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Research Article

On a boundary value problem with symmetric double well potential and spectral parameter in the boundary condition

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ABSTRACT

The asymptotic expansion of the eigenvalue of Sturm-Liouville problem is presented. The problem has a symmetric double well potential that is continuous, symmetrical to both the midpoint and quarter point of the related interval and non-increasing on the quarter interval.

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INTRODUCTION

We are interested in the following equation

$$y''(t) + [\lambda - q(t)]y(t) = 0, t \in [0, a]. \quad (1)$$

In (1), we accept that t is independent variable, y is dependent variable of t, real spectral parameter λ is independent of t, real potential function q is dependent of t and continuous. We consider (1) with the pair of following equations

$$\alpha_1 y(0) - \alpha_2 y'(0) = \lambda[\alpha_1' y(0) - \alpha_2' y'(0)] \quad (2)$$

and

$$y(a)\cos\beta + y'(a)\sin\beta = 0. \quad (3)$$

(2)-(3) are named as boundary conditions; (2) is composed of real α_1, α_2, α_1', α_2' constants and $\beta \in [0,\pi)$ in (3). The problem (1)-(3) is a boundary value problem. We noticed that spectral parameter λ (is also called an eigenvalue) seems not only in (1) but also in (2) and it is desirable to determine all values of λ. Problems of this type arise routinely in solving partial differential equations, but also come up in other applications (see [14], [15] and [18]). Walter [26] proves very important theorem for (1)-(3) that if we have

$$\delta_1 = \alpha_1'\alpha_2 - \alpha_1\alpha_2' > 0, \quad (4)$$

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(1)-(3) is a self-adjoint problem. The problem (1)-(3) described here is one of an eigenvalue problem (or called Sturm-Liouville problem) and this type problem is studied a lot of researchers (see [3,5,8-12,16,20,22]).

The expectation of this study is to achieve asymptotic estimates for eigenvalues of (1)-(3) with symmetric double well potential q. The symmetric single and double well potentials are very important and famous functions especially in quantum mechanics (see [1,2,6,7,13,19,21,23,24]). We note that, on the related interval, a symmetric double well potential means that the function is symmetric not only on the whole related interval but also on the half of the related interval and non-increasing on the quarter of the related interval. So we can write for our continuous q in (1) that $q(t) = q(a-t) = q\left(\frac{a}{2}-t\right)$ is satisfied, mathematically. We also take without loss of generality that $q(t)$ has a mean value zero, that is $\int_0^a q(t)dt = 0$ and (4) is provided by (2).

MATERIALS AND METHODS

We know that if a function is monotone on the related interval, the function is also differentiable almost everywhere on that interval [17], so first of all, it should be emphasized that the derivative of the potential of our problem exists.

Our method is based on [9]. If we reconstruct its main theorems for $N = 2$ in pursuit of our goal, we readily get the following results:

Theorem 1. The eigenvalues of (1)-(3) satisfy as $\lambda \to \infty$

(i) for $\alpha_2{}' \neq 0, \beta \neq 0$

$$\begin{aligned}(n+1)\pi &= \int_0^a [r_2(t,\lambda)+\rho_2(t,\lambda)]\,dt \\ &-tan^{-1}\left\{\frac{\alpha_1-\alpha_2[r_1(0,\lambda)+\rho_1(0,\lambda)]-\lambda[\alpha_1'-\alpha_2'(r_1(0,\lambda)+\rho_1(0,\lambda))]}{(\alpha_2-\lambda\alpha_2')[r_2(0,\lambda)+\rho_2(0,\lambda)]}\right\} \\ &-tan^{-1}\left\{\frac{cos\beta+[r_1(a,\lambda)+\rho_1(a,\lambda)]sin\beta}{sin\beta[r_2(a,\lambda)+\rho_2(a,\lambda)]}\right\}+O(\lambda^{-3/2}),\end{aligned}$$

