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Research Article

Characterizations of tzitzeica curves using FLC frame

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ABSTRACT

In this study, we investigate the condition that a given polynomial curve with respect to the Frenet like curve (Flc) frame is a Tzitzeica curve. We also show that any planar polynomial curve cannot be a Tzitzeica curve. Finally, the condition to be the Tzitzeica curve for each of the spherical indicatrix curves defined according to the Flc frame is expressed, separately.

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INTRODUCTION

The Tzitzeica curve is characterized by a unique condition that pertains to the proportion between the curve's torsion. This ratio remains a nonzero constant.The defining characteristic of the Tzitzeica curve is that the mentioned ratio, involving torsion and distance, is a constant value. It has been proven that no planar polynomial curve can be a Tzitzeica curve. This result provides insights into the limitations and specific conditions for the existence of Tzitzeica curves.

New results have been obtained for spherical indicatrix curves defined with respect to the Flc frame, demonstrating their qualification as Tzitzeica curves. This extends the understanding of Tzitzeica curves to specific instances in spherical geometry. Researchers have explored various aspects of Tzitzeica curves, including obtaining parametric, open, and closed equations for Tzitzeica hypersurfaces. The study involves mathematical formulations that describe the shape and behavior of these curves. Tzitzeica curves have been studied in different mathematical spaces, such as Minkowski space and Euclidean space. The characterization of Tzitzeica curves varies based on the geometric properties of the specific space in which they are examined. The geometric information about Tzitzeica curves has been investigated with respect to various frames of reference. Different frames provide different perspectives on the curve's behavior and properties.

In summary, the geometric information about Tzitzeica curves encompasses their defining characteristics, the invariance of certain ratios, non-existence for certain types of curves, and exploration of their behavior in different mathematical spaces and frames of reference, [1-10]. This study aims to examine Tzitzeica curves in Euclidean space using the Flc frame. The condition for a curve to be a Tzitzeica curve has been reinterpreted and defined for given polynomial curves. It has been established that no planar polynomial curve qualifies as a Tzitzeica curve. Additionally, new results have been obtained for spherical

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indicatrix curves defined with respect to the Flc frame, meeting the criteria to be Tzitzeica curves.

PRELIMINARIES

Suppose $\alpha = \alpha(u)$ is a regular space curve. Three mutually orthogonal vector fields, referred to as the Frenet frame, are defined as [11]

$$T=\frac{\alpha'}{\|\alpha'\|},\quad B=\frac{\alpha'\wedge\alpha''}{\|\alpha'\wedge\alpha''\|},\quad N=B\wedge T. \tag{1}$$

The Frenet apparatus are given by

$$T'=\kappa\nu N,\ N'=-\kappa\nu T+\tau\nu B,\ B'=-\tau\nu N,\ \|\alpha'\|=\nu, \tag{2}$$

$$\kappa=\frac{\|\alpha'\wedge\alpha''\|}{\alpha'''},\quad \tau=\frac{\langle\alpha'\wedge\alpha'',\alpha'''\rangle}{\|\alpha'\wedge\alpha''\|^2}. \tag{3}$$

In relation to these, Consider $\alpha = \alpha(u)$ as a polynomial space curve. The definition of the Flc frame for [12],

$$T=\frac{\alpha'}{\|\alpha'\|},\ D_1=\frac{\alpha'\wedge\alpha^{(n)}}{\|\alpha'\wedge\alpha^{(n)}\|},\ D_2=D_1\wedge T \tag{4}$$

Here $^{(n)}$ indicates the n^{th} order derivative of the curve. D_1 is called as binormal-like vector and D_2 is normal-like vector. The curvatures of the Flc-frame are given by

$$d_1=\frac{\langle T',D_2\rangle}{v},\ d_2=\frac{\langle T',D_1\rangle}{v},\ d_3=\frac{\langle D_2',D_1\rangle}{v}. \tag{5}$$

