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Research Article

On ψ-quantum fractional operators: Existence, uniqueness and Ulam-Hyers stability

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ABSTRACT

In this paper, we study the generalized concept of q-calculus with respect to another function. The ψ-quantum Riemann-Liouville fractional integral, ψ-quantum Riemann-Liouville fractional derivative, and ψ-quantum Caputo fractional derivative were introduced. The existence, uniqueness, and Ulam-Hyers stability of the solutions with the mentioned derivatives were established. Finally, some examples are considered to demonstrate the results obtained.

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INTRODUCTION

The theory of fractional calculus caught much attention in mathematical fields. Indeed, the theory of fractional calculus is another segment of mathematical analysis designed to observe real-world phenomena with its dominant non-integer order of operation. From a historical perspective, the first-order derivative, which is the beginning of fractional calculus, is developed from the h-calculus, where the h-derivative is defined by

$$D_h f(t) = \frac{f(t+h) - f(t)}{h}.$$

Then, taking the limit of $h \to 0$, we will obtain the well-known classical derivative as follows.

$$\frac{d}{dt} f(t) = \lim_{h \to 0} D_h f(t) = \frac{f(t+h) - f(t)}{h}.$$

Thus, the integer-order derivative only displays the rate of change of one function around the neighborhood of the inspected point. In other words, the displayed rate of change conforms to the time scale and is a homogeneous mixture. But, in reality, the rate of change, which implies most natural phenomena, possesses the time-retardation or the time-acceleration in itself. Preferably, the rate of change in the real world does not fully harmonize with the time scale. And the non-local operators, especially the fractional operators, are more suitable for dealing with this kind of problem. Examples of well-known fractional derivatives

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are given as follows: The first type is the Riemann-Liouville derivative given by

$$ {}_{a}^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{a+}^{t} (t-s)^{n-\alpha-1} f(s) ds \,, n-1 < \alpha \le n, $$

and

$$ {}_{a}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a+}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s) ds, $$

is called the left Caputo fractional operator of order α.

By these, fractional calculus became powerful tools for describing real world phenomena and caught many applications from various research fields such as engineering, viscoelastics, and modeling (see [1–4, 8, 9, 15, 18, 20–22, 25–27, 29, 30]). Moreover, fractional calculus also caught the attention of enormous numbers of mathematical researchers. Tremendous mathematicians proposed several useful methods to investigate the existence, uniqueness, and stability of the solution. Some examples are Banach's fixed point theorem, Krasnoselkii's fixed point theorem, the method of upper and lower solutions, and Ulam-Hyers stability (see [10, 11, 17, 23, 28]).

Since time has moved forward, the development of mathematical concepts has never stopped. Various mathematicians extended the concept of derivatives from their curiosity. Such a concept is called global derivative [7]. Firstly, the fundamental extension concept of rate of change is extended from the standard rate of change, which is in the form of

$$ r_g = \frac{f(t_2) - f(t_1)}{g(t_2) - g(t_1)}. $$

Clearly, if $g(t_2) = t$, the global rate of change is reduced to the normal one. Then, the global derivative is defined by

$$ D_g f(t) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{g(t) - g(t_1)}. $$

For example, set $g(t) = t^{\beta}$, the global derivative is said to be the fractional derivative as

$$ D_{t^{\beta}} f(t) = \frac{d}{dt^{\beta}} f(t) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^{\beta} - {t_1}^{\beta}}. $$

The physical meaning of the fractal derivative is that the rate of change is rescaled by the fractal. In this case, the rescaled rate respects the function $g(t) = t^{\beta}$. But, such a rate of change still fully conforms to the normal time-flow pattern. So, the fractal-fractional operators are the illustration of the rate of change that respects t^{β} But is not homogeneous with a time scale. The two definitions of fractal-fractional derivatives, where α ∈ (0, 1), are as follows. The fractal-fractional derivative of order α for the function $f(t)$ in the sense of Riemann-Liouville is given by

$$ {}_{a}^{RL}D_{t^{\beta}}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt^{\beta}} \int_{a+}^{t} (t-s)^{-\alpha} f(s) ds $$
$$ = \frac{1}{\beta t^{\beta-1}\Gamma(1-\alpha)} \frac{d}{dt} \int_{a+}^{t} (t-s)^{-\alpha} f(s) ds. $$

