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Research Article

Matrix variate skew laplace distribution

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ABSTRACT

In this study, we introduce a matrix variate skew Laplace distribution as a variance-mean mixture of the matrix variate normal and the scale inverse gamma distribution. The proposed distribution is a generalization of the multivariate skew Laplace distribution studied by [1]. We explore some distributional properties of the proposed distribution such as the probability density function and the characteristic function. Also, we study the estimation of the parameters and give an EM algorithm to obtain the estimates of the parameters. Then, we give a small simulation study to illustrate the performance of the proposed EM algorithm for finding the estimates.

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INTRODUCTION

Scale mixtures of normal distribution play a very important role to analyze data that have heavier tails than the normal distribution. The assumption of a scale mixture of the normal distribution is that the variance is not fixed for all members of the population. But, in some situations, in addition to non-constant variance problems, mean and variance can be related. These types of problems can be seen in financial data. To overcome such phenomena, Barndorff-Nielsen et al. proposed the normal variance-mean mixture distributions [2, 3]. The class of generalized hyperbolic (GH) distribution proposed by Barndorff-Nielsen, which is obtained as the variance mean a mixture of normal and the generalized inverse Gaussian distribution (GIG), is a widely known class of variance-mean mixture distribution [2, 3]. Recently, Gallaugher and McNicholas proposed skew matrix variate distributions using the variance-mean mixture approach [4].

The multivariate Laplace distribution is frequently used for the cases that have heavier tails than the multivariate normal tails. The multivariate Laplace distribution can be obtained as a scale mixture of multivariate normal distribution [5,6]. However, since the multivariate Laplace distribution is a member of the elliptical symmetric distribution family, it cannot be able to model skewness. To model skew data, Kozubowski and Podgorski proposed asymmetric multivariate Laplace distribution [7]. As an alternative to their asymmetric Laplace distribution, Arslan introduced multivariate skew Laplace distribution as the variance-mean mixture of multivariate normal distribution and the inverse gamma distribution [1]. The multivariate skew Laplace distribution given in Arslan has a simpler

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form than the asymmetric Laplace given by Kozubowski and Podgorski [1, 7]. Therefore, it is more applicable than the asymmetric Laplace distribution. One can see the paper by Arslan and Fang et al. for more details about these two skew versions of the Laplace distribution [1, 8].

Concerning the matrix variate case, Sanchez-Manzano et al. proposed the matrix variate power exponential distribution, and the matrix variate Laplace distribution is a special case of this family [9]. This matrix variate generalization of the Laplace distribution is an extension of the multivariate symmetric Laplace distribution. Recently, Yurchenko introduced matrix variate asymmetric Laplace and matrix variate generalized asymmetric Laplace distributions which are the matrix variate extensions of multivariate asymmetric Laplace and generalized asymmetric Laplace distributions [10]. After the work of the Yurchenko, Kozubowski et al. extend the matrix variate generalized asymmetric Laplace distribution using the matrix variate Gamma distribution as a mixing distribution instead of the univariate gamma distribution [11].

In this study, we give a matrix variate generalization of the multivariate skew Laplace distribution given in Arslan using the variance-mean mixture of the matrix variate normal distribution [1]. We give an explicit form of the density function and study some of its distributional properties such as characteristic function. The main aim of this paper is to propose an alternative matrix variate skew distribution to other matrix variate skew distributions which are previously proposed in the literature [4, 12, 13].

The paper is organized as follows. In Section 2, we give the definition of matrix variate skew Laplace distribution and study some of its properties. In Section 3, we give a parameter estimation procedure based on the EM algorithm and in Section 4, we give a small simulation study and the paper is finalized with a conclusion.

Matrix Variate Skew Laplace Distribution: Definition and Properties

In this section, we give the probability density function (pdf), characteristic function (cf), and some distributional properties of the proposed distribution.

Proposition 1 Let Z have a matrix variate normal distribution with the parameters $M = 0$, $\Sigma = I_n$, $\Psi = I_p$ $\left(Z \sim N_{n,p}(0, I_n, I_p)\right)$ and V independent *of* Z have a inverse Gamma distribution with the parameters $\frac{np+1}{2}, \frac{1}{2}$ $\left(V \sim InvG\left(\frac{np+1}{2}, \frac{1}{2}\right)\right)$. Then the random matrix

$$X = M + V^{-1}\Upsilon + V^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}Z\Psi^{\frac{1}{2}} \tag{1}$$

in $R^{n\times p}$ has a matrix variate skew Laplace distribution (MVSL) with the density function

$$f_X(X) = \frac{|\Sigma|^{-\frac{p}{2}}|\Psi|^{-\frac{n}{2}}}{2^{np}\pi^{\frac{np-1}{2}}\alpha\Gamma\left(\frac{np+1}{2}\right)} \exp\Big\{-\alpha\sqrt{tr\{\Sigma^{-1}(X-M)\Psi^{-1}(X-M)'\}} + tr\{\Sigma^{-1}(X-M)\Psi^{-1}\Upsilon'\}\Big\} \tag{2}$$

where $\alpha = \sqrt{1 + tr\{\Sigma^{-1}\Upsilon\Psi^{-1}\Upsilon'\}}$ and $tr(.)$ is a trace function. Here, $M \in R^{n\times p}$ *is a location matrix,* $\Upsilon \in R^{n\times p}$ is a skewness parameter, Σ and Ψ are variance-covariance matrices of size $n \times n$ and $p \times p$ respectively.