The Frenet-like apparatus are given by:

$$\begin{bmatrix}T'\\D_2'\\D_1'\end{bmatrix}=v\begin{bmatrix}0&d_1&d_2\\-d_1&0&d_3\\-d_2&-d_3&0\end{bmatrix}\begin{bmatrix}T\\D_2\\D_1\end{bmatrix}, \tag{6}$$

$$d_1=\kappa\cos\theta,\quad d_2=-\kappa\sin\theta,\quad d_3=\frac{d\theta}{v}+\tau.$$

Definition 2.1. Let the α polynomial curve $\{T, D_2, D_1\}$ be given with the Flc frame. The path followed by the vector T centered at unit sphere is called $T-$ is defined as

$$\alpha_T=T. \tag{7}$$

Definition 2.2. Let the α polynomial curve $\{T, D_2, D_1\}$ be given with the Flc frame. The path followed by the vector D_2 centered at unit sphere is called D_2- is defined as

$$\alpha_{D_2}=D_2. \tag{8}$$

Definition 2.3. Let the α polynomial curve $\{T, D_2, D_1\}$ be given with the Flc frame. The path followed by the vector D_1 centered at unit sphere is called D_1- is defined as

$$\alpha_{D_1}=D_1. \tag{9}$$

TZITZEICA CURVES USING FLC FRAME

In this section of the research, we present the characterization of the Tzitzeica curve employing the Flc frame within 3-dimensional Euclidean space. Essential and comprehensive conditions are articulated for the spherical indicatrix curves to qualify as Tzitzeica curves.

Theorem 3.1. Let $\alpha: I\subset IR\to E^3$ be a polynomial curve and $\{T,D_2,D_1,d_1,d_2,d_3\}$ be apparatus of the Flc frame, then there exists the following equation:

$$\frac{\langle\alpha'\wedge\alpha'',\alpha'''\rangle}{\langle\alpha,\alpha'\wedge\alpha''\rangle^2}=\frac{\left(d_1^{\,2}+d_2^{\,2}\right)\left(vd_3-\theta'\right)}{v\langle\alpha,d_2D_2-d_1D_1\rangle^2}.$$

Proof . The differentials of the polynomial curve α can be readily obtained as follows:

$$\begin{aligned}\alpha'&=vT,\\ \alpha''&=v'T+v^2d_1D_2+v^2d_2D_1,\\ \alpha'''&=(v''-d_1^{\,2}v^3-d_2^{\,2}v^3)T+(3vv'd_1+v^2d_1'-v^3d_2d_3)D_2\\ &\quad+(3vv'd_2+v^2d_2'+v^3d_1d_3)D_1.\end{aligned}$$

With the help of these equations, we get

$$\begin{aligned}\alpha'\wedge\alpha''&=\begin{vmatrix}T&D_2&D_1\\v&0&0\\v'&v^2d_1&v^2d_2\end{vmatrix}\\ &=-v^3d_2D_2+v^3d_1D_1,\\ \langle\alpha,\alpha'\wedge\alpha''\rangle^2&=\langle\alpha,-v^3d_2D_2+v^3d_1D_1\rangle^2,\\ \langle\alpha'\wedge\alpha'',\alpha'''\rangle&=\left\langle\begin{matrix}-v^3d_2D_2+v^3d_1D_1,\ (v''-d_1^{\,2}v^3-d_2^{\,2}v^3)T+\\(3vv'd_1+v^2d_1'-v^3d_2d_3)D_2+(3vv'd_2+v^2d_2'+v^3d_1d_3)D_1\end{matrix}\right\rangle,\\ &=v^5(d_1d_2'-d_1'd_2)+v^6d_3(d_1^{\,2}+d_2^{\,2}),\\ &=v^5(d_1^{\,2}+d_2^{\,2})\left(\arctan\left(\frac{d_2}{d_1}\right)\right)'+v^6d_3(d_1^{\,2}+d_2^{\,2}),\\ &=v^5(d_1^{\,2}+d_2^{\,2})(vd_3-\theta').\end{aligned}$$

Therefore, it is evident that the following equation exists:

$$\frac{\langle\alpha'\wedge\alpha'',\alpha'''\rangle}{\langle\alpha,\alpha'\wedge\alpha''\rangle^2}=\frac{\left(d_1^{\,2}+d_2^{\,2}\right)\left(vd_3-\theta'\right)}{v\langle\alpha,d_2D_2-d_1D_1\rangle^2}.$$

The demonstration is concluded.