Also, the fractal-fractional derivative of the Caputo type is defined as

$$ {}_{a}^{C}D_{t^{\beta}}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{a+}^{t} (t-s)^{-\alpha} \frac{d}{ds^{\beta}} f(s) ds $$
$$ = \frac{1}{\Gamma(1-\alpha)} \int_{a+}^{t} (t-s)^{-\alpha} \frac{f'(s)}{\beta s^{\beta-1}} ds. $$

Also, the fractal-fractional derivatives are used to visualize the slope in non-euclidean geometry and to display more complex physical problems. One of the examples is the fractal-fractional derivative, which was used to investigate the fluid flow (see [6, 12]). Also, in the mathematical aspect, the fractal-fractional derivatives are more general operators for the reason that such operators can be reduced to the fractional derivatives when β = 1. Since we already have the definition of the rescaled rate of change, which does not totally correspond to the time flow. The fractional calculus is developed again to explain the rescaled rate of change, which is not homogeneous with the rescaled time. Such an operator is called a ψ-fractional derivative [14]. Determine that ψ(t) is strictly increased, the ψ-Riemann-Lioville fractional derivative of order α for function $f(t)$ is define by

$$ {}_{a}^{RL}D_{\psi}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_{a+}^{t} (\psi(t) - \psi(s))^{n-\alpha-1} f(s) \psi'(s) ds. $$

Also, the left ψ- Caputo fractional derivative of order α is defined by

$$ {}_{a}^{C}D_{\psi}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a+}^{t} (\psi(t) - \psi(s))^{n-\alpha-1} \left(\psi'(s)\right)^{1-n} f^{(n)}(s) ds. $$

The physical explanation of such a derivative is an extension of the global derivative, where both rates of change and normal scale of time-flow are rescaled by $\psi(t)$. In particular, in this derivative, the rate of change with respect to the time flow from any $\psi(tk)$ to $\psi(tk+1)$. In addition, if the function $\psi(t) = t$, then ψ-fractional derivatives reduces to the classical fractional derivatives.

In 1909, Jackson [13] introduced the quantum calculus, where q-derivative is defined as

$$ D_q f(t) = \frac{f(qt) - f(t)}{qt - t}, $$

And $q-$ integral as

$$I_q f(t) = \int_0^t f(s) d_q s = (1-q) \sum_{n=0}^{\infty} t q^n f(t q^n).$$

The concepts of q-derivative and q-integral are studied through fractional approaches by numerous researchers (see [5, 16, 24]). However, the studies on fractional q-calculus are quite vast, especially for the q-derivative and q-integral with respect to another function.

In this work, we extend the concept of quantum calculus as well as introduce a novel definition of ψ-quantum fractional operators. Also, we study the existence and Ulam-Hyers stability of a nonlinear q-difference equation with a q-derivative with respect to another function. The main advantage of these novel operators is that they can be reduced to the q-fractional derivatives, the fractional derivatives, and the ψ-fractional derivatives by varying the value of q and the function ψ.

Concept of Normal q-Fractional Calculus

This section will introduce the fundamental concept of q-calculus. Firstly, for any q ∈ (0, 1), the q-analog structures are defined as follows.

$$(n-m)^{(k)} = \prod_{i=0}^{\infty} \frac{n - m q^i}{n - m q^{i+k}}, n \neq 0, k \in \mathbb{R},$$

and

$$\Gamma_q(t) = \frac{(1-q)^{(t-1)}}{(1-q)^{t-1}}, t \in \mathbb{R} - \{0, -1, -2, \dots\},$$

where $\Gamma_q(t+1) = [t]_q \Gamma_q(t)$ with

$$[m]_q = \frac{1-q^m}{1-q}, m \in \mathbb{R}.$$

Definition 2.1. [5] Let $p \in \mathbb{R}^+$, then $L^p\,[a,b]$ is the space of the functions on $(a,b]$. Thus, $L^p\,[a,b]$ with

$$\|f\|_p = \sup_{t \in (a,b]} \left(\int_a^b |f(t)|^p d_q t \right)^{\frac{1}{p}} < \infty,$$

is a Banach space.

If $p=1$, then the space reduces to $L_q(a,b)$.

Definition 2.2. [5] Let $C_q^n[a,b]$ such that $D_q^{n-1} f(t) \in C_q^n[a,b]$. Thus, $C_q^n[a,b]$ with

$$\|f\| = \sup_{t \in [a,b]} \sum_{i=0}^{n-1} \left| D_q^i f(t) \right| < \infty,$$

is a Banach space and for any $n=1$ it reduces to $C[a,b]$, for $q=1$ as $C^n[a,b]$.

