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Technical Note

The quasicompact-open topology on *KC*(*X,Y*)

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ABSTRACT

In this paper, we introduce quasicompact-open topology on $KC(X, Y)$, the set of all functions from X to Y, which are continuous on the compact subsets of X and compare this topology with the open-cover topology, the uniform topology and m-topology. Then, we examine metrizability, completeness and countability properties of the quasicompact-open topology on $KC(X, Y)$. Also, we obtain similar results for the open-cover topology and m- topology on $KC(X, Y)$.

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INTRODUCTION

It is a well-known fact that there are many topologies on $C(X, Y)$ of all con-tinuous functions from a Tychonoff space X to a metric space Y. A number of these natural topologies are the point-open topology, the compact-open topology, the open-cover topology, the uniform topology, the ne topology and the graph topology. The compact-open topology, which was introduced by Fox [1], is one of the commonly used topologies on function spaces, and has many applications in homotopy theory and functional analysis. Later on it was improved by Arens and Dugundji in [2, 3]. Since it is used to study uniformly convergent sequences of functions on compact subsets, it is also called the topology of uniform convergence on compact sets. Kundu and Garg [4] presented some results on the compact-open topology on $KC(X)$, the set of all real-valued functions on X, which are continuous on the compact subsets of X. Clearly $KC(X)$ = $C(X)$ if and only if X is a $k_{\mathbb{R}}$-space. Therefore, more general and bene cial results can be presented if $KC(X)$ is used instead of $C(X)$.

In the present paper, we introduce quasicompact-open topology on $KC(X, Y)$ and compare this topology with some other known topologies such as the compact-open topology and the uniform topology. We investigate the properties of the quasicompact-open topology on $KC(X)$ such as submetrizability, metrizability, separability, and second countability.

Unless otherwise stated clearly, throughout this paper, all spaces are assumed to be Tychonoff (completely regular Hausdorff).

If X and Y are any two topological spaces with the same underlying set, then we use the notation $X = Y$, $X \leq Y$, and $X < Y$ to indicate, respectively, that X and Y have the same topology, that the topology on Y is ner than or equal to the topology on X, and that the topology on Y is strictly ner than the topology on X. Topological space will be used as space. The topology of the space X will be represented

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by $\tau(X)$. If $A \subseteq X$ and $f \in C(X)$, then we use the notation $f|_A$ for the restriction of the function f to the set A. We denote by $\mathbb{R}$ the real line with the natural topology. Finally, the constant zero function in $C(X)$ is denoted by f_0.

Topologies on Function Spaces

In this section, we define the quasicompact-open topology on $KC(X, Y)$ and also give some equivalent definitions. Also we give the definitions of various function space topologies on the space $KC(X, Y)$ such as the open- cover topology, the uniform topology and m-topology.

A subset A of X is called a zero-set if there is a continuous real-valued function f defined on X such that $A = \{x \in X : f(x) = 0\}$. The complement of a zero-set is called a cozero set. A space X is said to be quasicompact [5] if every covering of X by cozero sets admits a finite subcollection which covers X, also known as z-compact space.

We recall that any compact space is quasicompact, any quasicompact space is pseudocompact and the continuous image of a quasicompact space is quasicompact. Also the closure of a quasicompact subset is quasicompact [6].

Let α be a nonempty collection of subsets of a space X.

Various topologies on $C(X, Y)$ has a subbase consisting of the sets $S(A, V) = \{f \in C(X, Y) : f(A) \subseteq V\}$, where $A \in \alpha$ and $V \in \tau(Y)$, and the function space $C(X, Y)$ endowed with these topologies is denoted by $C_\alpha(X, Y)$. The topologies defined in this way is called set-open topology. Let $QK(X)$ denote the collection of all quasicompact subsets of X. For the quasicompact-open topology on $C(X, Y)$, we take as subbase, the collection $\{S(A, V) : A \in QK(X), V \in \tau(Y)\}$ and we denote the corresponding space by $C_q(X, Y)$ [7].

