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Research Article

Rough convergent functions defined on amenable semigroups

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ABSTRACT

The authors in this study, firstly, identified the rough convergence and presented the set of rough limit points of a function defined on discrete countable amenable semigroups (DCASG) with some characteristics such as convexity, closedness and boundedness. Then, they introduced rough Cauchy sequence and also, examined the relations between rough Cauchy sequences and rough convergence of functions defined on discrete countable amenable semigroups.

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INTRODUCTION

In semigroups, which have an important place in algebra and number theory, the types of convergence (classical convergence, statistical convergence, ideal convergence, rough convergence, etc.) that form the basis of summability theory have not been sufficiently studied and are still not sufficiently addressed by scientists today. The concept of rough convergence, which was started to be studied for the first time in the 2000s in summability theory, has been handled only in normed spaces and not much work has been done on this subject. One of the most important criticisms against the scientists working in the theory of summability in mathematics in recent years is that this theory has almost never been studied except for metric spaces, normed spaces and topological spaces. In this context, the concept of rough convergence in amenable semigroups, which we intend to study in this article, has not been studied in summability theory before, and we think that the study of this concept will bring an innovation that will bring a vision to the field and eliminate the important deficiencies in the literature mentioned above.

The first studies on the concept of amenable semigroup (or simply ASG) were carried out by Day [1]. After taht, a few mathematicians [2-4] studied the concepts of convergence types in ASG. In [5] Douglas expanded the concept of arithmetic mean to ASG and obtained a characterization for almost convergence in ASG. Also in [6], Nuray and Rhoades gave the concepts of convergence and statistical convergence in ASG. And recently, some mathematicians studied on the new concepts in ASG (see, [7-10]).

Phu [11] introduced, firstly, the concepts of rough convergence and rough Cauchy sequence in finite-dimensional normed spaces and he examined some characteristics of LIMr_x such as convexity, closedness and boundedness. Phu [12,13] studied on rough convergence and some newly

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characteristics of this concept. In recently a few mathematicians [14,15] examined the rough convergence and rough statistical convergence types in different normed spaces.

The rough convergence of bounded but not convergent sequences differs from the convergence of sequences. In addition, this concept has not been studied before in amenable semigroups. In this sense, it differs from studies in any metric space or normed space. The target of this article is to identify the rough convergence and present the set of rough limit points of a function defined on discrete countable amenable semigroups with some characteristics such as convexity, closedness and boundedness. Next, we aim to introduce the rough Cauchy sequence and also to examine the relationships between rough Cauchy sequences and rough convergence in DCASG.

Firstly, let us recall the some important basic definitions and concepts such as amenable semigroups, rough convergence and rough Cauchy sequence that we will use in this study. (see, [1-6,11-13,16]).

After that, in this study let G be a DCASG with identity in which both left and right cancelation laws hold, and $w(G)$ denotes the space of all real valued functions on G.

If we let G as a countable amenable group, so there exists a sequence $\{S_n\}$ of finite subsets of G such that

i. $G = \bigcup_{n=1}^{\infty} S_n$,

ii. $S_n \subset S_{n+1} \quad (n = 1,2,\dots)$,

iii. $\lim_{n\to\infty} \frac{|S_n g \cap S_n|}{|S_n|} = 1, \ \lim_{n\to\infty} \frac{|gS_n \cap S_n|}{|S_n|} = 1,$ for all $g \in G$.

If a sequence consisting of finite subsets of G satisfies (i)-(iii), this sequence is called a Folner sequence (FS).

The sequence

$$S_n = \{0,1,2,\dots n\text{-}1\}$$

is a familiar FS leading to the classical Cesàro summability method.

Now, firstly, we give definitions of convergence and Cauchy sequence of function defined on DCASG.

If for every $\varepsilon > 0$ there exists a $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$|f(g) - f(h)| < \varepsilon,$$

for all $m > k_0$ and $g \in G\backslash S_m$, then the function $f \in w(G)$ is convergent to t for any FS $\{S_n\}$ of G.