Corollary 3.1.1. If α is a Tzitzeica curve, then

$$\frac{\left(d_1{}^2+d_2{}^2\right)\left(vd_3-\theta'\right)}{v\left\langle \alpha, d_2D_2-d_1D_1\right\rangle^2}$$

is a nonzero constant.

Corollary 3.1.2. If α is a planar polynomial curve, it is established that α cannot be the Tzitzeica curve. Now, let's determine the essential conditions for spherical indicatrix curves to meet the criteria for being Tzitzeica curves:

Theorem 3.2. Let $\alpha : I \subset IR \to E^3$ be a polynomial curve with the Flc frame. Then tangent spherical indicatrix curve α_T of the curve α is a Tzitzeica curve if and only if

$$\frac{<\alpha_T' \wedge \alpha_T'', \alpha_T'''>}{<\alpha_T, \alpha_T' \wedge \alpha_T''>^2} = \frac{\begin{array}{l} -v^2\left(v^2d_3(d_1^2+d_2^2)+d_1(vd_2)'-d_2(vd_1)'\right) \\ \left(2v'(d_1^2+d_2^2)+v(d_1^2+d_2^2)'+d_1(vd_1)'-d_2(vd_2)'\right) \\ +v^3(d_1^2+d_2^2)\left(d_1((vd_2)'+v^2d_1d_3)'-d_2((vd_1)'-v^2d_2d_3)'\right) \\ +v^4d_3(d_1^2+d_2^2)\left(d_1((vd_1)'-v^2d_2d_3)'+d_2((vd_2)'+v^2d_1d_3)'\right) \end{array}}{\begin{array}{l} <\alpha_T, \left(v^3d_3(d_1^2+d_2^2)+vd_1(vd_2)'-vd_2(vd_1)'\right)T \\ \left(v^3d_2(d_1^2+d_2^2)\right)D_2+\left(v^3d_1(d_1^2+d_2^2)\right)D_1>^2 \end{array}} = \lambda_1,$$

where $\lambda_1 \neq 0$.

Proof. Let α_T be a tangent spherical indicatrix curve of the curve α, then differentiations of α_T are

$$\begin{aligned} \alpha_T' &= v(d_1D_2+d_2D_1), \\ \alpha_T'' &= -v^2(d_1^2+d_2^2)T+\left((vd_1)'-v^2d_2d_3\right)D_2+\left((vd_2)'+v^2d_1d_3\right)D_1, \\ \alpha_T''' &= -v\left(2v'(d_1^2+d_2^2)+v(d_1^2+d_2^2)'+d_1(vd_1)'-d_2(vd_2)'\right)T \\ &\quad +\left(-v^3d_1(d_1^2+d_2^2)+((vd_1)'-v^2d_2d_3)'-vd_3((vd_2)'+v^2d_1d_3)\right)D_2 \\ &\quad +\left(-v^3d_2(d_1^2+d_2^2)+((vd_2)'+v^2d_1d_3)'+vd_3((vd_1)'-v^2d_2d_3)\right)D_1. \end{aligned}$$

Using these equations, we have

$$\alpha_T' \wedge \alpha_T'' = \begin{vmatrix} T & D_2 & D_1 \\ 0 & vd_1 & vd_2 \\ -v^2(d_1^2+d_2^2) & (vd_1)'-v^2d_2d_3 & (vd_2)'+v^2d_1d_3 \end{vmatrix}$$