Let X be a topological space and (Y, d) be a metric space. The topology of uniform convergence on members of α has as base at each point $f \in C(X, Y)$ the family of all sets of the form $B_A(f, \varepsilon) = \{g \in C(X, Y)$: $\sup_{x\in A} d(f(x), g(x)) < \varepsilon$, for all $x \in A\}$, where $A \in \alpha$ and $\varepsilon > 0$. The space $C(X, Y)$ having the topology of uniform convergence on α is denoted by $C_{\alpha,u}(X, Y)$. For $\alpha = QK(X)$, we denote the corresponding space by $C_{q,u}(X, Y)$. This topology is equivalent to the quasicompact-open topology on $C(X, Y)$ [7]. In the case that $\alpha = \{X\}$, the topology on $C(X, Y)$ is called the topology of uniform convergence or uniform topology and denoted by $C_u(X, Y)$.

Theorem 1. For any space X and any metric space Y, $C_q(X, Y) = C_{q,u}(X, Y) \leq C_u(X, Y)$ [7].

Let $\mathcal{O}$ is any open cover of Y and define $O(f) = \{g \in C(X, Y)$: for all $x \in X$ there is a $O \in \mathcal{O}$ such that $f(x)$, $g(x) \in O\}$. The open-cover topology on $C(X, Y)$ [8] is then generated by the subbase $\mathcal{O}(f) = \{O(f) : O \in \mathcal{O}, f \in C(X, Y)\}$ and is denoted by $C_\gamma(X, Y)$. Let any cozero cover of Y be $\mathcal{O}$. In that case, the cozero- cover topology on $C(X, Y)$ denoted by $C_{q,\gamma}(X, Y)$ is generated by the subbase $O(f)$.

Theorem 2. For any space X and Y, $C_{q,\gamma}(X, Y) \leq C_\gamma(X, Y)$.

Since the concepts of open cover and cozero cover are equivalent in metric space, the following theorem can be given.

Theorem 3. For any space X and any metric space Y, $C_{q,\gamma}(X, Y) = C_\gamma(X, Y)$.

For $f \in C(X, Y)$, the set $G(f) = \{(x, f(x)) : x \in X\}$ is called the graph of the function f. For $U \in \tau(X \times Y)$, let $N(U) = \{f \in C(X, Y) : G(f) \subseteq U\}$. The graph topology on $C(X, Y)$ [9] is then generated by the basis $\{N(U) : U \in \tau(X \times Y)\}$, the m-topology on $C(X, Y)$ is then generated by the basis $\{N(C) : C$ is a cozero set in $X \times Y\}$ andthis topologies on $C(X, Y)$denoted by $C_g(X, Y)$ and $C_m(X, Y)$, respectively.

Theorem 4. For any space X and Y, $C_m(X, Y) \leq C_g(X, Y)$

Note that in a perfectly normal space, every open set is a cozero-set.

Theorem 5. For perfectly normal space X and any space Y, $C_m(X, Y) = C_g(X, Y)$.

A function $f: X \to Y$ is called compact-continuous [10] if restriction function $f|_A: X \to Y$ is continuous whenever A is a compact subspace of X. Let $KC(X, Y)$ denote the set of all compact-continuous functions from X to. Since the restriction function of every continuous function is also continuous, every continuous function is compact-continuous. Therefore, it is seen that $C(X, Y) \subseteq KC(X, Y)$.

A space X is a $k_{\mathbb{R}}$-space if it is a Tychonoff space and if every mapping $f: X \to \mathbb{R}$, whose restriction to every compact set $K \subset X$ is continuous, is continuous on X [11]

Recall that $C(X) = KC(X)$ if and only if the space X is $k_{\mathbb{R}}$-space. Also submetrizable space is a $k_{\mathbb{R}}$- space. It is clearly seen that for compact space X, $C(X) = KC(X)$.

The quasicompact-open topology, the cozero-cover topology and m-topology on $KC(X, Y)$ is defined similarly and is denoted by $KC_q(X, Y)$, $KC_{q,\gamma}(X, Y)$) and $KC_m(X, Y)$, respectively.