(ii) for $\alpha_2{}' \neq 0, \beta = 0$

$$\begin{aligned}\frac{(2n+3)\pi}{2} &= \int_0^a [r_2(t,\lambda)+\rho_2(t,\lambda)]\,dt \\ &-tan^{-1}\left\{\frac{\alpha_1-\alpha_2[r_1(0,\lambda)+\rho_1(0,\lambda)]-\lambda[\alpha_1'-\alpha_2'(r_1(0,\lambda)+\rho_1(0,\lambda))]}{(\alpha_2-\lambda\alpha_2')[r_2(0,\lambda)+\rho_2(0,\lambda)]}\right\} \\ &+O(\lambda^{-3/2})\end{aligned}$$

where $r_1(t,\lambda)+\rho_1(t,\lambda)$ and $r_2(t,\lambda)+\rho_2(t,\lambda)$ are defined by

$$\begin{aligned}r_1(t,\lambda)+\rho_1(t,\lambda) &= \frac{1}{2}\lambda^{-1/2}\int_0^t q'(x)sin2\lambda^{1/2}(t-x)dx \\ &-\frac{1}{2}\lambda^{-1}\int_0^t q'(x)\left[\int_x^t q(s)ds\right]cos2\lambda^{\frac{1}{2}}(t-x)dx \\ &+\frac{1}{4}\lambda^{-1}\int_0^t q^2(x)cos2\lambda^{1/2}(t-x)dx+O(\lambda^{-3/2})\end{aligned} \tag{5}$$

and

$$\begin{aligned}r_2(t,\lambda)+\rho_2(t,\lambda) &= \lambda^{1/2}-\frac{1}{2}\lambda^{-1/2}q(t)+\frac{1}{2}\lambda^{-1/2}\int_0^t q'(x)cos2\lambda^{1/2}(t-x)dx \\ &+\frac{1}{2}\lambda^{-1}\int_0^t q'(x)\left[\int_x^t q(s)ds\right]sin2\lambda^{1/2}(t-x)dx \\ &-\frac{1}{4}\lambda^{-1}\int_0^t q^2(x)sin2\lambda^{1/2}(t-x)dx+O(\lambda^{-3/2}).\end{aligned} \tag{6}$$

Theorem 2. The eigenvalues of (1)-(3) satisfy as $\lambda \to \infty$

(i) for $\alpha_2{}' = 0, \beta \neq 0$

$$\begin{aligned}(n+1)\pi &= \int_0^a [r_2(t,\lambda)+\rho_2(t,\lambda)]\,dt \\ &-cot^{-1}\left\{\frac{\alpha_2[r_2(0,\lambda)+\rho_2(0,\lambda)]}{\alpha_1-\alpha_2[r_1(0,\lambda)+\rho_1(0,\lambda)]-\lambda\alpha_1'}\right\} \\ &-tan^{-1}\left\{\frac{cos\beta+[r_1(a,\lambda)+\rho_1(a,\lambda)]sin\beta}{sin\beta[r_2(a,\lambda)+\rho_2(a,\lambda)]}\right\}+O(\lambda^{-3/2})\end{aligned}$$

(ii) for $\alpha_2{}' = 0, \beta = 0$

$$\begin{aligned}\frac{(2n+3)\pi}{2} &= \int_0^a [r_2(t,\lambda)+\rho_2(t,\lambda)]\,dt \\ &-cot^{-1}\left\{\frac{\alpha_2[r_2(0,\lambda)+\rho_2(0,\lambda)]}{\alpha_1-\alpha_2[r_1(0,\lambda)+\rho_1(0,\lambda)]-\lambda\alpha_1'}\right\} \\ &+O(\lambda^{-3/2}).\end{aligned}$$

RESULTS AND DISCUSSION

Our aim is to find the following asymptotic approximations for eigenvalues λ_n of (1)-(3) with symmetric double well potential q:

Theorem 3. Let $q(t)$ be double symmetric in (1). Then, the eigenvalues λ_n of (1)-(3) satisfy as $n \to \infty$

(i) for $\alpha_2{}' \neq 0, \beta \neq 0$

$$\begin{aligned}\lambda_n^{1/2} &= \frac{(n+1)\pi}{a}+\frac{1}{(n+1)\pi}\left[\frac{\alpha_1'}{\alpha_2'}+cot\beta\right]-\frac{a}{2(n+1)^2\pi^2}[1+cos(n+1)\pi] \\ &\int_0^{a/4} q'(x)sin\left(\frac{2(n+1)\pi}{a}x\right)dx+O(n^{-3}),\end{aligned}$$

(ii) for $\alpha_2{}' \neq 0, \beta = 0$

$$\lambda_n^{1/2} = \frac{(2n+3)\pi}{2a}+\frac{2}{(2n+3)\pi}\frac{\alpha_1'}{\alpha_2'}+O(n^{-3}),$$

(iii) for $\alpha_2{}' = 0, \beta \neq 0$

$$\lambda_n^{1/2} = \frac{(2n+3)\pi}{2a}+\frac{2}{(2n+3)\pi}\left[\frac{\alpha_2}{\alpha_1'}+cot\beta\right]+O(n^{-3}),$$

(iv) for $\alpha_2{}' = 0, \beta = 0$

$$\begin{aligned}\lambda_n^{1/2} &= \frac{(n+2)\pi}{a}+\frac{1}{(n+2)\pi}\frac{\alpha_2}{\alpha_1'} \\ &+\frac{a}{2(n+2)^2\pi^2}[1+cosn\pi]\int_0^{a/4} q'(x)sin\left(\frac{2(n+2)\pi}{a}x\right)dx+O(n^{-3}).\end{aligned}$$

Proof. **(i)** We compute the terms in Theorem 1-i). Firstly, from (5) and (6), we write that

$$r_1(0,\lambda)+\rho_1(0,\lambda)=O\left(\lambda^{-3/2}\right),$$

$$r_2(0,\lambda)+\rho_2(0,\lambda)=\lambda^{1/2}-\frac{1}{2}\lambda^{-1/2}q(0)+O\left(\lambda^{-3/2}\right),$$

$$\begin{aligned}r_1(a,\lambda)+\rho_1(a,\lambda) &=\frac{1}{2}\lambda^{-1/2}\int_0^a q'(x)sin2\lambda^{1/2}(a-x)dx\\ &-\frac{1}{2}\lambda^{-1}\int_0^a q'(x)\left[\int_x^a q(s)ds\right]cos2\lambda^{1/2}(a-x)dx\\ &+\frac{1}{4}\lambda^{-1}\int_0^a q^2(x)cos2\lambda^{1/2}(a-x)dx+O\left(\lambda^{-3/2}\right)\end{aligned}$$

and

$$\begin{aligned}r_2(a,\lambda)+\rho_2(a,\lambda) &=\lambda^{1/2}-\frac{1}{2}\lambda^{-1/2}q(a)+\frac{1}{2}\lambda^{-1/2}\int_0^a q'(x)cos2\lambda^{1/2}(a-x)dx\\ &+\frac{1}{2}\lambda^{-1}\int_0^a q'(x)\left[\int_x^a q(s)ds\right]sin2\lambda^{1/2}(a-x)dx\\ &-\frac{1}{4}\lambda^{-1}\int_0^a q^2(x)sin2\lambda^{1/2}(a-x)dx+O\left(\lambda^{-3/2}\right).\end{aligned}$$