$$\begin{aligned} &= \left(v^3d_3(d_1^2+d_2^2)+vd_1(vd_2)'-vd_2(vd_1)'\right)T \\ &\quad -\left(v^3d_2(d_1^2+d_2^2)\right)D_2+\left(v^3d_1(d_1^2+d_2^2)\right)D_1, \\ <\alpha_T, \alpha_T' \wedge \alpha_T''>^2 &= <\alpha_T, \left(v^3d_3(d_1^2+d_2^2)+vd_1(vd_2)'-vd_2(vd_1)'\right)T \\ &\quad -\left(v^3d_2(d_1^2+d_2^2)\right)D_2+\left(v^3d_1(d_1^2+d_2^2)\right)D_1>^2, \end{aligned}$$

$$\begin{aligned} <\alpha_T' \wedge \alpha_T'', \alpha_T'''> &= -v^2\left(v^2d_3(d_1^2+d_2^2)+d_1(vd_2)'-d_2(vd_1)'\right) \\ &\quad \left(2v'(d_1^2+d_2^2)+v(d_1^2+d_2^2)'+d_1(vd_1)'-d_2(vd_2)'\right) \\ &\quad +v^3(d_1^2+d_2^2)\left(d_1((vd_2)'+v^2d_1d_3)'-d_2((vd_1)'-v^2d_2d_3)'\right) \\ &\quad +v^4d_3(d_1^2+d_2^2)\left(d_1((vd_1)'-v^2d_2d_3)'+d_2((vd_2)'+v^2d_1d_3)'\right). \end{aligned}$$

Thus, we can readily derive the condition for curve α_T to be a Tzitzeica curve

$$\frac{<\alpha_T' \wedge \alpha_T'', \alpha_T'''>}{<\alpha_T, \alpha_T' \wedge \alpha_T''>^2} = \frac{\begin{array}{l} -v^2\left(v^2d_3(d_1^2+d_2^2)+d_1(vd_2)'-d_2(vd_1)'\right) \\ \left(2v'(d_1^2+d_2^2)+v(d_1^2+d_2^2)'+d_1(vd_1)'-d_2(vd_2)'\right) \\ +v^3(d_1^2+d_2^2)\left(d_1((vd_2)'+v^2d_1d_3)'-d_2((vd_1)'-v^2d_2d_3)'\right) \\ +v^4d_3(d_1^2+d_2^2)\left(d_1((vd_1)'-v^2d_2d_3)'+d_2((vd_2)'+v^2d_1d_3)'\right) \end{array}}{\begin{array}{l} <\alpha_T, \left(v^3d_3(d_1^2+d_2^2)+vd_1(vd_2)'-vd_2(vd_1)'\right)T \\ \left(v^3d_2(d_1^2+d_2^2)\right)D_2+\left(v^3d_1(d_1^2+d_2^2)\right)D_1>^2 \end{array}}.$$

Theorem 3.3. Let $\alpha : I \subset IR \to E^3$ be a polynomial curve with the Flc frame. Then the normal like

spherical indicatrix curve α_{D_2} of the curve α is a Tzitzeica curve if and only if

$$\frac{<\alpha_{D_2}' \wedge \alpha_{D_2}'', \alpha_{D_2}'''>}{<\alpha_{D_2}, \alpha_{D_2}' \wedge \alpha_{D_2}''>^2} = \frac{\begin{array}{l} \left(v^3d_1(d_1^2+d_3^2)-((vd_1)'+d_2d_3v^2)'-vd_2((vd_3)'-d_1d_2v^2)\right)\left(v^3d_3(d_1^2+d_3^2)\right) \\ +\left((vd_1)((vd_1)'+d_2d_3v^2)+(vd_3)((vd_3)'-d_1d_2v^2)\right. \\ \left.+(v(d_1^2+d_3^2))'\right)\left(vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2)\right) \\ -\left(v^3d_3(d_1^2+d_3^2)-((vd_3)'-d_1d_2v^2)'+vd_2((vd_1)'+d_2d_3v^2)\right)\left(v^3d_1(d_1^2+d_3^2)\right) \end{array}}{\begin{array}{l} <\alpha_{D_2}, \left(v^3d_3(d_1^2+d_3^2)\right)T+\left(-vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2)\right)D_2 \\ +\left(v^3d_1(d_1^2+d_3^2)\right)D_1>^2 \end{array}} = \lambda_2,$$

where $\lambda_2 \neq 0$.