Theorem 6. The space $C_q(X, Y)$ is a subspace of $KC_q(X, Y)$ (Similar is the case of the space $C_{q,\gamma}(X, Y)$ and $C_m(X, Y)$.

Theorem 7. For compact space X and any space Y, $KC_q(X, Y) = C_q(X, Y)$ and $KC_{q,\gamma}(X, Y) = C_{q,\gamma}(X, Y)$

Theorem 8. For compact space X and any metric space Y, $KC_m(X, Y) = C_m(X, Y)$.

Comparison of Topologies

In this section, we compare the quasicompact-open topology with the open-cover topology, the uniform topology and m-topology.

Theorem 9. For any space X and any space Y, $C_q(X, Y) = C_{q,u}(X, Y) \leq C_u(X, Y)$ [7].

Theorem 10. For any space X and any metric space Y, $C_u(X, Y) \leq C_\gamma(X, Y)$ [12].

Corollary 1. For any space X and any space Y, $C_q(X, Y) \leq C_\gamma(X, Y)$.

Theorem 11. For any space X and Tychonoff space Y, $C_q(X, Y) \leq C_m(X, Y)$.

Proof. Let $f \in S(A, V)$. Then, $f(A) \subseteq V$. Since Y is Tychonoff, there is a continuous function $g: Y \to [0, 1]$ such that $g(f(A)) = 0$ and $g(V^c) = 1$. Here $\tilde{z}_1 = (g \circ f)^{-1}(0)$ is a zero set in X. Define the continuous functions $h: Y \to [0,1]$, $h(y) = 1 - g(y)$. Also, $\tilde{z}_2 = h^{-1}(0)$ is a zero set in Y. Hence, $z_1 = \tilde{z}_1 \times Y$ and $z_2 = X \times \tilde{z}_2$ are zero set in $X \times Y$. Therefore, $z = z_1 \cap z_2$ is a zero set in $X \times Y$ and so $C = z^c$ is a cozero set in $X \times Y$. If $(x, f(x)) \notin C$, then $(x, f(x)) \in z = z_1 \cap z_2$. That is, $x \in \tilde{z}_1$ and $f(x) \in \tilde{z}_2$. If $x \in \tilde{z}_1$, $(g \circ f)(x) = 0$. If $x \in \tilde{z}_2$, $h(f(x)) = 0$ and so $g(f(x)) = 1$; which is a contradiction. Then $(x, f(x)) \in C$ and $G(f) \subseteq C$. Thus, $f \in N(C)$.

We would like to conclude that $N(C) \subseteq S(A, V)$. Since $g(f(A)) = 0$ and $\tilde{z}_1 = (g \circ f)^{-1}(0)$, then $A \subseteq \tilde{z}_1$. Similarly, since $g(V^c) = 1$ and $\tilde{z}_2 = h^{-1}(0)$, then $V^c \subseteq \tilde{z}_2$ and that $(\tilde{z}_2)^c \subseteq V$. Let $\phi \in N(C)$ and $x \in A$. Hence, $(x, \phi(x)) \in C = (z_1 \cap z_2)^c = {z_1}^c \cup {z_2}^c$. Therefore, $(x, \phi(x)) \in {z_1}^c$ or $(x, \phi(x)) \in {z_2}^c$. If $(x, \phi(x)) \in {z_1}^c = (\tilde{z}_1 \times Y)^c$, then $x \notin \tilde{z_1}$. But $A \subseteq \tilde{z}_1$; which is a contradiction. So $(x, \phi(x)) \in {z_2}^c = (X \times \tilde{z}_2)^c$. Since $\phi(x) \in (\tilde{z}_2)^c \subseteq V$, then $\phi(x) \in V$. Thus, $\phi \in S(A, V)$.■

Corollary 2. For any space X and any metric space Y, $C_q(X, Y) = C_{q,u}(X, Y) \leq C_m(X, Y) \leq C_g(X, Y)$.