Therefore, if we define

$$\xi:=\frac{\alpha_1-\lambda\alpha_1'+O\left(\lambda^{-1/2}\right)}{-\lambda^{3/2}\alpha_2'+\lambda^{1/2}\alpha_2+\frac{1}{2}\lambda^{1/2}\alpha_2'q(0)-\frac{1}{2}\lambda^{-1/2}\alpha_2q(0)+O\left(\lambda^{-1/2}\right)},\tag{7}$$

and

$$\varpi:=\frac{cos\beta+sin\beta\left[\frac{1}{2}\lambda^{-1/2}S_1-\frac{1}{2}\lambda^{-1}C_2+\frac{1}{4}\lambda^{-1}C_3\right]+O\left(\lambda^{-3/2}\right)}{sin\beta\left[\lambda^{1/2}-\frac{1}{2}\lambda^{-1/2}q(a)+\frac{1}{2}\lambda^{-1/2}C_1+\frac{1}{2}\lambda^{-1}S_2-\frac{1}{4}\lambda^{-1}S_3\right]+O\left(\lambda^{-3/2}\right)},\tag{8}$$

where

$$\begin{aligned}S_1&:=\int_0^a q'(x)sin2\lambda^{\frac{1}{2}}(a-x)dx\\ C_1&:=\int_0^a q'(x)cos2\lambda^{1/2}(a-x)dx,\\ S_2&:=\int_0^a q'(x)\left[\int_x^a q(s)ds\right]sin2\lambda^{1/2}(a-x)dx,\\ C_2&:=\int_0^a q'(x)\left[\int_x^a q(s)ds\right]cos2\lambda^{1/2}(a-x)dx,\\ S_3&:=\int_0^a q^2(x)sin2\lambda^{1/2}(a-x)dx,\\ C_3&:=\int_0^a q^2(x)cos2\lambda^{1/2}(a-x)dx,\end{aligned}\tag{9}$$

we can rearrange Theorem 1-i) as follows:

$$(n+1)\pi=\int_0^a\left[r_2(t,\lambda)+\rho_2(t,\lambda)\right]dt-tan^{-1}(\xi)-tan^{-1}(\varpi)+O\left(\lambda^{-3/2}\right).\tag{10}$$

We will gain the asymptotic formula of the eigenvalue from (10). Also, we get ξ by using series expansion

$$\begin{aligned}\xi &=\frac{-\lambda\alpha_1'+\alpha_1+O\left(\lambda^{-1/2}\right)}{-\lambda^{3/2}\alpha_2'\left[1-\frac{\alpha_2}{\alpha_2'}\lambda^{-1}-\frac{1}{2}\lambda^{-1}q(0)+O(\lambda^{-2})\right]}\\ &=\left\{\lambda^{-1/2}\frac{\alpha_1'}{\alpha_2'}-\lambda^{-3/2}\frac{\alpha_1}{\alpha_2'}+O(\lambda^{-2})\right\}\left\{1+\lambda^{-1}\frac{\alpha_2}{\alpha_2'}+\frac{1}{2}\lambda^{-1}q(0)+O(\lambda^{-2})\right\}\\ &=\lambda^{-1/2}\frac{\alpha_1'}{\alpha_2'}+O\left(\lambda^{-3/2}\right).\end{aligned}$$

By using this ξ in inverse tangent expansion $tan^{-1}(\xi)=\xi-\frac{\xi^3}{3}+\cdots$, we obtain

$$tan^{-1}(\xi)=\lambda^{-1/2}\frac{\alpha_1'}{\alpha_2'}+O\left(\lambda^{-3/2}\right).\tag{11}$$