Proof Since Z and V are independent, we can write the joint density function as follows

$$f_{Z,V}(Z,v) = (2\pi)^{-\frac{1}{2}np}2^{-\frac{np+1}{2}}\Gamma^{-1}\left(\frac{np+1}{2}\right)v^{-\frac{np+1}{2}} \exp\left\{-\frac{1}{2v}\right\}etr\left\{-\frac{1}{2}ZZ'\right\}$$

Using the transformation $X = M + V^{-1}\Upsilon + V^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}Z\Psi^{\frac{1}{2}}$, we obtain joint density function of X and V as

$$f_{X,V}(X,v) = (2\pi)^{-\frac{1}{2}np}2^{-\frac{np+1}{2}}\Gamma^{-1}\left(\frac{np+1}{2}\right)v^{-\frac{3}{2}}\exp\Big\{-\frac{1}{2}v^{-1} - \frac{1}{2} vtr\Big\{\Sigma^{-1}\big((X-M)-v^{-1}\Upsilon\big)\Psi^{-1}\big((X-M)-v^{-1}\Upsilon\big)'\Big\}\Big\}$$

and the density function of X is given by

$$f(X) = \int_0^\infty f(X,v)dv = \frac{|\Sigma|^{-\frac{p}{2}}|\Psi|^{-\frac{n}{2}}etr\{\Sigma^{-1}(X-M)\Psi^{-1}\Upsilon'\}}{2^{np+\frac{1}{2}}\pi^{\frac{np}{2}}\Gamma\left(\frac{np+1}{2}\right)} \frac{2K_{-\frac{1}{2}}\left(\sqrt{\alpha^2 s}\right)}{\alpha(\alpha^2 s)^{-\frac{1}{4}}}$$

Since $K_{-\frac{1}{2}}\left(\sqrt{\alpha^2 s}\right) = K_{\frac{1}{2}}\left(\sqrt{\alpha^2 s}\right) = \frac{\sqrt{\pi}}{\sqrt{2}}(\alpha^2 s)^{-\frac{1}{4}}e^{-\sqrt{\alpha^2 s}}$, the pdf of X can be obtained as follows

$$f_X(X) = \frac{|\Sigma|^{-\frac{p}{2}}|\Psi|^{-\frac{n}{2}}}{2^{np}\pi^{\frac{np-1}{2}}\alpha\Gamma\left(\frac{np+1}{2}\right)} \exp\Big\{-\alpha\sqrt{tr\{\Sigma^{-1}(X-M)\Psi^{-1}(X-M)'\}} + tr\{\Sigma^{-1}(X-M)\Psi^{-1}\Upsilon'\}\Big\}.$$

Definition 1 A random matrix $X \in R^{n\times p}$ is said to have a $n \times p$ - *dimensional* matrix variate skew Laplace distribution $\left(X \sim MVSL_{n,p}(M, \Sigma, \Psi, \Upsilon)\right)$ if it has the density function given in Eq. (2).

Proposition 2 The conditional distribution of X given V is $N_{n,p}(M + v^{-1}\Upsilon, v^{-1}\Sigma, \Psi)$.

Proof The pdf of random matrix Z is

$$f_Z(Z) = (2\pi)^{-\frac{1}{2}np}etr\left\{-\frac{1}{2}ZZ'\right\} \tag{3}$$

when we use the transformation $X = M + V^{-1}\Upsilon + V^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}Z\Psi^{\frac{1}{2}}$ with the Jacobian $J(Z \to X) = |V^{-1}\Sigma|^{-\frac{p}{2}}|\Psi|^{-\frac{n}{2}}$, we achieve following pdf for X given V

$$f_{X|V}(X|v) = (2\pi)^{-\frac{1}{2}np}|v^{-1}\Sigma|^{-\frac{p}{2}}|\Psi|^{-\frac{n}{2}} etr\left\{-\frac{1}{2}(v^{-1}\Sigma)^{-1}\left(X-(M+v^{-1}\Upsilon)\right)\Psi^{-1}\left(X-(M+v^{-1}\Upsilon)\right)'\right\}.$$

So, we can see that the conditional distribution of X given V is the matrix variate normal distribution.

Proposition 3 Let $X \sim MVSL_{n,p}(M,\Sigma,\Psi,\Upsilon)$ then the characteristic function of X turn out to be

$$\phi_X(T) = etr\{iT'M\}[1+tr\{T'\Sigma T\Psi\} - 2itr\{T'\Upsilon\}]^{-\frac{np+1}{2}}, T \in R^{n\times p}.$$

Proof Since we know the conditional distribution of X given $V = v$, we obtain the characteristic function of X as follows

$$\begin{aligned}\phi_X(T) &= E_V\left[E_{X|V}(etr\{iXT'\})\right]\\ &= E_V\left[etr\left\{iT'(M+v^{-1}\Upsilon)-\frac{1}{2}v^{-1}T'\Sigma T\Psi\right\}\right]\\ &= etr\{iT'M\}\int_0^\infty \exp\left\{v^{-1}tr\left\{iT'\Upsilon-\frac{1}{2}T'\Sigma T\Psi\right\}\right\}g(v)d(v).\end{aligned}$$

If $V \sim InvG\left(\frac{np+1}{2},\frac{1}{2}\right)$ then

$$E(V^{-k}) = \frac{2^k\Gamma\left(\frac{np+1}{2}+k\right)}{\Gamma\left(\frac{np+1}{2}\right)} = 2^k\prod_{i=1}^{k}\left(\frac{np+1}{2}+k-i\right) \quad (4)$$

Using this moment, we can easily find the expectation the variance-covariance of $X \sim MVSL$ which is stated in the following proposition.