Corollary 3. For compact space X and any metric space Y, $KC_q(X, Y) = KC_m(X, Y) = KC_g(X, Y)$.

Theorem 12. For any space X and any space Y, $C_{q,\gamma}(X, Y) \leq C_m(X, Y)$.

Proof. Let $x \in X$ and $g \in O(f)$, then there is an open set $O_x \in \mathcal{O}$ such that $f(x), g(x) \in O_x$. By the continuity of f and g, there is an open neighborhood H_x of x such that if $y \in H_x$, then $f(y), g(y) \in O_x$. Let $U_x = H_x \times O_x$, then $(x, g(x)) \in U_x$ and let $U = \bigcup_{x \in X} U_x$, then $G(g) \subseteq U$. Hence, $g \in N(U)$.

Let $h \in N(U)$. So $G(h) \subseteq U$. For $x_0 \in X$, $(x_0, h(x_0)) \in U_{x0}$. Therefore, $h(x_0) \in O_{x0}$. S i nce $x_0 \in H_{x0}$, then $f(x_0) \in O_{x0}$. Thus, $h \in O(f)$.■

Corollary 4. For any space X and any metric space Y, $C_u(X, Y) \leq C_\gamma(X, Y) \leq C_g(X, Y)$.

Corollary 5. For any space X and any metric space Y, $KC_u(X, Y) \leq KC_\gamma(X, Y) \leq KC_g(X, Y)$.

Corollary 6. For compact space X, $KC_q(X) = KC_u(X) = KC_\gamma(X) = KC_m(X)$.

RESULT AND DISCUSSION

Metrizability and Completeness Properties

In this section, we study the submetrizability, metrizability and completeness properties of $KC_q(X)$. But in order to study the metrizability of $KC_q(X)$ in a broader perspective, first we show that a number of properties of $KC_q(X)$ are equivalent to submetrizability. We begin with the definition of submetrizability and some immediate consequences of this propert.

A space X is said to be submetrizable if it has a weaker metrizable topology, equivalently if there exists a metrizable space Y and a continuous bijection $f: X \to Y$ from the space X onto Y.

In a topological space a G_δ-set is a set which can be written as the intersection of a countable collection of open sets.

For any space X, if the set $\{(x, x): x \in X\}$ is G_δ-set (resp. Zero-set) in the product space $X \times X$, then X is said to have a G_δ-diagonal (resp. zero-set diagonal). Every submetrizable space X has a zero-set diagonal. Consequently, every submetrizable space X has a G_δ-diagonal since a zero-set is a G_δ-set.

A space X is called an E_0-space if every point in the space is a G_δ-set. The submetrizable spaces are E_0- spaces.

Finally, recall that in submetrizable space, the notions of compactness, countably compactness, quasicompactness and pseudocompactness coincide.

Theorem 13. For any space X, the following are equivalent.

(1) $KC_q(X)$ is submetrizable.
(2) Every quasicompact subset of $KC_q(X)$ is a G_δ-set in $KC_q(X)$.
(3) Every compact subset of $KC_q(X)$ is a G_δ-set in $KC_q(X)$.
(4) $KC_q(X)$ is an E_0-space.
(5) $KC_q(X)$ has a zero-set-diagonal.
(6) $KC_q(X)$ has a G_δ-diagonal.

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4) and (1) ⇒ (5) ⇒ (6) ⇒ (4) are all immediate.

(4) ⇒ (1) Let $X = \bigcup_{n=1}^{\infty} A_n$ where each A_n is quasicompact. Let $S = \bigoplus\{A_n: n \in \mathbb{N}\}$ be the topological sum of the A_n and let $\phi: S \to X$ be the natural function. Then, the induced function $\phi^*: KC_q(X) \to KC_q(S)$ defined by $\phi^*(f) = f \circ \phi$ is continuous. Now we shall show that ϕ^* is one-to-one. Let $\phi^*(g_1) = \phi^*(g_2)$. Then, g_1 and g_2 are equal on $\bigcup_{n=1}^{\infty} A_n$. So $g_1 - g_2 \in \bigcap_{n=1}^{\infty} B_A n\,(0, \varepsilon_n) = \{0\}$. Thus, $g_1 = g_2$ and consequently ϕ^* is one-to-one. By Theorem 2.2 and Theorem 2.3 in [4], $KC_q(\bigoplus\{A_n: n \in \mathbb{N}\})$ is homeomorphic to $\prod\{KC_q(A_n): n \in \mathbb{N}\}$. But each $KC_q(A_n) = C_q(A_n)$ is metrizable by Theorem 2.4 in [7]. Since $KC_q(S)$ is metrizable and ϕ^* is continuous injection, $KC_q(X)$ is submetrizable.■

