Now, applying Sobolev's Inequality and then using (9), we can write

$$\begin{aligned}&\gamma\left|\int_\Omega (u|u|^{p-1} - v|v|^{p-1}) w_t dx\right| \\ &\leq \gamma C_1 C_2 C_3 p(\|\nabla u\|^{p-1} + \|\nabla v\|^{p-1})\|\nabla w\|\|w_t\| \\ &\leq M_1\|\nabla w\|\|w_t\|,\end{aligned} \tag{30}$$

where $M_1 = \gamma C_1 C_2 C_3 p A_1^{\frac{p-1}{2}}$ and C_2, C_3 are Sobolev constants. Lastly by Cauchy's inequality, we have

$$\gamma\left|\int_\Omega (u|u|^{p-1} - v|v|^{p-1}) w_t dx\right| \leq M_2\|\nabla w\|^2 + \frac{1}{4}\|w_t\|^2, \tag{31}$$

where $M_2 = M_1^2$. Combining (26), (27), (31), we have

$$\frac{d}{dt}E_1(t) \leq |\bar{\alpha}|^2 A_2 + M_3 E_1(t), \tag{32}$$

where $M_3 = \max\left\{\frac{2M_2}{\alpha_1}, 1\right\}$. Solving differential equation (32) yields the desired result (24).

Continuous dependence on β

In this section, we establish the continuous dependence of the solution of problem (4), (5) and (6) on the coefficient β of the term $||\Delta^2 u||$. Let u and v be solutions of the following problems:

$$u_t - \alpha\Delta u + \beta_1\Delta^2 u + \gamma u|u|^{p-1} = 0, \quad x \in \Omega,\ t > 0 \tag{33}$$

$$u = \Delta u = 0, \quad x \in \partial\Omega \tag{34}$$

$$u(x,0) = f(x), \quad x \in \Omega, \tag{35}$$

and

$$v_t - \alpha\Delta v + \beta_2\Delta^2 v + \gamma v|v|^{p-1} = 0, \quad x \in \Omega,\ t > 0 \tag{36}$$

$$v = \Delta v = 0, \quad x \in \partial\Omega. \tag{37}$$

$$v(x,0) = f(x), \quad x \in \Omega, \tag{38}$$

respectively. Define $w = u - v$, and $\bar{\beta} = \beta_1 - \beta_2$. So w is the solution of

$$\begin{aligned}&w_t - \alpha\Delta w + \beta_1\Delta^2 w + \beta\Delta^2 v + \gamma[u|u|^{p-1} - v|v|^{p-1}] = 0, \\ &x \in \Omega, t > 0\end{aligned} \tag{39}$$

$$w = \Delta w = 0, \quad x \in \partial\Omega \tag{40}$$

$$w(x,0) = 0, \quad x \in \Omega. \tag{41}$$

Theorem 3. *For the solution w of problem (39),(40) and (41), one has*

$$\parallel w \parallel^2 \leq \frac{\bar{\beta}^2}{\beta_1}A_2 e^t. \tag{42}$$

Proof. Multiplying (39) by w and integrating over Ω, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w\|^2 + \alpha\|\nabla w\|^2 + \beta_1\|\Delta w\|^2 + \bar{\beta}(\Delta w, \Delta v) \\ &+\gamma\int_\Omega (u|u|^{p-1} - v|v|^{p-1}) w dx = 0.\end{aligned} \tag{43}$$

Then we can write (43) as following,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w\|^2 + \alpha\|\nabla w\|^2 + \beta_1\|\Delta w\|^2 \\ &\leq |\bar{\beta}||(\Delta w, \Delta v)| + \gamma\left|\int_\Omega (u|u|^{p-1} - v|v|^{p-1}) w dx\right|.\end{aligned} \tag{44}$$

Estimating first term on the right-hand side of (44) by Cauchy-Schwarz and ε-Cauchy inequalities, we have

$$|\bar{\beta}||(\Delta w, \Delta v)| \leq \varepsilon \parallel \Delta w \parallel^2 + \frac{\beta^2}{4\varepsilon} \parallel \Delta v \parallel^2. \quad (45)$$

Following the same steps with Theorem 2, we have the following estimate for the second term on the right-hand side of (44),

$$\gamma \left| \int_\Omega (u|u|^{p-1} - v|v|^{p-1}) w dx \right| \leq \frac{1}{4} \parallel w \parallel^2 + M_2 \parallel \nabla w \parallel^2. \quad (46)$$