Proposition 4 Let $X \sim MVSL_{n,p}(M,\Sigma,\Psi,\Upsilon)$ then expectation and variance-covariance of X is given as follows.

$$E(X) = M + (np+1)\Upsilon$$
$$Cov(X) = (np+1)(\Sigma\otimes\Psi + 2vec(\Upsilon)\mathrm{vec}(\Upsilon)').$$

Theorem 1 Let $X \sim MVSL_{n,p}(M,\Sigma,\Psi,\Upsilon)$ *and* $Y = A + BXC$ where A *is* $n\times p$ constant matrix, B and C are positive definite $n\times n$ and $p\times p$ matrices, respectively, then $Y \sim MVSL_{n,p}(A+BMC, B\Sigma B', C'\Psi C, B\Upsilon C)$.

Proof Using the characteristic function of matrix variate skew Laplace distribution, we obtain the following result

$$\begin{aligned}\phi_Y(T) &= E(etr\{iYT'\})\\ &= E(etr\{i(A+BXC)T'\})\\ &= etr\{iAT'\}E(etr\{iXCT'B\})\\ &= etr\{iAT'\}\phi_X(CT'B)\\ &= etr\{i(A+BMC)T'\}[1+tr\{T'(B\Sigma B')T(C'\Psi C)\} - 2itr\{T'B\Upsilon C\}]^{-\frac{np+1}{2}}.\end{aligned}$$

Parameter Estimation

In this section, we will give parameter estimation of the matrix variate skew Laplace distribution. Let $X_1, X_2, ..., X_l \in R^{n\times p}$ i.i.d. data matrices. Assume that these data come from a matrix variate skew Laplace distribution with the unknown parameters M, Σ, Ψ, and Υ. We will use the maximum likelihood estimation method to obtain the ML estimators of the unknown parameters. To get ML estimators, we have to maximize the following log-likelihood function

$$\begin{aligned}\ell(M,\Sigma,\Psi,\Upsilon) &= -npl\,In(2) - \left(\frac{np-1}{2}\right)l\,In(\pi)\\ &-\frac{l}{2}In(1+tr\{\Sigma^{-1}\Upsilon\Psi^{-1}\Upsilon'\}) - l\,In\left(\Gamma\left(\frac{np+1}{2}\right)\right)\\ &-\sqrt{1+tr\{\Sigma^{-1}\Upsilon\Psi^{-1}\Upsilon'\}}\sum_{i=1}^{l}\sqrt{tr\{\Sigma^{-1}(X_i-M)\Psi^{-1}(X_i-M)'\}}\\ &+\sum_{i=1}^{n}tr\{\Sigma^{-1}(X_i-M)\Psi^{-1}\Upsilon'\}.\end{aligned}$$

We can directly obtain the estimators of the parameters using the above log-likelihood function. However, using the normal variance-mean mixture representation of X will give us great advantages to obtain estimators via the EM algorithm.

Let X_i and V_i observed and missing values, respectively, and (X_i, V_i) as complete data. The joint density function of X and V is given in Proposition 1. So, for $i = 1,2, ..., n$, we can find the log-likelihood function for the complete data (X_i, V_i) as follows

$$\begin{aligned}L(M,\Sigma,\Psi,\Upsilon) &= -\frac{pl}{2}In|\Sigma| - \frac{nl}{2}In|\Psi| - \left(np+\frac{1}{2}\right)l\,In(2) - \frac{npl}{2}In(\pi)\\ &- l\,In\left(\Gamma\left(\frac{np+1}{2}\right)\right) - \frac{3}{2}\sum_{i=1}^{l}In(V_i)\\ &+\sum_{i=1}^{l}tr\{\Sigma^{-1}(X_i-M)\Psi^{-1}\Upsilon'\}\\ &-\frac{1}{2}\sum_{i=1}^{l}(1+tr\{\Sigma^{-1}\Upsilon\Psi^{-1}\Upsilon'\})V_i^{-1}\\ &-\frac{1}{2}\sum_{i}^{l}tr\{\Sigma^{-1}(X_i-M)\Psi^{-1}(X_i-M)'\}V_i.\end{aligned}$$