Definition 2.3. [5] Let $AC_q[a,b]$, then $f \in AC_q[a,b]$ if and only if $\exists\, \omega \in \mathbb{R}$ constant and $\Omega(t) \in L_q^p(a,b)$ such that

$$f(t) = \omega + \int_a^b \Omega(t) d_q s.$$

Moreover, for $q=1$, it reduces to $AC[a,b]$.

Definition 2.4. [5] The space $AC_q^{(n)}[a,b]$ is a space of function on $[a,b]$ such that $D_q^{n-1} f(t) \in AC_q[a,b]$. Also, for $q=1$, it reduces to $AC^{(n)}[a,b]$

Definition 2.5. [24] Let $0 < q < 1$ and $\alpha > 0$, then

$$q I_t^{\alpha} f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} f(s)\, d_q s,$$

is called the q-Riemann-Liouville fractional integral.

Now, if $0 \leq \alpha, \beta$ and $f(t)$ is a function on $[0. T]$,, then

1. $q I_t^{\alpha} q I_t^{\beta} f(t) = q I_t^{\alpha+\beta} f(t)$.
2. $q D_t^{\alpha} q I_t^{\alpha} f(t) = f(t)$.

Definition 2.6. [5] Let $n-1 < \alpha < n$, the q-Riemann-Liouville fractional derivative of the function $f(t)$ is given by $q D_t^{\alpha} f(t) = D_q^{\alpha} q I_t^{n-\alpha} f(t)$.

Definition 2.7. [5] Let $n-1 < \alpha < n$, the q-Caputo fractional derivative of the function $f(t)$ is given by $D_t^{\alpha} f(t) = q I_t^{n-\alpha} D_q^{\alpha} f(t)$.

Theorem 2.8. Suppose $n-1 < \alpha < n$ $f \in L\,[0,T]$ with $q I_t^{n-\alpha} f(t) \in AC_t^{(n)}[0,T]$, then

$$I_t^{\alpha} q D_t^{\alpha} f(t) = f(t) - \sum_{i=0}^{n-1} q I_t^{1+i-\alpha} f(0) \frac{t^{\alpha-i-1}}{\Gamma_q(\alpha-i)},$$

where

$$q I_t^{1+i-\alpha} f(0) = \lim_{t \to 0^+} q I_t^{1+i-\alpha} f(t).$$

Theorem 2.8. Suppose $n-1 < \alpha < n$ $f \in L\,[0,T]$ with $f(t) \in AC^{(n)}[0,T]$ with $f(t) \in AC_t^{(n)}[0,T]$, then

$$q I_t^{\alpha}\, {}_q^C D_t^{\alpha} f(t) = f(t) - \sum_{i=0}^{n-1} D_q^i f(0) \frac{t^i}{\Gamma_q(i+1)},$$

where

$$D_q^i f(0) = \lim_{t \to 0^+} D_q^i f(t).$$

Extension of q-Calculus

In the classical q-derivative, the rate of change in quantum sense is defined by

$$D_q f(t) = \frac{f(qt) - f(t)}{qt - t}.$$

In this case, the global q-derivative can be extended from $D_q f(t)$ as

$$qD_g f(t) = \frac{f(qt) - f(t)}{g(qt) - g(t)}.$$

Thus, the global q-integral with respect to qD_g is written by

$$qI_g f(t) = \int_{g(0)}^{g(t)} f(s) d_q s = \int_0^t f(s) D_q g(s)\, d_q s.$$

Now, according to the concept of iterated integral in quantum sense, the multiple q-integral follows [24].

$$qI_t^n f(t) = \int_0^t \int_0^{x_{n-1}} \dots \int_0^{x_1} f(s) d_q s d_q\, x_1 \dots d_q x_{n-1}$$
$$= \frac{1}{\Gamma_q(n)} \int_0^t (t - qs)^{(n-1)} f(s) d_q s.$$

As an analogous structure, the nth –times q-integral with respect to ψ, where $\psi(0) = 0$, is

$$qI_\psi^n f(t) = \int_{\psi(0)}^{\psi(t)} \int_{\psi(0)}^{\psi_{n-1}} \dots \int_{\psi(0)}^{\psi_1} f^*\big(s\psi_0(s)\big) d_q \psi_0 d_q\, \psi_1 \dots d_q \psi_{n-1},$$
$$= \frac{1}{\Gamma_q(n)} \int_0^t \big(\psi(t) - q\psi(s)\big)^{(n-1)} f(s) D_q \psi(s) d_q s.$$

Thus, we can define the ψ-quantum fractional integral, where $\psi(t)$ is continuous and stricly increase with $\psi(0) = 0$.