For ϖ, we manage similar manner, thus we find that

$$\begin{aligned}\varpi &=\frac{cot\beta+\frac{1}{2}\lambda^{-1/2}S_1-\frac{1}{2}\lambda^{-1}C_2+\frac{1}{4}\lambda^{-1}C_3+O\left(\lambda^{-3/2}\right)}{\lambda^{1/2}\left[1-\frac{1}{2}\lambda^{-1}q(a)+\frac{1}{2}\lambda^{-1}C_1+\frac{1}{2}\lambda^{-3/2}S_2-\frac{1}{4}\lambda^{-3/2}S_3+O(\lambda^{-2})\right]}\\ &=\left\{\lambda^{-1/2}cot\beta+\frac{1}{2}\lambda^{-1}S_1-\frac{1}{2}\lambda^{-3/2}C_2+\frac{1}{4}\lambda^{-3/2}C_3+O(\lambda^{-2})\right\}\\ &\times\left\{1+\frac{1}{2}\lambda^{-1}q(a)-\frac{1}{2}\lambda^{-1}C_1-\frac{1}{2}\lambda^{-3/2}S_2+\frac{1}{4}\lambda^{-3/2}S_3+O(\lambda^{-2})\right\}\\ &=\lambda^{-1/2}cot\beta+\frac{1}{2}\lambda^{-1}S_1+O\left(\lambda^{-3/2}\right).\end{aligned}$$

By putting this calculated ϖ with S_1 in (9) into inverse tangent expansion, we have that

$$tan^{-1}(\varpi)=\lambda^{-1/2}cot\beta+\frac{1}{2}\lambda^{-1}\int_0^a q'(x)sin2\lambda^{1/2}(a-x)dx+O\left(\lambda^{-3/2}\right).$$

The potential is symmetric double well in our problem, i.e. $q(x)$ is symmetric, $q'(x)$ exists and $q'(x) = -q'(a-x)$. So we can write

$$\begin{aligned}\int_{a/2}^a q'(x)sin2\lambda^{1/2}(a-x)dx &=-\int_{a/2}^0 q'(a-u)sin2\lambda^{1/2}udu\\ &=\int_{a/2}^0 q'(u)sin2\lambda^{1/2}udu.\end{aligned}$$

And then,

$$\begin{aligned}\int_0^a q'(x)sin2\lambda^{1/2}(a-x)dx &=\int_0^{a/2} q'(x)sin2\lambda^{1/2}(a-x)dx+\int_{a/2}^a q'(x)sin2\lambda^{1/2}(a-x)dx\\ &=\int_0^{a/2} q'(x)sin2\lambda^{1/2}(a-x)dx-\int_0^{a/2} q'(x)sin2\lambda^{1/2}xdx\\ &=sin2\lambda^{1/2}a\int_0^{a/2} q'(x)cos2\lambda^{1/2}x\,dx\\ &-cos2\lambda^{1/2}a\int_0^{a/2} q'(x)sin2\lambda^{1/2}xdx\\ &-\int_0^{a/2} q'(x)sin2\lambda^{1/2}xdx\end{aligned}$$

that is

$$\begin{aligned}\int_0^a q'(x)sin2\lambda^{1/2}(a-x)dx &=sin2\lambda^{1/2}a\int_0^{a/2} q'(x)cos2\lambda^{1/2}xdx\\ &-\left[1+cos2\lambda^{1/2}a\right]\int_0^{a/2} q'(x)sin2\lambda^{1/2}xdx.\end{aligned}\tag{12}$$

Also, since $q(x)$ is double symmetric and $q'(x)$ exists, we can compose $q'(x)=-q'\left(\frac{a}{2}-x\right)$, then

$$\begin{aligned}\int_{a/4}^{a/2} q'(x)cos2\lambda^{1/2}xdx &= -\int_{a/4}^{a/2} q'\left(\frac{a}{2}-x\right)cos2\lambda^{1/2}xdx\\ &= \int_{a/4}^{0} q'(u)cos2\lambda^{1/2}\left(\frac{a}{2}-u\right)du\\ &= -cos\lambda^{1/2}a\int_0^{a/4} q'(u)cos2\lambda^{1/2}udu\\ &-sin\lambda^{1/2}a\int_0^{a/4} q'(u)sin2\lambda^{1/2}udu,\end{aligned}$$

hence, we gain

$$\begin{aligned}\int_0^{a/2} q'(x)cos2\lambda^{1/2}xdx &= \int_0^{a/4} q'(x)cos2\lambda^{1/2}xdx + \int_{a/4}^{a/2} q'(x)cos2\lambda^{1/2}xdx\\ &= \left[1-cos\lambda^{1/2}a\right]\int_0^{a/4} q'(x)cos2\lambda^{1/2}xdx\\ &-sin\lambda^{1/2}a\int_0^{a/4} q'(x)sin2\lambda^{1/2}xdx.\end{aligned} \tag{13}$$