In the complete data log-likelihood function, since there are some terms in the complete data log-likelihood function that do not include unknown parameters, we can ignore them. Ignoring the irrelevant terms and taking the conditional expectation of complete data log-likelihood function given X and the current estimates of the parameters, we reach the following function to be maximized to get the estimates for the parameters

$$\begin{aligned}Q(M,\Sigma,\Psi,\Upsilon) &= E\left(L(M,\Sigma,\Psi,\Upsilon)|X_i,\widehat{M},\widehat{\Sigma},\widehat{\Psi},\widehat{\Upsilon}\right)\\ &= -\frac{pl}{2}In|\Sigma| - \frac{nl}{2}In|\Psi|\\ &+\sum_{i=1}^{l}tr\{\Sigma^{-1}(X_i-M)\Psi^{-1}\Upsilon'\}\\ &-\frac{1}{2}tr\{\Sigma^{-1}\Upsilon\Psi^{-1}\Upsilon'\}\sum_{i=1}^{l}E\left(V_i^{-1}|X_i,\widehat{M},\widehat{\Sigma},\widehat{\Psi},\widehat{\Upsilon}\right)\\ &-\frac{1}{2}\sum_{i=1}^{l}E\left(V_i|X_i,\widehat{M},\widehat{\Sigma},\widehat{\Psi},\widehat{\Upsilon}\right)tr\{\Sigma^{-1}(X_i-M)\Psi^{-1}(X_i-M)'\}\end{aligned}$$

where $E(V_i|X_i,\hat{M},\hat{\Sigma},\hat{\Psi},\hat{\Upsilon})$ and $E(V_i^{-1}|X_i,\hat{M},\hat{\Sigma},\hat{\Psi},\hat{\Upsilon})$ are the conditional expectation of V_i and V_i^{-1}, respectively, given the observed data and current estimates of the parameters. To obtain these conditional expectations, we have to obtain the conditional distribution of V given X. Doing some straightforward algebra, the density function of the conditional distribution of V given X can be obtained as follows. Note that this conditional distribution is the inverse Gaussian distribution with the parameters $\alpha = \sqrt{1 + tr\{\Sigma^{-1}\Upsilon\Psi^{-1}\Upsilon'\}}$ and $s = tr\{\Sigma^{-1}(X_i - M)\Psi^{-1}(X_i - M)'\}$.

$$f_{V|X}(v|X) = \frac{\alpha}{\sqrt{2\pi}}\exp\{\alpha\sqrt{s}\}v^{-\frac{3}{2}} \exp\left\{-\frac{1}{2}(\alpha^2 v^{-1} + sv)\right\} \quad for\ v > 0$$

Using this conditional distribution, the conditional expectations given above become as

$$w_i = E(V_i|X_i,\hat{M},\hat{\Sigma},\hat{\Psi},\hat{\Upsilon}) = \frac{\sqrt{1 + tr\{\hat{\Sigma}^{-1}\hat{\Upsilon}\hat{\Psi}^{-1}\hat{\Upsilon}'\}}}{\sqrt{tr\{\hat{\Sigma}^{-1}(X_i - \hat{M})\hat{\Psi}^{-1}(X_i - \hat{M})'\}}} \quad (6)$$

$$u_i = E(V_i^{-1}|X_i,\hat{M},\hat{\Sigma},\hat{\Psi},\hat{\Upsilon}) = \frac{1 + \sqrt{(1 + tr\{\hat{\Sigma}^{-1}\hat{\Upsilon}\hat{\Psi}^{-1}\hat{\Upsilon}'\})tr\{\hat{\Sigma}^{-1}(X_i - \hat{M})\hat{\Psi}^{-1}(X_i - \hat{M})'\}}}{1 + tr\{\hat{\Sigma}^{-1}\hat{\Upsilon}\hat{\Psi}^{-1}\hat{\Upsilon}'\}} \quad (7)$$

for

$i = 1,2, \ldots, l$. When we rewrite the Q function using w_i and u_i, we get

$$\begin{aligned} Q(M,\Sigma,\Psi,\Upsilon|X_i,\hat{M},\hat{\Sigma},\hat{\Psi},\hat{\Upsilon}) = &-\frac{pl}{2}In|\Sigma| - \frac{nl}{2}In|\Psi| \\ &+ \sum_{i=1}^{l} tr\{\Sigma^{-1}(X_i - M)\Psi^{-1}\Upsilon'\} \\ &- \frac{1}{2}tr\{\Sigma^{-1}\Upsilon\Psi^{-1}\Upsilon'\}\sum_{i=1}^{l} u_i \\ &- \frac{1}{2}\sum_{i=1}^{l} w_i tr\{\Sigma^{-1}(X_i - M)\Psi^{-1}(X_i - M)'\}. \end{aligned} \quad (8)$$

When we take the derivatives of this objective function with respect to the parameters Σ, Ψ and Υ and setting to zero, we obtain the following estimators

$$\hat{M} = \frac{ave\{w_i X_i\} - \hat{\Upsilon}}{ave\{w_i\}} \quad (9)$$

$$\hat{\Upsilon} = \frac{ave\{w_i(X - X_i)\}}{ave\{u_i\}ave\{w_i\} - 1} \quad (10)$$

$$\hat{\Sigma} = \frac{1}{p}\left[ave\{w_i(X_i - \hat{M})\hat{\Psi}^{-1}(X_i - \hat{M})'\} - ave\{u_i\}\hat{\Upsilon}\hat{\Psi}^{-1}\hat{\Upsilon}'\right] \quad (11)$$

$$\hat{\Psi} = \frac{1}{n}\left[ave\{w_i(X_i - \hat{M})'\hat{\Sigma}^{-1}(X_i - \hat{M})\} - ave\{u_i\}\hat{\Upsilon}\hat{\Sigma}^{-1}\hat{\Upsilon}'\right] \quad (12)$$

Where $ave(.)$ denote the average over $i = 1,2, \ldots, l$ and $\overline{X} = ave\{X_i\}$.