If for every $\varepsilon > 0$ there exists a $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$|f(g) - f(h)| < \varepsilon,$$

for all $m > k_0$ and $g, h \in G\backslash S_m$, then the function $f \in w(G)$ is Cauchy sequence for any FS $\{S_n\}$ of G.

Now, for sequences of real numbers, we will note the definitions and properties of rough convergence with basic characteristics, which is an important topic of our study.

Let $r \in \mathbb{R}$ and $r \geq 0$. And also, with the norm $\|.\|$, let $\mathbb{R}^n$ (the real n-dimensional space). Suppose that a sequence $x = (x_n)$ in $\mathbb{R}^n$.

The sequence (x_n) is rough convergent (r-convergent) to ξ, on condition that

$$\forall\varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N}: n \geq n_\varepsilon \Rightarrow \| x_n - \xi \| < r + \varepsilon.$$

This convergence is indicated by

$$x_n \xrightarrow{r} \xi.$$

For the sequence $x = (x_n)$, the rough limit set is showed by

$$\mathrm{LIM}^r x = \{\xi \in \mathbb{R}^n : x_n \xrightarrow{r} \xi\}.$$

If we let

$$\mathrm{LIM}^r x \neq \emptyset,$$

then the sequence $x = (x_n)$ is rough convergent and r is named the convergence degree of (x_n). For $r = 0$, the rough convergence coincides with the usual convergence.

Now we will give the definition of rough Cauchy sequence in normed space, which is an important concept in our study.

For $\wp \geq 0$, if

$$\forall\varepsilon > 0 \ \exists n_\varepsilon : m, n \geq n_\varepsilon \Rightarrow \| x_m - x_n \| < \wp + \varepsilon,$$

then the sequence (x_n) is rough Cauchy sequence. Here $\wp$ is roughness Cauchy degree of (x_n). Shortly (x_n) is called a $\wp$-Cauchy sequence. Also, $\wp$ is a Cauchy degree of (x_n).

Lemma 1 *[11] Let $x = (x_n)$ be r-convergent, that is,*

$$\mathrm{LIM}^r x \neq \emptyset.$$

Then, for every $\wp \geq 2r$, (x_n) is a $\wp$-Cauchy sequence. For the Cauchy degree, this bound cannot be generally reduced.

MAIN RESULTS

In summability theory, the relation between convergence and boundedness of a sequence is important. Regarding the convergence of bounded sequences, the concept of rough convergence and its properties have been studied in recent years. In this sense, firstly, we give definition of convergence in rough sense and $\mathrm{LIM}^r f$ of function defined on DCASG.

Definition 1 For all $g \in G\backslash S_m$, if

$$\forall\varepsilon > 0 \ \exists k_\varepsilon \in \mathbb{N}: m \geq k_\varepsilon \Rightarrow |f(g) - t| < r + \varepsilon \qquad (1)$$

or equivalently on condition that

$$limsup|f(g) - t| \leq r, \qquad (2)$$

for all $g \in G\backslash S_m$, then the function $f \in w(G)$ is rough convergent (r-convergent) to t for any FS $\{S_n\}$ *of* G. This convergence is indicated by

$$r - limf(g) = t \quad or \quad f(g) \xrightarrow{r} t.$$

With r as roughness degree, this convergence is the rough convergence. For $r = 0$, the rough convergence coincides with the usual convergence. However, our main area of interest is situation $r > 0$. For this situation there are various reasons. For example, since an orginally convergent function $h \in w(G)$ (with $h(g) \to t$) usually cannot be determined (i.e., measured or calculated) fully, one has to do with an approximated function $f \in w(G)$ satisfying

$$|f(g) - h(g)| \leq r,$$

for all $g \in G \setminus S_m$, with $r > 0$ an upper bound of approximation error. Hence, $f \in w(G)$ is no longer classical convergent, but for all $g \in G \setminus S_m$,

$$|f(g) - t| \leq |f(g) - h(g)| + |h(g) - t| \\ \leq r + |h(g) - t|$$

implies that is r-convergent for any FS $\{S_n\}$ of G in the sense of (1).