Proof. Let α_{D_2} be a normal spherical indicatrix curve of the curve α, then differentiations of the curve α_{D_2} are found as

$$\begin{aligned} \alpha_{D_2}' &= v(-d_1T+d_3D_1), \\ \alpha_{D_2}'' &= -\left((vd_1)'+d_2d_3v^2\right)T-v^2(d_1^2+d_3^2)D_2+\left((vd_3)'-d_1d_2v^2\right)D_1, \\ \alpha_{D_2}''' &= \left(v^3d_1(d_1^2+d_3^2)-((vd_1)'+d_2d_3v^2)'-vd_2((vd_3)'-d_1d_2v^2)\right)T \\ &\quad -\left((vd_1)((vd_1)'+d_2d_3v^2)+(vd_3)((vd_3)'-d_1d_2v^2)+(v(d_1^2+d_3^2))'\right)D_2 \\ &\quad -\left(v^3d_3(d_1^2+d_3^2)-((vd_3)'-d_1d_2v^2)'+vd_2((vd_1)'+d_2d_3v^2)\right)D_1. \end{aligned}$$

Utilizing these equations, we can readily compute the subsequent expressions

$$\alpha_{D_2}' \wedge \alpha_{D_2}'' = \begin{vmatrix} T & D_2 & D_1 \\ -vd_1 & 0 & vd_3 \\ -(vd_1)'-v^2d_2d_3 & -v^2(d_1^2+d_3^2) & (vd_3)'-^2vd_1d_2 \end{vmatrix},$$

$$\alpha_{D_2}' \wedge \alpha_{D_2}'' = \left(v^3d_3(d_1^2+d_3^2)\right)T+\left(-vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2)\right)D_2+\left(v^3d_1(d_1^2+d_3^2)\right)D_1,$$

$$\begin{aligned} <\alpha_{D_2}, \alpha_{D_2}' \wedge \alpha_{D_2}''>^2 &= <\alpha_{D_2}, \left(v^3d_3(d_1^2+d_3^2)\right)T+\left(-vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2)\right)D_2 \\ &\quad +\left(v^3d_1(d_1^2+d_3^2)\right)D_1>^2, \end{aligned}$$

$$\begin{aligned} <\alpha_{D_2}' \wedge \alpha_{D_2}'', \alpha_{D_2}'''> &= \left(v^3d_1(d_1^2+d_3^2)-((vd_1)'+d_2d_3v^2)'-vd_2((vd_3)'-d_1d_2v^2)\right)\left(v^3d_3(d_1^2+d_3^2)\right) \\ &\quad +\left((vd_1)((vd_1)'+d_2d_3v^2)+(vd_3)((vd_3)'-d_1d_2v^2)\right. \\ &\quad \left.+(v(d_1^2+d_3^2))'\right)\left(vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2)\right) \\ &\quad -\left(v^3d_3(d_1^2+d_3^2)-((vd_3)'-d_1d_2v^2)'+vd_2((vd_1)'+d_2d_3v^2)\right)\left(v^3d_1(d_1^2+d_3^2)\right). \end{aligned}$$

Therefore, the condition for curve α_{D_2} to be a Tzitzeica curve is articulated as:

$$\frac{<\alpha_{D_2}^{'}\wedge\alpha_{D_2}^{''},\alpha_{D_2}^{'''}>}{<\alpha_{D_2},\alpha_{D_2}^{'}\wedge\alpha_{D_2}^{''}>^2}=\frac{\begin{array}{l}\left(v^3d_1(d_1^2+d_3^2)-((vd_1)'+d_2d_3v^2)'-vd_2((vd_3)'-d_1d_2v^2)\right)\left(v^3d_3(d_1^2+d_3^2)\right)\\+\left((vd_1)((vd_1)'+d_2d_3v^2)+(vd_3)((vd_3)'-d_1d_2v^2)\right.\\+(v(d_1^2+d_3^2))'\Big)\left(vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2\right)\\-\left(v^3d_3(d_1^2+d_3^2)-((vd_3)'-d_1d_2v^2)'+vd_2((vd_1)'+d_2d_3v^2)\right)\left(v^3d_1(d_1^2+d_3^2)\right)\end{array}}{\begin{array}{l}<\alpha_{D_2},\left(v^3d_3(d_1^2+d_3^2)\right)T+\left(-vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2)\right)D_2\\+\left(v^3d_1(d_1^2+d_3^2)\right)D_1>^2\end{array}}.$$