Definition 3.1. Let $q \in (0,1)$, $\alpha > 0$, and $\psi'(t) \neq 0$, then the $\psi - q$-Riemann-Liouvelle fractional integral is define as

$$qI_\psi^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t \big(\psi(t) - q\psi(s)\big)^{(\alpha-1)} f(s) D_q \psi(s) d_q s.$$

By substitution $\psi(s) = u$, we gain

$$qI_\psi^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (\psi(t) - qu)^{(\alpha-1)} f^*(s,u) d_q u. \quad (1)$$

Since it is analogous structure to the operator qI_t^α, it is clear that qI_ψ^α also holds the following properties:

1. $qI_\psi^\alpha qI_\psi^\beta f(t) = qI_\psi^{\alpha+\beta} f(t)$.
2. $qD_\psi^\alpha qI_\psi^\beta f(t) = f(t)$.

It can be seen that the operator is reduced to the definition 2.5 when $\psi(t) = 1$

Definition 3.2. Let $n - 1 < \alpha < n$, the $\psi - q$ –Riemann-Liouville fractional derivative of the function $f(t)$ is given by $qD_\psi^\alpha f(t) = D_\psi^n qI_\psi^{n-\alpha} f(t)$.

It can be seen that the operator is reduced to the definition 2.6 when $\psi(t) = 1$.

Definition 3.3. Let $n - 1 < \alpha < n$, the $\psi - q$ –Caputo fractional derivative of the function $f(t)$ is defined by ${}_q^C D_\psi^\alpha f(t) = qI_\psi^{n-\alpha} D_\psi^\alpha f(t)$.

It can be seen that the operator is reduced to the definition 2.7 when $\psi(t) = 1$.

To go further than this, it is essential to know the following spaces.

Definition 3.4. The space $AC_{q,\psi}^n[a,b]$ is the space of function on $[a, b]$ such that $qD_\psi^{n-1} f(t) \in AC_q[a,b]$. Moreover, for $q = 1$, gives $AC_\psi^n[a,b]$..

Definition 3.5. The space $C_{q,\psi}^{(\delta)}[a,b]$ is a space of functions on $[a,b]$, such that $\big(\psi(t) - \psi(a)\big)^\delta f(t) \in [a,b]$.

Definition 3.6. The space $C_{q,\psi}^{n,(\delta)}[a,b]$ is a space of continuous function on $[a, b]$ such that $qD_\psi^{n-1} f(t) \in C[a,b]$ and $qD_\psi^n f(t) \in C_{q,\psi}^{(\delta)}[a,b]$.

Theorem 3.7. Suppose $n - 1 < \alpha < n$ $f \in \mathbb{L}\,[0,T]$ with $qI_\psi^{n-\alpha} f(t) \in AC_{q,\psi}^{(n)}[0,T]$, then

$$qI_\psi^\alpha {}_q^C D_\psi^\alpha f(t) = f(t) - \sum_{i=0}^{n-1} qI_\psi^{1+i-\alpha} f(0) \frac{\big(\psi(t) - \psi(0)\big)^i}{\Gamma_q(\alpha + 1)}.$$

Where

$$qI_\psi^{1+i-\alpha} f(0) = \lim_{t \to 0^+} qI_\psi^{1+i-\alpha} f(t).$$

Proof: The proof is trivial. By substituting the integral to the form of (3.1) and applying the theorem 2.8, we obtained the illustrated result.

Theorem 3.8. Suppose $n - 1 < \alpha < n$, $f \in \mathbb{L}q[0,T]$ with $f(t) \in AC_{q,\psi}^{(n)}[0,T]$, then

$$qI_\psi^\alpha {}_q^C D_\psi^\alpha f(t) = f(t) - \sum_{i=0}^{n-1} D_\psi^i f(0) \frac{\big(\psi(t) - \psi(0)\big)^i}{\Gamma_q(i + 1)},$$

where

$$D_\psi^i f(0) = \lim_{t \to 0^+} D_\psi^i f(t).$$

Proof: The proof is trivial. By substituting the integral to the form of (3.1) and applying the theorem 2.9, we obtained the illustrated result.