Similarly,

$$\begin{aligned}\int_{a/4}^{a/2} q'(x)sin2\lambda^{1/2}xdx &= -\int_{a/4}^{a/2} q'\left(\frac{a}{2}-x\right)sin2\lambda^{1/2}xdx\\ &= \int_{a/4}^{0} q'(u)sin2\lambda^{1/2}\left(\frac{a}{2}-u\right)du\\ &= -sin\lambda^{1/2}a\int_0^{a/4} q'(u)cos2\lambda^{1/2}udu\\ &+cos\lambda^{1/2}a\int_0^{a/4} q'(u)sin2\lambda^{1/2}udu,\end{aligned}$$

so

$$\begin{aligned}\int_0^{a/2} q'(x)sin2\lambda^{1/2}xdx &= \int_0^{a/4} q'(x)sin2\lambda^{1/2}xdx + \int_{a/4}^{a/2} q'(x)sin2\lambda^{1/2}xdx\\ &= \left[1+cos\lambda^{1/2}a\right]\int_0^{a/4} q'(x)sin2\lambda^{1/2}xdx\\ &-sin\lambda^{1/2}a\int_0^{a/4} q'(x)cos2\lambda^{1/2}xdx.\end{aligned} \tag{14}$$

By substituting (13) and (14) in (12), we obtain

$$\begin{aligned}\int_0^{a} q'(x)sin2\lambda^{1/2}(a-x)dx &= sin2\lambda^{1/2}a\int_0^{a/4} q'(x)cos2\lambda^{1/2}xdx\\ &-2cos\lambda^{1/2}a\left[1+cos\lambda^{1/2}a\right]\int_0^{a/4} q'(x)sin2\lambda^{1/2}xdx.\end{aligned} \tag{15}$$

If we use the last equality in $tan^{-1}(\varpi)$, we can write

$$\begin{aligned}tan^{-1}(\varpi) &= \lambda^{-1/2}cot\beta + \frac{1}{2}\lambda^{-1}sin2\lambda^{1/2}a\int_0^{a/4} q'(x)cos2\lambda^{1/2}xdx\\ &-\lambda^{-1}cos\lambda^{1/2}a\left[1+cos\lambda^{1/2}a\right]\int_0^{a/4} q'(x)sin2\lambda^{1/2}xdx.\end{aligned} \tag{16}$$

Now, we should calculate $\int_0^a [r_2(t,\lambda)+\rho_2(t,\lambda)]\,dt$ to compute asymptotic eigenvalues of (10). From (6)

$$\begin{aligned}\int_0^a [r_2(t,\lambda)+\rho_2(t,\lambda)]\,dt &= \lambda^{1/2}\int_0^a 1\,dt - \frac{1}{2}\lambda^{-1/2}\int_0^a q(t)dt\\ &+\frac{1}{2}\lambda^{-1/2}I_1 + \frac{1}{2}\lambda^{-1}I_2 - \frac{1}{4}\lambda^{-1}I_3 + O\left(\lambda^{-3/2}\right)\end{aligned} \tag{17}$$

where

$$I_1 := \int_0^a \left\{\int_0^t q'(x)cos2\lambda^{1/2}(t-x)dx\right\}dt,$$

$$I_2 := \int_0^a \left\{\int_0^t q'(x)\left[\int_x^t q(s)ds\right]sin2\lambda^{1/2}(t-x)dx\right\}dt$$