We can give the general description of the ECM algorithm as follows:

1) Initialization: Parameters M, Υ, Σ, and Ψ and are initialized and set $k = 0$.

2) E Step: Update w_i and u_i as follows:

$$w_i^{(k+1)} = E(V_i|X_i,\hat{M},\hat{\Sigma},\hat{\Psi},\hat{\Upsilon}) = \frac{\sqrt{1 + tr\{\hat{\Sigma}^{(k)-1}\hat{\Upsilon}^{(k)}\hat{\Psi}^{(k)-1}\hat{\Upsilon}^{(k)\prime}\}}}{\sqrt{tr\{\hat{\Sigma}^{(k)-1}(X_i - \hat{M}^{(k)})\hat{\Psi}^{(k)-1}(X_i - \hat{M}^{(k)})'\}}} \quad (13)$$

$$u_i^{(k+1)} = E(V_i^{-1}|X_i,\hat{M},\hat{\Sigma},\hat{\Psi},\hat{\Upsilon}) = \frac{1 + \sqrt{(1 + tr\{\hat{\Sigma}^{(k)-1}\hat{\Upsilon}^{(k)}\hat{\Psi}^{(k)-1}\hat{\Upsilon}^{(k)\prime}\})tr\{\hat{\Sigma}^{(k)-1}(X_i - \hat{M}^{(k)})\hat{\Psi}^{(k)-1}(X_i - \hat{M}^{(k)})'\}}}{1 + tr\{\hat{\Sigma}^{(k)-1}\hat{\Upsilon}^{(k)}\hat{\Psi}^{(k)-1}\hat{\Upsilon}^{(k)\prime}\}} \quad (14)$$

3) First CM Step: Parameter Υ is updated as follows:

$$\hat{\Upsilon}^{(k)} = \frac{ave\left\{w_i^{(k+1)}(\overline{X} - X_i)\right\}}{ave\left\{u_i^{(k+1)}\right\}ave\left\{w_i^{(k+1)}\right\} - 1} \quad (15)$$

4) Second CM Step: Updates M parameter as follows:

$$\hat{M}^{(k+1)} = \frac{ave\left\{w_i^{(k+1)}X_i\right\} - \hat{\Upsilon}^{(k+1)}}{ave\left\{w_i^{(k+1)}\right\}} \quad (16)$$

5) Third CM Step: Σ parameter updates as follows:

$$\begin{aligned} \hat{\Sigma}^{(k+1)} = \frac{1}{p}\Big[&ave\left\{w_i^{(k+1)}(X_i - \hat{M}^{(k+1)})\hat{\Psi}^{(k)-1}(X_i - \hat{M}^{(k+1)})'\right\} \\ &- ave\left\{u_i^{(k+1)}\right\}\hat{\Upsilon}^{(k+1)}\hat{\Psi}^{(k)-1}\hat{\Upsilon}^{(k+1)\prime}\Big] \end{aligned} \quad (17)$$

6) Fourth CM Step: Update Ψ parameter:

$$\begin{aligned} \hat{\Psi}^{(k+1)} = \frac{1}{n}\Big[&ave\left\{w_i^{(k+1)}(X_i - \hat{M}^{(k+1)})'\hat{\Sigma}^{(k+1)-1}(X_i - \hat{M}^{(k+1)})\right\} \\ &- ave\left\{u_i^{(k+1)}\right\}\hat{\Upsilon}^{(k+1)\prime}\hat{\Sigma}^{(k+1)-1}\hat{\Upsilon}^{(k+1)}\Big] \end{aligned} \quad (18)$$

7) Fifth CM Step: Check the convergence: If convergence is not achieved, set $t = t + 1$and return step 2.

Simulation

In this section, we conduct two simulation studies to illustrate the performance of the proposed estimation method. We use the variance-mean mixture representation given in Eq. 1 to generate data. For Case I, M and Υ matrices are taken as follows. Also, to eliminate the identifiability issue, we take the diagonal elements of Σ and Ψ matrices as 1. The same idea is used in [4].

$$M=\begin{bmatrix}1 & 0 & -1 & 2\\ -1 & 3 & 4 & 1\\ 1 & -4 & -1 & 2\end{bmatrix},\quad \Upsilon=\begin{bmatrix}0.5 & -0.5 & 0 & 1\\ 0.5 & -0.5 & 0 & 1\\ 0.5 & -0.5 & 0 & 1\end{bmatrix}$$

For Case II, We take M and ϒ as follows

$$M=\begin{bmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 5\\ 3 & 4 & 5 & 6\end{bmatrix},\quad \Upsilon=\begin{bmatrix}0.5 & 0 & 0 & 0.5\\ 0 & 0.5 & 0 & 0\\ 0 & 0 & 0.5 & 0\end{bmatrix}$$

In both simulation, Σ and Ψ matrices are taken as follows

$$\Sigma=\begin{bmatrix}1 & 0.6 & 0.3\\ 0.6 & 1 & 0\\ 0.3 & 0 & 1\end{bmatrix},\quad \Psi=\begin{bmatrix}1 & 0 & 0.8 & 0\\ 0 & 1 & 0 & 0.4\\ 0.8 & 0 & 1 & 0.2\\ 0 & 0.4 & 0.2 & 1\end{bmatrix}$$