Theorem 14. For any space X, the spaces $KC_m(X)$ and $KC_g(X)$ are always submetrizable.

Proof. It follows from Corollary 5.

In order to study the metrizability of the space $KC_q(X)$, there is a need the followingdefinitions.

A space X is a q-space if for each point $x \in X$, there exists a sequence $\{U_n: n \in \mathbb{N}\}$ of neighborhoods of x such that if $x_n \in U_n$ for each n, then $\{x_n: n \in \mathbb{N}\}$ has a cluster point.

A topological space is said to be hemi quasicompact if there exists a sequence of quasicompact sets $\{A_n\}$ in X such that for any quasicompact subset A of X, $A \subseteq A_n$ holds for some n.

Theorem 15. For any space X, the following are equivalent.

(1) $KC_q(X)$ is metrizable.
(2) $KC_q(X)$ is first countable.
(3) $KC_q(X)$ *is* a q-space.
(4) X is hemiquasicompact.
(5) $C_q(X)$ is metrizable.
(6) $C_q(X)$ is first countable.

Proof. (1) ⇒ (2) ⇒ (3) are all immediate.

(3) ⇒ (4) Suppose that $KC_q(X)$ is a q-space. Hence, there exists a sequence $\{U_n: n \in \mathbb{N}\}$ of neighborhoods of the zero function f_0 in $KC_q(X)$ such that if for each n, $f_n \in U_n$ then $\{f_n: n \in \mathbb{N}\}$ has a cluster point in $KC_q(X)$. Now for each n, there exist a quasicompact subset An of X and $\varepsilon n > 0$ such that $f0 \in BAn(f0, \varepsilon n) \subseteq Un$. Let A be a quasicompact subset of X. If possible, suppose that A is not a subset of A_n for any $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, there exists $a_n \in A \backslash A_n$. So for each $n \in \mathbb{N}$ there exists a continuous function $f_n: X \to [0, n]$ such that $f_n(a_n) = n$ and $f_n(x) = 0$ for all $x \in A_n$. It is clear that $f_n \in B_A n(f_0, \varepsilon_n)$. But the sequence $\{f_n\}_{n\in\mathbb{N}}$ does not have a cluster point in $KC_q(X)$. Suppose that this sequence has a cluster point f in $KC_q(X)$. Then, for each $k \in \mathbb{N}$, there exists a positive integer $n_k > k$ such that $f_n k \in B_A(f, 1)$. Thus, for all $k \in \mathbb{N}$, $f(a_n k) > f_n k(a_n k) - 1 = n_k - 1 \geq k$. But this means that f is unbounded on the quasicompact set A. Hence, the sequence $\{f_n\}_{n\in\mathbb{N}}$ cannot

have a cluster point in $KC_q(X)$ and consequently, $KC_q(X)$ fails to be a q-space. Thus, X must be hemiquasicompact.

(4) ⇒ (1) Here we need the well-known result which says that if the topology of a locally convex Hausdorff space is generated by a countable family of seminorms, then it is metrizable (see page 119 in [13]). Now the locally convex topology on $KC(X)$ generated by the countable family of seminorms $\{p_A n: n \in \mathbb{N}\}$ is metrizable and weaker than the quasicompact-open topology. But since for each quasicompact set A in X, there exists A_n such that $A \subseteq A_n$, the locally convex topology generated by the family of seminorms p_A: $A \in QC(X)$, that is, the quasicompact-open topology is weaker than the topology generated by the family of seminorms $\{p_A n: n \in \mathbb{N}\}$. Hence, $KC_q(X)$ is metrizable.