Combining (44), (45) and (46), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \parallel w \parallel^2 + (\alpha - M_2) \parallel \nabla w \parallel^2 + (\beta_1 - \varepsilon) \parallel \Delta w \parallel^2 \\ &\leq \frac{\beta^2}{4\varepsilon} \parallel \Delta v \parallel^2 + \frac{1}{4} \parallel w \parallel^2. \end{aligned} \quad (47)$$

Choosing $\varepsilon = \frac{\beta_1}{2}$ and assuming that $\alpha > M_2$, we have

$$\frac{d}{dt} \parallel w \parallel^2 \leq \frac{\bar{\beta}^2}{\beta_1} \parallel \Delta v \parallel^2 + \frac{1}{2} \parallel w \parallel^2 \leq \frac{\bar{\beta}^2}{\beta_1} \parallel \Delta v \parallel^2 + \parallel w \parallel^2. \quad (48)$$

Solving this inequality, we get (42).

Continuous Dependence On γ

Let u and v be solutions of the following problems:

$$u_t - \alpha \Delta u + \beta \Delta^2 u + \gamma_1 u |u|^{p-1} = 0, \ x \in \Omega, \ t > 0. \quad (49)$$

$$u = \Delta u = 0, \ x \in \partial\Omega \quad (50)$$

$$u(x, 0) = f(x), \ x \in \Omega, \quad (51)$$

and

$$v_t - \alpha \Delta v + \beta \Delta^2 v + \gamma_2 v |v|^{p-1} = 0, \ x \in \Omega, \ t > 0 \quad (52)$$

$$v = \Delta v = 0, \ x \in \partial\Omega \quad (53)$$

$$v(x, 0) = f(x), \ x \in \Omega, \quad (54)$$

respectively. Define $w = u - v$, and $\bar{\gamma} = \gamma_1 - \gamma_2$. So w is the solution of

$$\begin{gathered} w_t - \alpha \Delta w + \beta \Delta^2 w + \bar{\gamma} v |v|^{p-1} + \gamma_1 [u|u|^{p-1} - v|v|^{p-1}] = 0, \\ x \in \Omega, t > 0 \end{gathered} \quad (55)$$

$$w = \Delta w = 0, \ x \in \partial\Omega. \quad (56)$$

$$w(x, 0) = 0, \ x \in \Omega. \quad (57)$$

Theorem 4. *Assume (8). One has*

$$\frac{\alpha}{2} \parallel \nabla w \parallel^2 + \frac{\beta}{2} \parallel \Delta w \parallel^2 \leq A_4 A_1^p \bar{\gamma}^2 e^{M_5 t}, \quad (58)$$

where A_4 depends on initial data (51), (54) and parameters of (55).

Proof. Multiplying (55) by w_t and integrating over Ω, we obtain

$$\begin{aligned} &\frac{d}{dt}\left[\frac{\alpha}{2} \parallel \nabla w \parallel^2 + \frac{\beta}{2} \parallel \Delta w \parallel^2\right] + \parallel w_t \parallel^2 \\ &+ \gamma_1 \int_\Omega (u|u|^{p-1} - v|v|^{p-1}) w_t dx + \bar{\gamma} \int_\Omega v |v|^{p-1} w_t dx = 0. \end{aligned} \quad (59)$$

Then we can write (59) as

$$\begin{aligned} &\frac{d}{dt} E_2(t) + \parallel w_t \parallel^2 \\ &\leq \gamma_1 \left| \int_\Omega (u|u|^{p-1} - v|v|^{p-1}) w_t dx \right| + \bar{\gamma} \left| \int_\Omega v |v|^{p-1} w_t dx \right|, \end{aligned} \quad (60)$$

where $E_2(t) = \frac{\alpha}{2} \parallel \nabla w \parallel^2 + \frac{\beta}{2} \parallel \Delta w \parallel^2$. Similar to (31), we can estimate the first term on the right-hand side of (60) as following

$$\gamma_1 \left| \int_\Omega (u|u|^{p-1} - v|v|^{p-1}) w_t dx \right| \leq \frac{1}{4} \parallel w_t \parallel^2 + M_4 \parallel \nabla w \parallel^2, \quad (61)$$

where M_4 depends on γ_1, Sobolev constants and other parameters of equation (55). Next, we estimate the second term on the right-hand side of (60). We will use Young's inequality with $\varepsilon = 1$:

$$\bar{\gamma} \left| \int_\Omega v |v|^{p-1} w_t dx \right| \leq \bar{\gamma}^2 \parallel v \parallel_{2p}^{2p} + \frac{1}{4} \parallel w_t \parallel^2. \quad (62)$$