In both simulations, we take sample sizes as l = 50, 100, 200 and 400. The simulation is replicated 500 times. We compute the mean Euclidean distance to show the performance of the estimators. In all simulations, the stopping rule ε is taken as 10^{-10}. Mean Euclidean distance are computed as follows.

$$\left\|\widehat{M}-M\right\|=\left(\sum_{i=1}^{n}\sum_{j=1}^{p}(\widehat{m}_{ij}-m_{ij})^2\right)^{\frac{1}{2}},$$

$$\left\|\widehat{\Upsilon}-\Upsilon\right\|=\left(\sum_{i=1}^{n}\sum_{j=1}^{p}(\widehat{\upsilon}_{ij}-\upsilon_{ij})^2\right)^{\frac{1}{2}},$$

$$\left\|\widehat{\Sigma}-\Sigma\right\|=\left(\sum_{i=1}^{n}\sum_{j=1}^{p}(\widehat{\sigma}_{ij}-\sigma_{ij})^2\right)^{\frac{1}{2}} \text{ and}$$

$$\left\|\widehat{\Psi}-\Psi\right\|=\left(\sum_{i=1}^{n}\sum_{j=1}^{p}(\widehat{\psi}_{ij}-\psi_{ij})^2\right)^{\frac{1}{2}}.$$

Simulation results are given in Tables 1-8. In tables, we give estimated matrices, the mean Euclidean distance and the mean iteration number. For all simulations, when the sample size is getting bigger, the mean Euclidean distance, and the mean iteration number are getting smaller. These results imply that the proposed estimation procedure is working accurately.

Table 1. Estimated matrices, mean Euclidean distances and mean iteration number for n=50 in Case I

Estimated matrix	Mean Euclidean distance	Mean iteration number
$\widehat{M}=\begin{bmatrix}1.1295 & -0.1171 & -1.0201 & 2.3387\\ -0.9269 & 2.8835 & 3.9506 & 1.3927\\ 1.1933 & -4.1851 & -1.0127 & 2.2465\end{bmatrix}$	$\left\|\widehat{M}-M\right\|=5.1696$	359.97
$\widehat{\Upsilon}=\begin{bmatrix}0.4927 & -0.4929 & 0.0025 & 0.9682\\ 0.4948 & -0.4919 & 0.0034 & 0.9642\\ 0.4881 & -0.4856 & 0.0031 & 0.9693\end{bmatrix}$	$\left\|\widehat{\Upsilon}-\Upsilon\right\|=0.4330$	
$\widehat{\Sigma}=\begin{bmatrix}0.8856 & 0.5335 & 0.2635\\ 0.5335 & 0.8923 & -0.0025\\ 0.2635 & -0.0025 & 0.8884\end{bmatrix}$	$\left\|\widehat{\Sigma}-\Sigma\right\|=0.3673$	
$\widehat{\Psi}=\begin{bmatrix}1.0747 & 0.00001 & 0.8572 & -0.0064\\ 0.00001 & 1.0847 & -0.0040 & 0.4288\\ 0.8572 & -0.0040 & 1.0731 & 0.2142\\ -0.0064 & 0.4288 & 0.2142 & 1.0824\end{bmatrix}$	$\left\|\widehat{\Psi}-\Psi\right\|=0.4884$	

Table 2. Estimated matrices, mean Euclidean distances and mean iteration number for n=100 in Case I

Estimated matrix	Mean Euclidean distance	Mean iteration number
$\widehat{M}=\begin{bmatrix}1.0432 & -0.1241 & -1.0438 & 2.1538\\ -0.9726 & 2.9042 & 3.9362 & 1.1315\\ 1.0004 & -4.0798 & -1.0432 & 2.1898\end{bmatrix}$	$\left\|\widehat{M}-M\right\|=3.6290$	346.324
$\widehat{\Upsilon}=\begin{bmatrix}0.4981 & -0.4922 & 0.0036 & 0.9903\\ 0.4975 & -0.4949 & 0.0028 & 0.9918\\ 0.5023 & -0.4950 & 0.0045 & 0.9884\end{bmatrix}$	$\left\|\widehat{\Upsilon}-\Upsilon\right\|=0.3102$	
$\widehat{\Sigma}=\begin{bmatrix}0.9204 & 0.5518 & 0.2761\\ 0.5518 & 0.9195 & -0.0011\\ 0.2761 & -0.0011 & 0.9185\end{bmatrix}$	$\left\|\widehat{\Sigma}-\Sigma\right\|=0.2635$	
$\widehat{\Psi}=\begin{bmatrix}1.0729 & -0.0042 & 0.8577 & -0.0044\\ -0.0042 & 1.0723 & -0.0023 & 0.4299\\ 0.8577 & -0.0023 & 1.0712 & 0.2103\\ -0.0044 & 0.4299 & 0.2103 & 1.0694\end{bmatrix}$	$\left\|\widehat{\Psi}-\Psi\right\|=0.3662$	

Table 3. Estimated matrices, mean Euclidean distances and mean iteration number for n=200 in Case I