If (1) is valid, then t is an r-limit point of the function $f \in w(G)$, (for $r > 0$) usually no longer unique. Thus, we must think the so-named rough limit set (r-limit set) of the function $f \in w(G)$ indicated by

$$\mathrm{LIM}^{r} f := \{t : f(g) \xrightarrow{r} t\}. \tag{3}$$

For $r = 0$, if

$$\mathrm{LIM}^{r} f \neq \emptyset,$$

then the function $f \in w(G)$ is rough convergent for any FS $\{S_n\}$ for G and r is named the convergence degree of $f \in w(G)$.

First, let us translate some of the characteristics of classical convergence into rough convergence. The fact that a convergent sequence has unique limit is well known. With roughness degree $r > 0$, this property is not conserved for rough convergence but only has the following analogy.

Theorem 1 *For any FS $\{S_n\}$ of G, $diam(\mathrm{LIM}^{r} f) \leq 2r$ for a function $f \in w(G)$. Generally, $diam(\mathrm{LIM}^{r} f)$ has no smaller bound.*

Proof.

We must show that

$$diam(\mathrm{LIM}^{r} f) = sup\{|s - t| : s, t \in \mathrm{LIM}^{r} f\} \leq 2r, \tag{4}$$

for $f \in w(G)$. On the contrary, suppose that

$$diam(\mathrm{LIM}^{r} f) > 2r$$

then, there exist

$$s, t \in \mathrm{LIM}^{r} f$$

satisfying

$$d := |s - t| > 2r.$$

For an arbitrary

$$\varepsilon \in (0, d/2 - r),$$

from (1) and (3), there is an $k_\varepsilon \in \mathbb{N}$ such that for $m \geq k_\varepsilon$,

$$|f(g) - s| < r + \varepsilon$$

and

$$|f(g) - t| < r + \varepsilon,$$

for all $g \in G \setminus S_m$. This implies

$$|s - t| \leq |f(g) - s| + |f(g) - t| \\ < 2(r + \varepsilon) \\ < 2r + 2\left(\frac{d}{2} - r\right) \\ = d,$$

which conflicts with

$$d = |s - t|.$$

Therefore, (4) has to be true. For any FS $\{S_n\}$ of G, let a convergent function $f \in w(G)$ with

$$\lim f(g) = s.$$

Then, for

$$\overline{B}_r(s) := \{t : |t - s| \leq r\},$$

it follows from

$$|f(g) - t| \leq |f(g) - s| + |s - t| \\ \leq |f(g) - s| + r,$$

for $t \in \overline{B}_r(s)$, (1) and (3) that

$$\mathrm{LIM}^{r} f = \overline{B}_r(s).$$

Since

$$diam(\overline{B}_r(s)) = 2r,$$

this generally indicates that the upper bound $2r$ of the diameter of an rough limit set of $f \in w(G)$ cannot no longer be reduced.

Definition 2 *For $M > 0$ and all $g \in G$, if*

$$|f(g)| < M,$$

then the function $f \in w(G)$ is bounded for any FS $\{S_n\}$ of G.

Now the following theorem will give the relation between the boundedness of a function $f \in w(G)$ and $\mathrm{LIM}^{r} f$.

Theorem 2 *The function $f \in w(G)$ is bounded for any FS $\{S_n\}$ of G iff*

$$\mathrm{LIM}^{r} f \neq \emptyset \text{ for } r \geq 0.$$

Proof.

Let

$$p := \sup\{|f(g)| : g \in G\} < \infty,$$

then $\mathrm{LIM}^{p} f$ contains the origin of G. On the contrary, suppose that

$$\mathrm{LIM}^{r} f \neq \emptyset$$

for $r \geq 0$, hence some ball with any radius greater than r contain all but finite elements $f(g)$. Hence, $f \in w(G)$ is bounded for any FS $\{S_n\}$ of G.

Now, some interesting geometrical and topological properties related to the concept of rough convergence will be given.

Theorem 3 *For a function $f \in w(G)$, $\mathrm{LIM}^{r} f$ is closed (for all $r \geq 0$).*

Proof.