On ψ –Quantum Fractional Difference Equation

The existence and uniqueness of solutions to the following equations will be investigated in this part. Now, consider the equation,

$$qD_\psi^\alpha x(t) = f\big(t, x(t)\big),\ 0 < \alpha < 1,$$
$$qI_\psi^{1-\alpha} x(0) = x_0, \quad t \in (0,T]. \quad (2)$$

The mild solution of the equation (2) is written as

$$x(t) = x_0\big(\psi(t)\big)^{\alpha-1} + \frac{1}{\Gamma_q(\alpha)} \int_0^t \big(\psi(t) - q\psi(s)\big)^{(\alpha-1)} f\big(s, x(s)\big) D_q \psi(s) d_q s. \quad (3)$$

Such a solution is obtained by applying the theorem 3.7 on (4.1). Next, consider the equation

$$ {}_{q}^{C}D_{\psi}^{\alpha} y(t) = f\big(t, y(t)\big), 0 < \alpha < 1, \quad y(0) = y_0, \quad t \in (0, T]. \tag{4}$$

The mild solution of the equation (2) is written as

$$ y(t) = y_0 + \frac{1}{\Gamma_q(\alpha)} \int_0^t \big(\psi(t) - q\psi(s)\big)^{(\alpha-1)} f\big(s, y(s)\big) D_q\psi(s) d_q s. \tag{5}$$

Theorem 4.1. [5] Given that $\phi: 0, \alpha] \to \mathbb{R}$ is a function, such that $\phi \in \mathbb{L}_q[0, \alpha]$, then $I_t^\alpha \phi \in \mathbb{L}_q[0, \alpha]$, and $\|qI_t^\alpha \phi\|_1 \leq \frac{\alpha^\alpha}{\Gamma_q(\alpha+1)} \|\phi\|_1$.

Theorem 4.2. [19] Given that $0 < q < 1$ and $0 < s < 1$, the inequality of q −gamma function for any $z > 0$ is satisfied,

$$ \left(\frac{1 - q^{z+\frac{s}{2}}}{1-q}\right)^{1-s} < \frac{\Gamma_q(z+1)}{\Gamma_q(z+s)} < \left(\frac{1 - q^{z+s}}{1-q}\right)^{1-s}. $$

Theorem 4.3. Given that $0 < q < 1$ and $0 < \alpha < 1$, the inequality of q−gamma function is true,

$$ 1 < \frac{1}{\Gamma_q(\alpha+1)} < \left(\frac{1-q}{1 - q^{\frac{\alpha+1}{2}}}\right)^\alpha. $$

Proof: Suppose $z = \alpha$ and $s = 1 - \alpha$ into inequality of theorem 4.2; gives

$$ \left(\frac{1 - q^{\alpha+\frac{1-\alpha}{2}}}{1-q}\right)^\alpha < \Gamma_q(\alpha+1) < 1. $$

Rearrange the inequality, the new inequality holds:

$$ 1 < \frac{1}{\Gamma_q(\alpha+1)} < \left(\frac{1-q}{1 - q^{\frac{\alpha+1}{2}}}\right)^\alpha. $$

The proof is completed.

To establish the uniqueness of the solution, the following assumption is important.

(A0) There exists $M > 0$, such that

$$ \|f(t, u) - f(t, v)\| \leq M\|u - v\|, $$

for all $u, v \in \mathbb{L}_q[0, T]$.

Theorem 4.2. Suppose that the assumption (A0) holds. The equation (3) is a unique solution of (2), and the equation (5) is a unique solution of (4) if there exists the contraction constant $M\big(\psi(T)\big)^{\alpha+1} \left(\frac{1-q}{1-q^{\frac{\alpha+1}{2}}}\right)^\alpha < 1$.

Proof: Firstly, Define $\mathbb{T}: \mathbb{L}_q[0, T] \to \mathbb{L}_q[0, T]$ by $\mathbb{T}z = z$. The fixed point equation of (3) is written as

$$ \mathbb{T}x(t) = x_0\big(\psi(t)\big)^{\alpha-1} + \frac{1}{\Gamma_q(\alpha)} \int_0^t \big(\psi(t) - q\psi(s)\big)^{(\alpha-1)} f\big(s, x(s)\big) D_q\psi(s) d_q s. $$