(4) ⇔ (5) ⇔ (6) was given Theorem 3.8 in [7].■

Corollary 7. For submetrizable space X, the following are equivalent.

(1) $KC_m(X)$ is metrizable.
(2) $KC_m(X)$ is first countable.
(3) $KC_\gamma(X)$ is metrizable.
(4) $KC_\gamma(X)$ is first countable.
(5) $KC_q(X)$ is metrizable.
(6) $KC_q(X)$ is first countable.
(7) X is countably compact.

Proof. Note that if X is submetrizable space, then $C(X) = KC(X)$ and also in a submetrizable space, all these kinds of compactness coincide. Hence, the proof of theorem follows from Theorem 15.■

Now we examine the complete metrizability of $KC_q(X)$. First we recall the definitions of various kinds of completeness properties.

A uniform space X with an uniformity $\mathcal{U}$ is called uniformly complete if the uniformity $\mathcal{U}$ is complete. We say that the uniformity $\mathcal{U}$ on X is complete if every Cauchy net in X converges.

A space X is called Cech-complete if X is a G_δ-set in βX, the Stone-Cech compactication of X. A space X is called locally Cech-complete if every point $x \in X$ has a Cech-complete neighborhood.

Proposition 1. For $k_{\mathbb{R}}$-space X, $C_q(X)$ is complete.

Proof. Let A be a compact subset of X and let $f \in C(X)$. Then, the set $f(A)$ is a bounded subset of $\mathbb{R}$. Hence, by Theorem 4.6 in [14], $C_q(X)$is complete.■

Theorem 16. For any space X, $KC_q(X)$ is uniformly complete.

Proof. Let (f_n) be a Cauchy net in $KC_q(X)$. If A is a compact subset of X, then the net $(f_n|_A)$ is Cauchy in $KC_q(A) = C_q(A)$. But since $C_q(A)$ is uniformly complete [7], the net $(f_n|_A)$ converges to some f_A in $C_q(A)$. Define $f: X \to \mathbb{R}$ by $f(x) = f_A(x)$ if $x \in A$. It can easily be seen that f is well defined and $f|_A = f_A$ for A for each compact subset A of X. Clearly $f \in KC(X)$. Also it is easy to see that (f_n) converges to f.■

Considering Theorem 15 and Proposition 1, it can give the following results.

Corollary 8. For any space X, the following are equivalent.

(1) $KC_q(X)$ is complete metrizable.
(2) $KC_q(X)$ is Cech-complete.
(3) $KC_q(X)$ is locally Cech-complete.
(4) $KC_q(X)$ is an open continuous image of Cech-complete space.
(5) $KC_q(X)$ is metrizable.
(6) X is hemiquasicompact.

Corollary 9. For any space X, the following are equivalent.

(1) $C_q(X)$ is complete metrizable.
(2) $C_q(X)$ is Cech-complete.
(3) $C_q(X)$ is locally Cech-complete.
(4) $C_q(X)$ is an open continuous image of Cech-complete space.
(5) $C_q(X)$ is metrizable.
(6) X is hemiquasicompact $k\mathbb{R}$-space.

Corollary 10. For submetrizable space X, the following are equivalent.

(1) $KC_m(X)$ is completely metrizable.
(2) $KC_m(X)$ is Cech-complete.
(3) $KC_\gamma(X)$ is completely metrizable.
(4) $KC\gamma(X)$ is Cech-complete.
(5) $KC_q(X)$ is completely metrizable.
(6) $KC_q(X)$ is Cech-complete.
(7) X is countably compact

Countability Properties

In this section, we study some countability properties such as $\aleph_0$-space, cosmic, separability and second countability.