Let $h \in w(G)$ be an arbitrary function in $\mathrm{LIM}^{r} f$ which converges to some point s for any FS $\{S_n\}$ of G. For each $\varepsilon > 0$, by definition, there are an $k_{\frac{\varepsilon}{2}}$ such that for all $g, k \in G \setminus S_m$

$$|h(k) - s| < \frac{\varepsilon}{2}$$

and

$$|f(g) - h(k)| < r + \frac{\varepsilon}{2},$$

whenever $m \geq k_{\frac{\varepsilon}{2}}$. Consequently, if $m \geq k_{\frac{\varepsilon}{2}}$

$$|f(g) - s| \leq |f(g) - h(k)| + |h(k) - s| < r + \varepsilon.$$

That means $s \in \mathrm{LIM}^r f$, too. Hence, $\mathrm{LIM}^r f$ is closed.

Theorem 4 *For the function $f \in w(G)$ let*

$$s_0 \in \mathrm{LIM}^{r_0} f \text{ and } s_1 \in \mathrm{LIM}^{r_1} f.$$

Then,

$$s_\alpha := (1-\alpha)s_0 + \alpha s_1 \in \mathrm{LIM}^{(1-\alpha)r_0 + \alpha r_1} f, \quad for\ \alpha \in [0,1].$$

Proof.

From definition, for each $\varepsilon > 0$, r_0, $r_1 > 0$ there exists an k_ε such that for all $m > k_\varepsilon$ implies

$$|f(g) - s_0| < r_0 + \varepsilon$$

and

$$|f(g) - s_1| < r_1 + \varepsilon,$$

for any FS $\{S_n\}$ of G and all $g \in G\backslash S_m$, which yields also

$$\begin{aligned}|f(g) - s_\alpha| &\leq (1-\alpha)|f(g) - s_0| + \alpha|f(g) - s_1| \\ &< (1-\alpha)(r_0 + \varepsilon) + \alpha(r_1 + \varepsilon) \\ &= (1-\alpha)r_0 + \alpha r_1 + \varepsilon.\end{aligned}$$

Hence, we have

$$s_\alpha \in \mathrm{LIM}^{(1-\alpha)r_0 + \alpha r_1} f.$$

Theorem 5 *For any FS $\{S_n\}$ of G, let a function $f \in w(G)$. $\mathrm{LIM}^r f$ is convex.*

Proof.

In particular, if we let $r = r_0 = r_1$, then Theorem 4 yields immediately that $\mathrm{LIM}^r f$ is convex.

Theorem 6 *For any FS $\{S_n\}$ of G, if*

$$f_1(g) \overset{r}{\to} s \text{ and } f_2(g) \overset{r}{\to} t,$$

then

i. $(f_1(g) + f_2(g)) \overset{2r}{\longrightarrow} (s+t)$ and

ii. $cf_1(g) \overset{|c|r}{\longrightarrow} cs, \quad (c \in \mathbb{R})$.

Proof.

i. For any FS $\{S_n\}$ of G, let

$$f_1(g) \overset{r}{\to} s \text{ and } f_2(g) \overset{r}{\to} t.$$

From definition

$$\forall \varepsilon > 0\ \exists i_\varepsilon \in \mathbb{N}: m \geq i_\varepsilon \Rightarrow |f_1(g) - s| \leq r + \frac{\varepsilon}{2}$$

and

$$\forall \varepsilon > 0\ \exists j_\varepsilon \in \mathbb{N}: k \geq j_\varepsilon \Rightarrow |f_2(h) - t| \leq r + \frac{\varepsilon}{2},$$

for all $g \in G\backslash S_m$ and all $h \in G\backslash S_k$. Let

$$p := \max\{i_\varepsilon, j_\varepsilon\}.$$

For every $m > p$, we have

$$\begin{aligned}|(f_1(g) + f_2(g)) - (s+t)| &= |f_1(g) - s| + |f_2(g) - t| \\ &\leq r + \frac{\varepsilon}{2} + r + \frac{\varepsilon}{2} \\ &= 2r + \varepsilon\end{aligned}$$

and so

$$(f_1(g) + f_2(g)) \overset{2r}{\longrightarrow} (s+t),$$

for all $g \in G\backslash S_m$.