Then,

$$ \|\mathbb{T}x - \mathbb{T}u\|_1 \leq \big\|f\big(t, x(t)\big) - f\big(t, u(t)\big)\big\|_1 \frac{\big(\psi(T)\big)^\alpha}{\Gamma_q(\alpha+1)} \big\|qI_\psi^\alpha 1\big\|_1, $$
$$ \leq \frac{M\big(\psi(T)\big)^{\alpha+1}}{\Gamma_q(\alpha+1)} \|x - u\|_1, $$
$$ \leq M\big(\psi(T)\big)^{\alpha+1} \left(\frac{1-q}{1 - q^{\frac{\alpha+1}{2}}}\right)^\alpha \|x - u\|_1. $$

Also, the fixed point equation of (5) is written as

$$ \mathbb{T}y(t) = y\big(\psi(t)\big)^{\alpha-1} + \frac{1}{\Gamma_q(\alpha)} \int_0^t \big(\psi(t) - q\psi(s)\big)^{(\alpha-1)} f\big(s, y(s)\big) D_q\psi(s) d_q s. $$

Then

$$ \|\mathbb{T}y - \mathbb{T}v\|_1 \leq \big\|f\big(t, y(t)\big) - f\big(t, v(t)\big)\big\|_1 \frac{\big(\psi(T)\big)^\alpha}{\Gamma_q(\alpha+1)} \big\|qI_\psi^\alpha 1\big\|_1, $$
$$ \leq \frac{M\big(\psi(T)\big)^{\alpha+1}}{\Gamma_q(\alpha+1)} \|y - v\|_1, $$
$$ \leq M\big(\psi(T)\big)^{\alpha+1} \left(\frac{1-q}{1 - q^{\frac{\alpha+1}{2}}}\right)^\alpha \|y - v\|_1. $$

It is clear that the contraction constants of both equations are the same. Thus, by Banach contraction theorem, equation (3) is a unique of (2), and equation (5) is a unique solution of (4), since $M\big(\psi(T)\big)^{\alpha+1} \left(\frac{1-q}{1-q^{\frac{\alpha+1}{2}}}\right)^\alpha < 1$. The proof is completed.

Ulam-Hyers Stability of Solutıon

In this part, we discuss the Ulam-Hyers stability.

Definition 5.1. Equation (2) is Ulam-Hyers stable if for any ϵ and for solution $x \in \mathbb{L}_q[0, T]$ of the inequality

$$ {}_{q}D_{\psi}^{\alpha} x(t) - f\big(t, x(t)\big)| \leq \varepsilon, $$

there exists a constant.$c_1 > 0$ and a solution $u \in \mathbb{L}_q[0, T]$ of the equation (2) with $|x(t) - u(t)| \leq c_1\varepsilon$.

Definition 5.2. Equation (4) is Ulam-Hyers stable if for anyϵ and for solution $y \in \mathbb{L}_q[0, T]$ of the inequality

$$ |{}_{q}^{C}D_{\psi}^{\alpha} y(t) - f\big(t, y(t)\big)| \leq \varepsilon, $$

there exists a constant$c_2 > 0$ and a solution $u \in \mathbb{L}_q[0,T]$ of the equation (4) with $|y(t) - u(t)| \leq c_2\varepsilon$.

Theorem 5.3. Suppose the assumption (A0) is satisfied, then equation (2) is Ulan-Hyers stable.

Proof. Let $x \in \mathbb{L}_q[0,T]$ the first inequality in the definition (5.1). It follows that

$$\left|x(t) - x_0(\psi(t))^{\alpha-1} - \frac{1}{\Gamma_q(\alpha)}\int_0^t (\psi(t) - q\psi(s))^{(\alpha-1)} f(s, x(s)) D_q\psi(s) d_q s\right| \leq \varepsilon \frac{(\psi(T))^{\alpha+1}}{\Gamma_q(\alpha+1)}.$$

Now, suppose $u \in \mathbb{L}_q[0,T]$ be a solution of the equation (2). Hence, it satisfies

$$u(t) = x_0(\psi(t))^{\alpha-1} + \frac{1}{\Gamma_q(\alpha)}\int_0^t (\psi(t) - q\psi(s))^{(\alpha-1)} f(s, u(s)) D_q\psi(s) d_q s.$$