and

$$I_3 := \int_0^a \left\{\int_0^t q^2(x)sin2\lambda^{1/2}(t-x)dx\right\}dt.$$

In the equation (17), since $q(t)$ has a mean value zero, the term $\frac{1}{2}\lambda^{-1/2}\int_0^a q(t)dt$ is zero. We need to calculate I_1, I_2, I_3. Let us adapt Leibniz Formula for these integrals, right away:

$$\begin{aligned}I_1 &= \frac{1}{2\lambda^{1/2}}\left\{\int_0^t q'(x)sin2\lambda^{1/2}(t-x)dx\right\}\Bigg|_{t=0}^{a}\\ &= \frac{1}{2}\lambda^{-1/2}\int_0^a q'(x)sin2\lambda^{1/2}(a-x)dx,\end{aligned} \tag{18}$$

$$\begin{aligned}I_2 &= \left\{-\frac{1}{2\lambda^{1/2}}\int_0^t q'(x)\left[\int_x^t q(s)ds\right]cos2\lambda^{1/2}(t-x)dx\right\}\Bigg|_{t=0}^{a}\\ &+\left\{\frac{1}{2\lambda^{1/2}}\int_0^t q(t)q'(x)cos2\lambda^{1/2}(t-x)dx\right\}\Bigg|_{t=0}^{a}\\ &= -\frac{1}{2}\lambda^{-1/2}\int_0^a q'(x)\left[\int_x^a q(s)ds\right]cos2\lambda^{1/2}(a-x)dx\\ &+\frac{1}{2}\lambda^{-1/2}q(a)\int_0^a q'(x)cos2\lambda^{1/2}(a-x)dx\end{aligned} \tag{19}$$

and since we know $q(t) = q(a\text{-}t)$

$$\begin{aligned}I_3 &= \left\{-\frac{1}{2\lambda^{1/2}}\int_0^t q^2(x)cos2\lambda^{1/2}(t-x)dx\right\}\Bigg|_{t=0}^{a}\\ &+\left\{\frac{1}{2\lambda^{1/2}}q^2(t)\right\}\Bigg|_{t=0}^{a}\\ &= -\frac{1}{2}\lambda^{-1/2}\int_0^a q^2(x)cos2\lambda^{1/2}(a-x)dx.\end{aligned} \tag{20}$$

Consequently, in the equation (17), the terms $\frac{1}{2}\lambda^{-1}I_2$ and $\frac{1}{4}\lambda^{-1}I_3$ get into error term $O\left(\lambda^{-3/2}\right)$ because of (19) and (20). So by using (15) and (18), we reorganize (17) as following:

$$\begin{aligned}\int_0^a [r_2(t,\lambda)+\rho_2(t,\lambda)]\,dt &= \lambda^{1/2}a + \frac{1}{4}\lambda^{-1}sin2\lambda^{1/2}a\int_0^{a/4} q'(x)cos2\lambda^{1/2}xdx\\ &-\frac{1}{2}\lambda^{-1}cos\lambda^{1/2}a\left[1+cos\lambda^{1/2}a\right]\int_0^{a/4} q'(x)sin2\lambda^{1/2}xdx\\ &+O\left(\lambda^{-3/2}\right).\end{aligned} \tag{21}$$

Finally, substituting (11), (16) and (21) into (10) and using reversion, we demonstrate the theorem.

(ii) Similar to (i), we write Theorem 1-ii) as follows:

$$\frac{(2n+3)\pi}{2} = \int_0^a [r_2(t,\lambda)+\rho_2(t,\lambda)]\,dt - tan^{-1}(\xi) + O\left(\lambda^{-\frac{3}{2}}\right).$$

Theorem 3-ii) eventually is proved by using substitution of (11), (21) and ξ is defined by (7) into the this equation, and then reversion.