Now under the condition that $p \leq \frac{n}{n-2}$, we use the Holder's inequality on the first term of the right-hand side of (62) and so we can use Sobolev-Poincaré's inequality (7) to obtain

$$\bar{\gamma} \left| \int_\Omega v |v|^{p-1} w_t dx \right| \leq \bar{\gamma}^2 C_5 \parallel \nabla v \parallel^{2p} + \frac{1}{4} \parallel w_t \parallel^2, \quad (63)$$

with $C_5 > 0$ depends on Sobolev constant and the space Ω.

Then combining (60), (61), (63) together with (9), we have

$$\frac{d}{dt} E_2(t) \leq M_5 E_2(t) + \bar{\gamma}^2 C_5 A_1^p, \quad (64)$$

where $M_5 = \max\left\{\frac{2M_4}{\alpha}, 1\right\}$. Then, by solving (64), we get (58).

CONCLUSION

In this paper, we have shown that the fourth-order nonlinear parabolic equation (4), with three parameters α, β and γ, has solutions that depend continuously on these parameters. Our investigation primarily focuses on a homogeneous Dirichlet problem, and our analysis relies on energy estimates for the derivatives of the solutions. This analysis depends on *a priori* estimates for ∇u and Δu. While our findings significantly contribute to understanding the behavior of solutions to the given equation, they come with certain limitations and can serve as an outline for future research in this field. For instance, we have focused on Dirichlet boundary conditions, but exploring different conditions, such as Neumann or Robin types, could provide valuable insights. Additionally, future studies could further investigate the sensitivity of solutions to variations in the parameters α, β and γ. For example, one could discuss the structural stability concerning these coefficients. Furthermore, integrating numerical simulations or experimental validation could strengthen the validity of our conclusions and offer practical insights for real-world applications.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

Data availability is not applicable to this article as no new data were created or analyzed in this study.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