Thus we obtain

$$\begin{aligned}|x(t) - u(t)| &= \left|x(t) - x_0(\psi(t))^{\alpha-1} - \frac{1}{\Gamma_q(\alpha)}\int_0^t (\psi(t) - q\psi(s))^{(\alpha-1)} f(s, u(s)) D_q\psi(s) d_q s\right|, \\ &\leq \varepsilon \frac{(\psi(T))^{\alpha+1}}{\Gamma_q(\alpha+1)} + \left|\frac{1}{\Gamma_q(\alpha)}\int_0^t (\psi(t) - q\psi(s))^{(\alpha-1)} [f(s, x(s)) - f(s, u(s))] D_q\psi(s) d_q s\right|, \\ &\leq \varepsilon \frac{(\psi(T))^{\alpha+1}}{\Gamma_q(\alpha+1)} + \frac{M(\psi(T))^{\alpha+1}}{\Gamma_q(\alpha+1)} |x(t) - u(t)|, \\ &\leq \varepsilon(\psi(T))^{\alpha+1}\left(\frac{1-q}{1-q^{\frac{\alpha+1}{2}}}\right)^{\alpha} + M(\psi(T))^{\alpha+1}\left(\frac{1-q}{1-q^{\frac{\alpha+1}{2}}}\right)^{\alpha} |x(t) - u(t)|.\end{aligned}$$

Denote that $L = (\psi(T))^{\alpha+1}\left(\frac{1-q}{1-q^{\frac{\alpha+1}{2}}}\right)^{\alpha}$, then

$$x(t) - u(t)| \leq \epsilon\left(\frac{L}{1 - ML}\right) = c_1\varepsilon.$$

Therefore, the equation (2) is Ulam-Hyers stable.

Theorem 5.4. Suppose the assumption (A0) is satisfied, then equation (4) is Ulan-Hyers stable.

Proof. Let $y \in \mathbb{L}_q[0,T]$ the first inequality in the definition 5.2. It follows that

$$\left|y(t) - y_0 - \frac{1}{\Gamma_q(\alpha)}\int_0^t (\psi(t) - q\psi(s))^{(\alpha-1)} f(s, y(s)) D_q\psi(s) d_q s\right| \leq \varepsilon \frac{(\psi(T))^{\alpha+1}}{\Gamma_q(\alpha+1)}.$$

Now, suppose $v \in \mathbb{L}_q[0,T]$ be a solution of the equation (4). Hence, it satisfies

$$v(t) = y_0 + \frac{1}{\Gamma_q(\alpha)}\int_0^t (\psi(t) - q\psi(s))^{(\alpha-1)} f(s, v(s)) D_q\psi(s) d_q s.$$

This implies that

$$\begin{aligned}|y(t) - v(t)| &= \left|y(t) - y_0(\psi(t))^{\alpha-1} - \frac{1}{\Gamma_q(\alpha)}\int_0^t (\psi(t) - q\psi(s))^{(\alpha-1)} f(s, v(s)) D_q\psi(s) d_q s.\right|, \\ &\leq \varepsilon \frac{(\psi(T))^{\alpha+1}}{\Gamma_q(\alpha+1)} + \left|\frac{1}{\Gamma_q(\alpha)}\int_0^t (\psi(t) - q\psi(s))^{(\alpha-1)} [f(s, y(s)) - f(s, v(s))] D_q\psi(s) d_q s\right|, \\ &\leq \varepsilon \frac{(\psi(T))^{\alpha+1}}{\Gamma_q(\alpha+1)} + \frac{M(\psi(T))^{\alpha+1}}{\Gamma_q(\alpha+1)} |y(t) - v(t)|, \\ &\leq \varepsilon(\psi(T))^{\alpha+1}\left(\frac{1-q}{1-q^{\frac{\alpha+1}{2}}}\right)^{\alpha} + M(\psi(T))^{\alpha+1}\left(\frac{1-q}{1-q^{\frac{\alpha+1}{2}}}\right)^{\alpha} |y(t) - v(t)|.\end{aligned}$$

Denote that $L = (\psi(T))^{\alpha+1}\left(\frac{1-q}{1-q^{\frac{\alpha+1}{2}}}\right)^{\alpha}$, then

$$y(t) - v(t)| \leq \epsilon\left(\frac{L}{1 - ML}\right) = c_2\varepsilon.$$

Therefore, equation (4) is Ulam-Hyers stable.

EXAMPLE

In this section, we give the examples to illustrate our result. Consider the following equations.