(iii) We can reformulate Theorem 2-i) as following:

$$(n+1)\pi = \int_0^a [r_2(t,\lambda)+\rho_2(t,\lambda)]\,dt - cot^{-1}(\varsigma) - tan^{-1}(\varpi) + O(\lambda^{-3/2}) \quad (22)$$

where ϖ is defined by (8) and ς is defined by

$$\varsigma := \frac{\lambda^{1/2}\alpha_2 - \frac{1}{2}\lambda^{-1/2}\alpha_2 q(0) + O(\lambda^{-3/2})}{\alpha_1 - \lambda\alpha_1' + O(\lambda^{-3/2})}. \quad (23)$$

By using series expansion, ς is found as

$$\begin{aligned}\varsigma &= \frac{\lambda^{1/2}\alpha_2 - \frac{1}{2}\lambda^{-1/2}\alpha_2 q(0) + O(\lambda^{-3/2})}{-\lambda\alpha_1'\left[1-\lambda^{-1}\frac{\alpha_1}{\alpha_1'}+O(\lambda^{-5/2})\right]}\\ &= \left\{-\lambda^{-1/2}\frac{\alpha_2}{\alpha_1'}+\frac{1}{2}\lambda^{-3/2}\frac{\alpha_2}{\alpha_1'}q(0)+O(\lambda^{-5/2})\right\}\\ &\times\left\{1+\lambda^{-1}\frac{\alpha_1}{\alpha_1'}+\lambda^{-2}\frac{\alpha_1^2}{(\alpha_1')^2}+O(\lambda^{-5/2})\right\}\\ &= -\lambda^{-1/2}\frac{\alpha_2}{\alpha_1'}+O(\lambda^{-3/2}).\end{aligned}$$

By using this ς in inverse cotangent expansion $cot^{-1}(\varsigma) = \frac{\pi}{2} - \varsigma + \frac{\varsigma^3}{3} + \cdots$, we write that

$$cot^{-1}(\varsigma) = \frac{\pi}{2} + \lambda^{-1/2}\frac{\alpha_2}{\alpha_1'} + O(\lambda^{-3/2}). \quad (24)$$

Substituting (16), (21) and (24) into (22), we verify the theorem.

(iv) We reduce Theorem 2-ii) as following:

$$\frac{(2n+3)\pi}{2} = \int_0^a [r_2(t,\lambda)+\rho_2(t,\lambda)]\,dt - cot^{-1}(\varsigma) + O\left(\lambda^{-\frac{3}{2}}\right).$$

Here ς is given by (23). Theorem 3-iv) is a result of substituting of (21) and (24) into the last equation and using reversion.

An Example. $q(t) = cos2t$ is an important, double symmetric function and used with different types in differential equations as potential. For example, Equation (1) with $q(t) = \epsilon cos2t$ is named as Mathieu Equation. We note that ϵ is a real parameter and independent of t. Mathieu equation occurs in a broad spectrum of physical (for example, see [25]). If we express our conclusions Theorem 3 for $q(t) = cos2t$ on $[0,2\pi]$, we get as $n \to \infty$

(i) for $\alpha_2' \neq 0, \beta \neq 0$

$$\lambda_n^{1/2} = \frac{n+1}{2} + \frac{1}{n+1}\left[\frac{\alpha_1'}{\alpha_2'} + cot\beta\right] + O(n^{-3}),$$

(ii) for $\alpha_2' \neq 0, \beta = 0$

$$\lambda_n^{1/2} = \frac{(2n+3)}{2} + \frac{2}{(2n+3)}\frac{\alpha_1'}{\alpha_2'} + O(n^{-3}),$$

(iii) for $\alpha_2' = 0, \beta \neq 0$

$$\lambda_n^{1/2} = \frac{(2n+3)}{4} + \frac{2}{(2n+3)}\left[\frac{\alpha_2}{\alpha_1'} + cot\beta\right] + O(n^{-3}),$$

(iv) for $\alpha_2' = 0, \beta = 0$

$$\lambda_n^{1/2} = \frac{n+2}{2} + \frac{1}{(n+2)}\frac{\alpha_2}{\alpha_1'} + O(n^{-3}).$$

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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