[1] Cahn JW, Hilliard JE. Free energy of a nonuniform system. i. interfacial free energy. J Chem Phys 1958;28:258–267. [CrossRef]
[2] Calderón CP, Kwembe, TA. Dispersal models. Rev Un Mat Argentina 1991;37:212–229.
[3] Elliott CM, Garcke H. On the cahn-hilliard equation with degenerate mobility. SIAM J Math Anal 1996;27:404–423. [CrossRef]
[4] Elliott CM, Songmu Z. On the cahn-hilliard equation. Arch Rat Mech Anal 1986;96:339–357. [CrossRef]
[5] Liang B, Peng X, Shen H. Study of solutions to a fourth order parabolic equation. Math Model Anal 2016;21:1–15. [CrossRef]
[6] Myers TG. Thin films with high surface tension. SIAM Rev 1998;40:441–462. [CrossRef]
[7] Xu M, Zhou S. Existence and uniqueness of weak solutions for a generalized thin film equation. Nonlinear Anal 2005;60:755–774. [CrossRef]
[8] Xu M, Zhou S. Stability and regularity of weak solutions for a generalized thin film equation. J Math Anal Appl 2008;337:49–60. [CrossRef]
[9] Zangwill A. Some causes and a consequence of epitaxial roughening. J Cryst Growth 1996;163:8–21. [CrossRef]
[10] Das Sarma S, Ghaisas SV. Solid-on-solid rules and models for nonequilibrium growth in 2+1 dimensions. Phys Rev Lett 1992;69:3762–3765. [CrossRef]
[11] Ortiz M, Repetto EA, Si H. A continuum model of kinetic roughening and coarsening in thin films. J Mech Phys Solids 1999;47:697–730. [CrossRef]
[12] King JR. Two generalisations of the thin film equation. Math Comput Model 2001;34:737–756. [CrossRef]
[13] King BB, Stein O, Winkler M. A fourth-order parabolic equation modeling epitaxial thin film growth. J Math Anal Appl 2003;286:459–490. [CrossRef]
[14] Edwards SF, Wilkinson, DR. The surface statistics of a granular aggregate. Math Phys Sci 1982;381:17–31. [CrossRef]
[15] Mullins WW. Theory of thermal grooving. J Appl Phys 1957;28:333–339. [CrossRef]
[16] Herring C. Surface Tension as a Motivation for Sintering. Fundamental Contributions to the Continuum Theory of Evolving Phase Interfaces in Solids 1999. p. 33–69. [CrossRef]
[17] Liu C. Regularity of solutions for a fourth order parabolic equation. Bull Belg Math Soc Simon Stevin 2006;13:527–535. [CrossRef]
[18] Polat, M. A blow-up result for nonlocal thin-film equation with positive initial energy. Turkish J Math 2019;43:1797–1807. [CrossRef]
[19] Qu C, Zhou W. Blow-up and extinction for a thin-film equation with initial-boundary value conditions. J Math Anal Appl 2016;436:796–809. [CrossRef]
[20] Bertsch M, Giacomelli L, Karali G. Thin-film equations with "partial wetting" energy: existence of weak solutions. Phys D Nonlinear Phenom 2005;209:17–27. [CrossRef]
[21] Zhang C, Zhou S. A fourth-order degenerate parabolic equation with variable exponent. J Partial Differ Equ 2009;22:376–392. [CrossRef]
[22] Antontsev S, Shmarev S. Blow-up of solutions to parabolic equations with nonstandard growth conditions. J Comput Appl Math 2010;234:2633–2645. [CrossRef]
[23] Shangerganesh L, Gurusamy A, Balachandran K. Weak solutions for nonlinear parabolic equations with variable exponents. Commun Math 2017;25:55–70. [CrossRef]
[24] Philippin GA, Piro SV. Behaviour in time of solutions to a class of fourth order evolution equations. J Math Anal Appl 2016;436:718–728. [CrossRef]
[25] Zhang H, Cui Z, Xiao X. Decay estimate and blow-up for a fourth order parabolic equation modeling epitaxial thin film growth. AIMS Math 2023;8:11297–11311. [CrossRef]
[26] Han Y. Blow-up phenomena for a fourth-order parabolic equation with a general nonlinearity. J Dyn Control Syst 2021;27:261–270. [CrossRef]

[27] Jansen J, Lienstromberg C, Nik K. Long-time behavior and stability for quasilinear doubly degenerate parabolic equations of higher order. SIAM J Math Anal 2023;55:674–700. [CrossRef]

[28] Liang B, Wang Y, Qu C. Study of weak solutions to a nonlinear fourth-order parabolic equation with boundary degeneracy. Math Methods Appl Sci 2022;45:5892–5907. [CrossRef]

[29] Mishra VN, Rajawat RS, Sharma V. On generalized quantum bernstein polynomials. Advances in Pure and Applied Algebra: Proceedings of the CONIAPS XXVII International Conference 2023. p. 149–160. [CrossRef]

[30] Mittal RC, Rohila R. A study of one dimensional nonlinear diffusion equations by Bernstein polynomial based differential quadrature method. J Math Chem 2016;55:673–695. [CrossRef]

[31] Pandey SH, Rajawat RS, Mishra VN. Approximation properties of modified Jain-Gamma operators preserving linear function. Palestine J Math 2023;12:169–182.

[32] Rajawat RS, Singh KK, Mishra VN. Approximation by modified Bernstein polynomials based on real parameters. Math Found Comput 2023;7:297–309. [CrossRef]

[33] Raiz M, Rajawat RS, Mishra LN, Mishra VN. Approximation on bivariate of Durrmeyer operators based on beta function. J Anal 2024;32:311–333. [CrossRef]

[34] Raiz M, Rajawat RS, Mishra VN. Schurer Durrmeyer operators and their approximation properties. An Univ Craiova Ser Mat Inform 2023;50:189–204. [CrossRef]

[35] Liang B, Li Q, Zhang J, Wang Y. Existence of solutions to a doubly degenerate fourth-order parabolic equation. Appl Math Comput 2022;413:126650. [CrossRef]

[36] Liang B, Su C, Wang Y, Li X, Zhang Z. On a viscous fourth-order parabolic equation with boundary degeneracy. Bound Value Probl 2022;2022:29. [CrossRef]

[37] Ramazanova A, Mehreliyev Y. On solvability of inverse problem for one equation of fourth order. Turk J Math 2020;44:19.