A k-network for a space X is a family $\mathcal{K}$ of subsets of X such that whenever compact K is contained in open U, then there is a nite subset $\mathcal{K}_0 \subseteq \mathcal{K}$ such that $K \subseteq \cup \mathcal{K}_0 \subseteq \mathcal{K}$. A space X is called a $\aleph_0$-space [15] if it has a countable k-network.

A space X is said to have a countable network if there exists a countable family $\{A_n: n \in \mathbb{N}\}$ of subsets of X such that for each $x \in X$ and for each open set U containing x, there exists an A_n such that $x \in A_n \subseteq U$. A space X is called a cosmic space [15] if it has a countable network.

Recall that any $\aleph_0$-space is cosmic, any cosmic space is Lindelöf and separable. Also in metrizable space, the notions of second countability, $\aleph_0$-space and cosmic property coincide [15].

Proposition 2. For separable metrizable space X, $KC_q(X)$ *is* $\aleph_0$-space.

Proof. If X is separable metrizable, then $C_q(X) = C_k(X)$ by Theorem 3.9 in [7], $C_q(X) = KC_q(X)$ and also $C_k(X)$ is $\aleph_0$-space by Lemma 2.3.6 in [15]. Consequently, $KCq(X)$ is $\aleph_0$-space.■

Since separable metrizable space is $\aleph_0$-space [15], then the following results can be given.

Corollary 11. For $\aleph_0$-space *X*, $C_q(X) = KC_q(X)$.

Corollary 12. For $\aleph_0$-space *X*, $KC_q(X)$ is $\aleph_0$-space.

Theorem 17. For any space X, the following are equivalent.

(1) $KC_q(X)$ is $\aleph_0$-space.
(2) $KC_q(X)$ is cosmic space.
(3) $C_q(X)$ is $\aleph_0$-space.
(4) $C_q(X)$ is cosmic space.
(5) X is $\aleph_0$-space

Proof. (1) ⇒ (2) and (3) ⇒ (4) it clear. By Proposition 2, (5) ⇒ (1) and (5) ⇒ (3).

(1) $\Rightarrow$ (5) Let $\mathcal{F}$ be a countable network for the space $KC_q(X)$. For $F \in \mathcal{F}$, let's define the set $F^* = \{x \in X : f(x) > 0\}$ and the class $\mathcal{F}^* = \{F^*: F \in \mathcal{F}\}$. Let us show that class $\mathcal{F}^*$ is a k-network for the space X. Let U be the open and A compact subsets in X. Since the space X is Tychonoff, there is a function $f \in C(X)$ such that $f(A) = \{1\}$ and $f(X \backslash U) = \{0\}$. Therefore $f \in S(A, (0, \infty))$. Since $\mathcal{F}$ is a countable network for the space $KC_q(X)$, $F \in \mathcal{F}$ exists as $f \in F \subseteq S(A, (0, \infty))$. Hence, it is seen that $A \subseteq F^*$. It is sufficient to show that $F^* \subseteq U$. Let $\in F^* \backslash U$. Since $x \notin U$, then $f(X) = 0$. But this contradicts the fact that for $x \in F^*$ and $f \in F$, $f(x) > 0$. Therefore, $F^* \subseteq U$. Thus, X *is* $\aleph_0$-space. Consequently, (4) $\Rightarrow$ (5).■

Theorem 18. For σ-quasicompact space X, the following are equivalent.

(1) $C_q(X)$ is separable.
(2) $KC_q(X)$ is separable.
(3) Every quasicompact subset of X is metrizable.
(4) X is a cosmic space.
(5) X is submetrizable.

Proof. (1) $\Rightarrow$ (2) If $C_q(X)$ is separable, then X is submetrizable by Theorem 3.10 in [7] and so $C(X) = KC(X)$. It follows that $KC_q(X)$ is separable.

(2) $\Rightarrow$ (3) Let A be quasicompact subset of X. It is easy to see that $C(X)$ is dense in $KC_q(X)$. Hence $C_q(A)$ is separable and so A is submetrizable. Since every quasicompact completely regular submetrizable space is metrizable [16, Corollary 2.7], then A is metrizable

(3) $\Rightarrow$ (4) Since X is σ-quasicompact, there exists a countable family $\{A_n: n \in \mathbb{N}\}$ of quasicompact subsets of X such that $X = \bigcup_{n=1}^{\infty} A_n$. Each A_n, being compact and metrizable, is second countable and consequently, each A_n has a countable network $\mathcal{B}_n$. It is easy to show that $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ is a network for X, that is, X is a cosmic space.