Theorem 3.4. Let $\alpha: I\subset IR\rightarrow E^3$ be a polynomial curve with the Flc frame. Then the binormal like

spherical indicatrix curve α_{D_1} of the curve α is a Tzitzeica curve if and only if

$$\frac{<\alpha_{D_1}^{'}\wedge\alpha_{D_1}^{''},\alpha_{D_1}^{'''}>}{<\alpha_{D_1},\alpha_{D_1}^{'}\wedge\alpha_{D_1}^{''}>^2}=\frac{\begin{array}{l}\left(v^3d_2(d_2^2+d_3^2)+vd_1((vd_3)'+vd_1d_2)-((vd_2)'-vd_1d_3)'\right)\left(v^3d_3(d_2^2+d_3^2)\right)\\-\left(v^3d_3(d_2^2+d_3^2)-vd_1((vd_2)'-vd_1d_3)-((vd_3)'+vd_1d_2)'\right)\left(v^3d_2(d_2^2+d_3^2)\right)\\-\left(vd_2((vd_3)'+vd_1d_2)-vd_3((vd_2)'-vd_1d_3)\right)\Big((vd_2((vd_2)'-vd_1d_3)+(vd_3((vd_3)'\\+vd_1d_2)+(v^2(d_2^2+d_3^2))'\Big)\end{array}}{\begin{array}{l}<\alpha_{D_1},\left(v^3d_3(d_2^2+d_3^2)\right)T-\left(v^3d_2(d_2^2+d_3^2)\right)D_2\\+\left(vd_2((vd_3)'-vd_3(vd_2)'+v^2d_1(d_2^2+d_3^2)\right)D_1>^2\end{array}}=\lambda_3,$$

where $\lambda_3\neq 0$.

Proof. Let α_{D_1} be a binormal spherical indicatrix curve of the curve α, then differentiations of the curve α_{D_1} are found as

$$\begin{aligned}\alpha_{D_1}^{'}&=-v(d_2T+d_3D_2),\\ \alpha_{D_1}^{''}&=-\left((vd_2)'-vd_1d_3\right)T-\left((vd_3)'+vd_1d_2\right)D_2-v^2(d_2^2+d_3^2)D_1,\\ \alpha_{D_1}^{'''}&=\left(v^3d_2(d_2^2+d_3^2)-((vd_2)'-vd_1d_3)'+vd_1((vd_3)'+vd_1d_2)\right)T\\&\quad+\left(v^3d_3(d_2^2+d_3^2)-(vd_1)((vd_2)'-vd_1d_3)-(vd_3)((vd_3)'+vd_1d_2)'\right)D_2\\&\quad-\left((vd_2)((vd_2)'-vd_1d_3)+vd_3((vd_3)'+vd_1d_2)+(v^2(d_2^2+d_3^2))'\right)D_1.\end{aligned}$$

So, we obtain the following equations

$$\alpha_{D_1}^{'}\wedge\alpha_{D_1}^{''}=\begin{vmatrix}T & D_2 & D_1\\ -vd_2 & -vd_3 & 0\\ -(vd_2)'+vd_1d_3 & -(vd_3)'-vd_1d_2 & -v^2(d_2^2+d_3^2)\end{vmatrix}$$