Estimated matrix	Mean Euclidean distance	Mean iteration number
$\widehat{M} = \begin{bmatrix} 1.0479 & -0.0153 & -0.9830 & 2.0947 \\ -0.9814 & 2.9691 & 4.0025 & 1.1042 \\ 1.0245 & -4.0506 & -1.0072 & 2.0825 \end{bmatrix}$	$\left\|\widehat{M} - M\right\| = 2.5520$	
$\widehat{\Upsilon} = \begin{bmatrix} 0.4959 & -0.5002 & -0.0023 & 0.9913 \\ 0.4978 & -04998 & -0.0007 & 0.9915 \\ 0.4976 & -04958 & -0.0011 & 0.9922 \end{bmatrix}$	$\left\|\widehat{\Upsilon} - \Upsilon\right\| = 0.2172$	
$\widehat{\Sigma} = \begin{bmatrix} 0.9285 & 0.5564 & 0.2801 \\ 0.5564 & 0.9276 & 0.0010 \\ 0.2801 & 0.0010 & 0.9298 \end{bmatrix}$	$\left\|\widehat{\Sigma} - \Sigma\right\| = 0.2075$	337.884
$\widehat{\Psi} = \begin{bmatrix} 1.0667 & 0.0048 & 08527 & 0.0006 \\ 0.0048 & 1.0722 & 0.0034 & 0.4279 \\ 0.8527 & 0.0034 & 1.0668 & 0.2141 \\ 0.0006 & 0.4279 & 0.2141 & 1.0664 \end{bmatrix}$	$\left\|\widehat{\Psi} - \Psi\right\| = 0.2807$	

Table 4. Estimated matrices, mean Euclidean distances and mean iteration number for n=400 in Case I

Estimated matrix	Mean Euclidean distance	Mean iteration number
$\widehat{M} = \begin{bmatrix} 1.0394 & -0.0240 & -0.9949 & 2.0394 \\ -0.9472 & 2.9888 & 4.0173 & 1.0650 \\ 1.0212 & -4.0335 & -1.0025 & 2.0237 \end{bmatrix}$	$\left\|\widehat{M} - M\right\| = 1.7507$	
$\widehat{\Upsilon} = \begin{bmatrix} 0.4963 & -0.4975 & -0.0004 & 0.9971 \\ 0.4955 & -0.4991 & -0.0009 & 0.9948 \\ 0.4986 & -0.4970 & -0.0006 & 0.9979 \end{bmatrix}$	$\left\|\widehat{\Upsilon} - \Upsilon\right\| = 0.1532$	
$\widehat{\Sigma} = \begin{bmatrix} 0.9318 & 0.5600 & 0.2785 \\ 0.5600 & 0.9344 & -0.0005 \\ 0.2785 & -0.0005 & 0.9323 \end{bmatrix}$	$\left\|\widehat{\Sigma} - \Sigma\right\| = 0.1685$	333.956
$\widehat{\Psi} = \begin{bmatrix} 1.0693 & 0.0022 & 0.8544 & 0.0018 \\ 0.0022 & 1.0671 & 0.0026 & 0.4277 \\ 0.8544 & 0.0026 & 1.0655 & 0.2143 \\ 0.0018 & 0.4277 & 0.2143 & 1.0657 \end{bmatrix}$	$\left\|\widehat{\Psi} - \Psi\right\| = 0.2277$	

Table 5. Estimated matrices, mean Euclidean distances and mean iteration number for n=50 in Case II

Estimated matrix	Mean Euclidean distance	Mean iteration number
$\widehat{M} = \begin{bmatrix} 1.1624 & 2.0441 & 3.0405 & 4.1986 \\ 2.0042 & 3.0893 & 4.0644 & 5.0336 \\ 3.0553 & 4.0424 & 5.1319 & 6.0381 \end{bmatrix}$	$\left\|\widehat{M} - M\right\| = 4.7335$	
$\widehat{\Upsilon} = \begin{bmatrix} 0.4865 & -0.0064 & -0.0049 & 0.4836 \\ -0.0002 & 0.4909 & -0.0061 & -0.0049 \\ -0.0034 & -0.0058 & 0.4898 & -0.0024 \end{bmatrix}$	$\left\|\widehat{\Upsilon} - \Upsilon\right\| = 0.3911$	
$\widehat{\Sigma} = \begin{bmatrix} 0.8068 & 0.4857 & 0.2454 \\ 0.4857 & 0.8110 & 0.0038 \\ 0.2454 & 0.0038 & 0.8178 \end{bmatrix}$	$\left\|\widehat{\Sigma} - \Sigma\right\| = 0.4509$	284.432
$\widehat{\Psi} = \begin{bmatrix} 1.1756 & -0.0020 & 0.9388 & -0.0037 \\ -0.0020 & 1.1728 & -0.0007 & 0.4690 \\ 0.9388 & -0.0007 & 1.1837 & 0.2396 \\ -0.0037 & 0.4690 & 0.2396 & 1.1779 \end{bmatrix}$	$\left\|\widehat{\Psi} - \Psi\right\| = 0.6871$	

Table 6. Estimated matrices, mean Euclidean distances and mean iteration number for n=100 in Case II