Example 1.

$$\begin{aligned} {}_{\frac{1}{2}}D^{\frac{1}{2}}_{\sqrt{t}} x(t) &= \frac{\sin(x(t))}{10}, \\ {}^{RL}_{\frac{1}{2}}I^{\frac{1}{2}}_{\sqrt{t}} x(0) &= 0 \quad t \in (0,1]. \end{aligned} \tag{6}$$

The mild solution of (6) is written as

$$x(t) = \frac{\sqrt{2} - 1}{5\sqrt{2}\Gamma_{\frac{1}{2}}(\frac{1}{2})}\int_0^t \left(\sqrt{t} - \frac{1}{2}\sqrt{s}\right)^{\left(\frac{-1}{2}\right)} \frac{\sin(x(s))}{\sqrt{s}} d_{\frac{1}{2}}s. \tag{7}$$

It can be seen that $M = \frac{1}{10}$. By the mean value theorem, we get

$$\begin{aligned} f(t,x) - f(t,y)| &= \left|\frac{\sin(x)}{10} - \frac{\sin(y)}{10}\right| \\ &\leq \frac{1}{10}\|x - y\|. \end{aligned}$$

This means the contraction constant

$$\frac{1}{10}\left(\frac{1 - 0.5}{1 - 0.5^{\frac{3}{4}}}\right)^{\frac{1}{2}} \approx 0.1111 < 1.$$

Thus, the equation (6) has (7) as a unique solution and is Ulam-Hyers stable.

Example 2.

$$\begin{aligned} {}_{\frac{1}{2}}D^{\frac{1}{2}}_{\sqrt{t}} x(t) &= \frac{x(t) + x^2(t)\sin(x(t)) + \sin(x(t))}{20x^2(t) + 20}, \\ {}^{RL}_{\frac{1}{2}}I^{\frac{1}{2}}_{\sqrt{t}} x(0) &= \frac{1}{2} \quad t \in (0,1]. \end{aligned} \tag{8}$$

The mild solution of (8) is written as:

$$x(t) = \frac{2}{3} + \frac{\sqrt{2} - 1}{10\sqrt{2}\Gamma_{\frac{1}{2}}(\frac{1}{2})}\int_0^t \left(\sqrt{t} - \frac{1}{2}\sqrt{s}\right)^{\left(\frac{-1}{2}\right)} \frac{x(t) + x^2(t)\sin(x(s)) + \sin(x(s))}{\sqrt{s}x^2(s) + \sqrt{s}} d_{\frac{1}{2}}s. \tag{9}$$

It can be seen that $M = \frac{1}{10}$. By the mean value theorem, we get

$$f(t,x) - f(t,y)| = \frac{1}{20}\left(\left|\frac{x}{x^2+1} - \frac{y}{y^2+1}\right| + |\sin(x) - \sin(y)|\right)$$
$$\le \frac{1}{10}\|x-y\|.$$

This means the contraction constant

$$\frac{1}{10}\left(\frac{1-0.5}{1-0.5^{\frac{3}{4}}}\right)^{\frac{1}{2}} \approx 0.1111 < 1.$$

Thus, the equation (8) has (9) as a unique solution and is Ulam-Hyers stable.

Example 6.3.

$$ {}^{C}_{\frac{1}{2}}D^{\frac{1}{2}}_{t^2} y(t) = \frac{\tan^{-1}(x(t))}{2}$$
$$y(0) = 0 \;\; t \in (0,1]. \quad (10)$$

The mild solution of (10) is written as:

$$y(t) = \frac{1}{2} + \frac{3}{4\Gamma_{\frac{1}{2}}(\frac{1}{2})}\int_0^t \left(t^2 - \frac{1}{2}s^2\right)^{\left(\frac{-1}{2}\right)} \tan^{-1}(y(s))d_{\frac{1}{2}}s. \quad (11)$$

According to the mean value theorem, it can be seen that $M = \frac{1}{2}$ as

$$f(t,x) - f(t,y)| = \left|\frac{\tan^{-1}(x)}{2} - \frac{\tan^{-1}(y)}{2}\right|,$$
$$\le \frac{1}{2}\|x-y\|.$$

This means the contraction constant

$$\frac{1}{2}\left(\frac{1-0.5}{1-0.5^{\frac{3}{4}}}\right)^{\frac{1}{2}} \approx 0.5553 < 1.$$

Thus, the equation (10) has (11) as a unique solution and is Ulam-Hyers stable.

CONCLUSION

This paper present the extension concept of the q-calculus and introduce a new operators, qI^{α}_{ψ}, qD^{α}_{ψ} and ${}^{C}_{q}D^{\alpha}_{\psi}$. The novel-introduced operators are the more general version than the classical q-fractional operators. The existence and uniqueness of solutions to the quantum fractional difference equations are proved by the Banach contraction theorem under Lipschitz conditions. Therefore, the Ulam-Hyers stability of solutions is demonstrated by the examples considered.

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AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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