(4) $\Rightarrow$ (5) Follows from Theorem 4.3.4 in [17].

(5) $\Rightarrow$ (1) First recall that since X σ-quasicompact, X is submetrizable if and only if X has a separable metrizable compression, that is, X has a weaker separable metrizable topology (see [18, Example 3.8.C]). Follows from Theorem 3.10 in [7].■

Theorem 19. For any space X, the following are equivalent.

(1) $C_q(X)$ is second countable.
(2) $KC_q(X)$ is second countable.
(3) X is hemiquasicompact and submetrizable.
(4) X is hemiquasicompact and $\aleph_0$-space.
(5) X is hemiquasicompact and cosmic space.

Proof. (1) $\Rightarrow$ (2) If $C_q(X)$ is second countable, then it is separable and submetrizable by Theorem 3.10 in [7]. Thus, $C(X) = KC(X)$. It follows that $KC_q(X)$ is second countable.

(2) $\Rightarrow$ (3) If $KCq(X)$ is second countable, then it is metrizable as well as seprable. But then by Theorem 15, X is hemicompact and consequently by Theorem 18, X is submetrizable also.

By Theorem 3.16 in [7], (3) $\Rightarrow$ (1) and the proof of (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) is given Theorem 2.4.1 in [19].■

Since locally compact $\aleph_0$-space is separable and metrizable [15], then the folloing result can be given.

Corollary 13. For locally compact space X, the following are equivalent.

(1) $C_q(X)$ is second countable.
(2) $KC_q(X)$ is second countable.
(3) X is Lindeloöf and submetrizable.
(4) X is $\aleph_0$-space.
(5) X is cosmic space.
(6) X is second countable.

Theorem 20. For any space X, the following are equivalent.

(1) $C_q(X)$ is separable.
(2) $KC_q(X)$ is separable.
(3) X has *a* $\aleph_0$-space compression.
(4) X has a cosmic compression.
(5) X has a separable metrizable compression.

Proof. (1) $\Leftrightarrow$ (2) It follows from Theorem 18.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) It follows from Lemma 10.1 and Proposition 10.2 in [15].

(1) $\Leftrightarrow$ (5) It follows from Theorem 3.10 in [7]. ■

Theorem 21. For any space X, the following are equivalent.

(1) $KC_m(X)$ is $\aleph_0$-space.
(2) $KC_m(X)$ is cosmic space.
(3) $KC_m(X)$ is second countable.
(4) $KC_m(X)$ is separable.
(5) $KC_m(X)$ is Lindeloöf.
(6) $KC_\gamma(X)$ is $\aleph_0$-space.
(7) $KC_\gamma(X)$ is cosmic space.
(8) $KC_\gamma(X)$ is second countable.
(9) $KC_\gamma(X)$ is separable.
(10) $KC_\gamma(X)$ is Lindeloöf.
(11) X is compact and metrizable.

Proof. It is clear from Corollary 6 and Theorem 20.■

CONCLUSION

In this study, which we think contributes to studies on function spaces, we introduced quasicompact-open topology on $KC(X, Y)$, the set of all functions from X to Y, which are continuous on the compact subsets of X and compared this topology with the open-over topology, the uniform topology and m-topology. Then, we examined the metrizability, completeness, and countability properties of the quasicompact-open topology on $KC(X, Y)$. Also, we obtained similar results for the open-cover topology and m-topology on $KC(X, Y)$.

$KC(X)$ kümesi üzerindeki topolojiler, $C(X)$ üzerindeki topolojilere genel bir bakış kazandırmaktadır. For follow-up studies of this study, topological features not examined within the scope of this study can be analyzed for the spaces given in the article.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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