Estimated matrix	Mean Euclidean distance	Mean iteration number
$\hat{M}=\begin{bmatrix}1.1119 & 1.9406 & 3.0494 & 4.1163\\ 2.0310 & 3.0602 & 4.0573 & 5.0371\\ 2.9834 & 3.9618 & 5.0770 & 5.9969\end{bmatrix}$	$\left\|\hat{M}-M\right\| = 3.2976$	266.340
$\hat{\Upsilon}=\begin{bmatrix}0.4909 & 0.0037 & -0.0046 & 0.4891\\ -0.0028 & 0.4958 & -0.0049 & -0.0043\\ 0.0007 & 0.0013 & 0.4932 & 0.0011\end{bmatrix}$	$\left\|\hat{\Upsilon}-\Upsilon\right\| = 0.2757$	
$\hat{\Sigma}=\begin{bmatrix}0.8344 & 0.5002 & 0.2469\\ 0.5002 & 0.8362 & -0.0023\\ 0.2469 & -0.0023 & 0.8304\end{bmatrix}$	$\left\|\hat{\Sigma}-\Sigma\right\| = 0.3660$	
$\hat{\Psi}=\begin{bmatrix}1.1683 & -0.0008 & 0.9338 & 0.0008\\ -0.0008 & 1.1705 & -0.0002 & 0.4687\\ 0.9338 & -0.0002 & 1.1702 & 0.2379\\ 0.0008 & 0.4687 & 0.2379 & 1.1787\end{bmatrix}$	$\left\|\hat{\Psi}-\Psi\right\| = 0.5407$	

Table 7. Estimated matrices, mean Euclidean distances and mean iteration number for n=200 in Case II

Estimated matrix	Mean Euclidean distance	Mean iteration number
$\hat{M}=\begin{bmatrix}1.0456 & 1.9916 & 3.0033 & 4.0104\\ 2.0329 & 3.0552 & 4.0256 & 5.0110\\ 3.0078 & 3.9329 & 5.0192 & 5.9365\end{bmatrix}$	$\left\|\hat{M}-M\right\| = 2.3592$	263.048
$\hat{\Upsilon}=\begin{bmatrix}0.4952 & -0.0004 & -0.0014 & 0.4996\\ -0.0038 & 0.4942 & -0.0035 & -0.0014\\ -0.0019 & 0.0046 & 0.4979 & 0.0055\end{bmatrix}$	$\left\|\hat{\Upsilon}-\Upsilon\right\| = 0.1903$	
$\hat{\Sigma}=\begin{bmatrix}0.8474 & 0.5087 & 0.2526\\ 0.5087 & 0.8484 & -0.0018\\ 0.2526 & -0.0018 & 0.8445\end{bmatrix}$	$\left\|\hat{\Sigma}-\Sigma\right\| = 0.3214$	
$\hat{\Psi}=\begin{bmatrix}1.1701 & 0.0018 & 0.9363 & 0.0017\\ 0.0018 & 1.1730 & -0.0020 & 0.4709\\ 0.9363 & -0.0020 & 1.1708 & 0.2361\\ 0.0017 & 0.4709 & 0.2361 & 1.1709\end{bmatrix}$	$\left\|\hat{\Psi}-\Psi\right\| = 0.4674$	

Table 8. Estimated matrices, mean Euclidean distances and mean iteration number for n=400 in Case II

Estimated matrix	Mean Euclidean distance	Mean iteration number
$\hat{M}=\begin{bmatrix}1.0131 & 2.0110 & 3.0060 & 4.0306\\ 1.9800 & 3.0337 & 3.9849 & 5.0173\\ 2.9825 & 4.0189 & 5.0039 & 6.0030\end{bmatrix}$	$\left\|\hat{M}-M\right\| = 1.6025$	258.862
$\hat{\Upsilon}=\begin{bmatrix}0.4980 & -0.0006 & -0.0009 & 0.4975\\ 0.0022 & 0.4969 & 0.0016 & -0.0019\\ 0.0007 & -0.0007 & 0.4989 & 0.0003\end{bmatrix}$	$\left\|\hat{\Upsilon}-\Upsilon\right\| = 0.1345$	
$\hat{\Sigma}=\begin{bmatrix}0.8463 & 0.5069 & 0.2545\\ 0.5069 & 0.8485 & -0.0006\\ 0.2545 & -0.0006 & 0.8484\end{bmatrix}$	$\left\|\hat{\Sigma}-\Sigma\right\| = 0.3047$	
$\hat{\Psi}=\begin{bmatrix}1.1654 & 0.0010 & 0.9306 & 0.0011\\ 0.0010 & 1.1749 & 0.0001 & 0.4716\\ 0.9306 & 0.0001 & 1.1652 & 0.2359\\ 0.0011 & 0.4716 & 0.2359 & 1.1722\end{bmatrix}$	$\left\|\hat{\Psi}-\Psi\right\| = 0.4121$	

CONCLUSION

We have introduced a new matrix variate distribution using the variance-mean mixture approach. Also, we have given some distributional properties of the newly proposed distribution and proposed an estimation procedure based on the EM algorithm. Then, we have conducted a small simulation study to show the performance of the proposed algorithm. This distribution has been used as an alternative distribution to model matrix variate skew data. As we compare the proposed distribution with the previously proposed distribution, the newly proposed distribution has a simpler form than the other matrix variate Laplace distribution given in the literature.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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