$$\begin{aligned}\alpha_{D_1}^{'}\wedge\alpha_{D_1}^{''}&=\left(v^3d_3(d_2^2+d_3^2)\right)T-\left(v^3d_2(d_2^2+d_3^2)\right)D_2\\&\quad+\left(vd_2((vd_3)'-vd_3(vd_2)'+v^2d_1(d_2^2+d_3^2)\right)D_1,\\ <\alpha_{D_1},\alpha_{D_1}^{'}\wedge\alpha_{D_1}^{''}>^2&=<\alpha_{D_1},\left(v^3d_3(d_2^2+d_3^2)\right)T-\left(v^3d_2(d_2^2+d_3^2)\right)D_2\\&\quad+\left(vd_2((vd_3)'-vd_3(vd_2)'+v^2d_1(d_2^2+d_3^2)\right)D_1>^2,\end{aligned}$$

$$\begin{aligned}<\alpha_{D_1}^{'}\wedge\alpha_{D_1}^{''},\alpha_{D_1}^{'''}>&=\left(v^3d_2(d_2^2+d_3^2)+vd_1((vd_3)'+vd_1d_2)-((vd_2)'-vd_1d_3)'\right)\left(v^3d_3(d_2^2+d_3^2)\right)\\&\quad-\left(v^3d_3(d_2^2+d_3^2)-vd_1((vd_2)'-vd_1d_3)-((vd_3)'+vd_1d_2)'\right)\left(v^3d_2(d_2^2+d_3^2)\right)\\&\quad-\left(vd_2((vd_3)'+vd_1d_2)-vd_3((vd_2)'-vd_1d_3)\right)\Big((vd_2((vd_2)'-vd_1d_3)+(vd_3((vd_3)'\\&\quad+vd_1d_2)+(v^2(d_2^2+d_3^2))'\Big).\end{aligned}$$

From here, the condition to be the Tzitzeica curve of the curve α_{D_1} is given by

$$\frac{<\alpha_{D_1}^{'}\wedge\alpha_{D_1}^{''},\alpha_{D_1}^{'''}>}{<\alpha_{D_1},\alpha_{D_1}^{'}\wedge\alpha_{D_1}^{''}>^2}=\frac{\begin{array}{l}\left(v^3d_2(d_2^2+d_3^2)+vd_1((vd_3)'+vd_1d_2)-((vd_2)'-vd_1d_3)'\right)\left(v^3d_3(d_2^2+d_3^2)\right)\\-\left(v^3d_3(d_2^2+d_3^2)-vd_1((vd_2)'-vd_1d_3)-((vd_3)'+vd_1d_2)'\right)\left(v^3d_2(d_2^2+d_3^2)\right)\\-\left(vd_2((vd_3)'+vd_1d_2)-vd_3((vd_2)'-vd_1d_3)\right)\Big((vd_2((vd_2)'-vd_1d_3)+(vd_3((vd_3)'\\+vd_1d_2)+(v^2(d_2^2+d_3^2))'\Big)\end{array}}{\begin{array}{l}<\alpha_{D_1},\left(v^3d_3(d_2^2+d_3^2)\right)T-\left(v^3d_2(d_2^2+d_3^2)\right)D_2\\+\left(vd_2((vd_3)'-vd_3(vd_2)'+v^2d_1(d_2^2+d_3^2)\right)D_1>^2\end{array}}.$$

CONCLUSION

In this study, the condition for a polynomial curve given according to the Flc framework to be a Tzitzeica curve expressed. It was then shown that no planar polynomial curve can be a Tzitzeica curve. As a result, each of the spherical indicatrix curves defined according to the Flc framework is the Tzitzeica curve condition was investigated.

AUTHORSHIP CONTRIBUTIONS

Concept: Ş.S., E.K., A.K.H; Design: Ş.S., E.K., A.K.H; Supervision: Ş.S., E.K., A.K.H; Materials: Ş.S., E.K., A.K.H; Analysis: Ş.S., E.K., A.K.H; Literature search: Ş.S., E.K., A.K.H; Writing: Ş.S., E.K., A.K.H; Critical revision: Ş.S., E.K., A.K.H.

DATA AVAILABILITY STATEMENT

The published document incorporates all visuals and data gathered or generated throughout the course of the study.

CONFLICT OF INTEREST

The author has disclosed no potential conflicts of interest concerning the research, authorship, and/or publication of this article.

ETHICS

No ethical issues are associated with the publication of this manuscript.

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