ii. Proof is clear for $c = 0$. Suppose $c \neq 0$. Since

$$f_1(g) \overset{r}{\longrightarrow} s,$$

for each $\varepsilon > 0$ $\exists k_\varepsilon \in \mathbb{N}$ such that for every $m \geq k_\varepsilon$, we have

$$|f_1(g) - s| \leq r + \varepsilon,$$

for all $g \in G\backslash S_m$. According to this, for all $m \geq k_\varepsilon$ we can write

$$\begin{aligned}|cf_1(g) - cs| &= |c||f_1(g) - s| \\ &\leq |c|(r+\varepsilon) \\ &= |c|r + |c|\varepsilon\end{aligned}$$

and so

$$cf_1(g) \overset{|c|r}{\longrightarrow} cs,$$

for all $g \in G\backslash S_m$.

Finally, by giving the definition of Cauchy sequence in rough sense with some properties, the relation between the rough convergence and Cauchy sequence in rough sense will be analysed.

Definition 3 *For $\wp > 0$ and all $g, h \in G\backslash S_m$, if*

$$\forall \varepsilon > 0\ \exists k_\varepsilon: m \geq k_\varepsilon \Rightarrow |f(g) - f(h)| \leq \wp + \varepsilon$$

is hold then the function $f \in w(G)$ for any FS $\{S_n\}$ of G is rough Cauchy sequence with roughness degree $\wp$. Here, $\wp$ is named Cauchy degree for $f \in w(G)$.

Proposition 1

i. Monotonicity: *Suppose $\wp' > \wp$. For any FS $\{S_n\}$ of G, if $\wp$ is a Cauchy degree of a given function $f \in w(G)$ so $\wp'$ is a Cauchy degree of $f \in w(G)$, too.*
ii. Boundedness: T*he function $f \in w(G)$, for any FS $\{S_n\}$ of G, is bounded iff there exists a $\wp \geq 0$ such that $f \in w(G)$ is a $\wp$-Cauchy.*

Theorem 7 *For any FS $\{S_n\}$ of G, the function $f \in w(G)$ is rough convergent (i.e., $\mathrm{LIM}^r f \neq \emptyset$) iff for every $\wp \geq 2r$, $f \in w(G)$ is a $\wp$-Cauchy sequence for any FS $\{S_n\}$ of G. For the Cauchy degree, this bound cannot be generally reduced.*

Proof.

For any FS $\{S_n\}$ of G, let t be any point in $\mathrm{LIM}^r f$. Then, for all $\varepsilon > 0$ there exists an $k_\varepsilon \in \mathbb{N}$ such that for all $m \geq k_\varepsilon$

$$|f(g)-t| \leq r+\frac{\varepsilon}{2}$$

and

$$|f(h)-t| \leq r+\frac{\varepsilon}{2},$$

for all $g, h \in G \backslash S_m$. Therefore, for all $m \geq k_\varepsilon$ we have

$$\begin{aligned} |f(g)-f(h)| &= |f(g)-t+t-f(h)| \\ &\leq |f(g)-t|+|f(h)-t| \\ &\leq r+\frac{\varepsilon}{2}+r+\frac{\varepsilon}{2} \\ &= 2r+\varepsilon, \end{aligned}$$

for all $g, h \in G \backslash S_m$. Hence for $\wp = 2r$, $f \in w(G)$ is a $\wp$-Cauchy sequence for any FS $\{S_n\}$ of G. By Proposition 1, every $\wp > 2r$ is also a Cauchy degree of $f \in w(G)$.

Let $f \in w(G)$ be a Cauchy sequence in rough sense for any FS $\{S_n\}$ of G. Since $f \in w(G)$ be a Cauchy sequence in rough sense, then for $\wp > 0$, f is bounded and as a result f is r-convergent for any FS $\{S_n\}$ of G. Clearly for the function f, generally this bound $2r$ can not be reduced (see Lemma 1).

CONCLUSION

We defined the rough convergence and the set of rough limit points of a sequence of functions defined on discrete countable amenable semigroups with some properties such as boundedness, closedness and convexity. Then, we introduced rough Cauchy sequence and also, examined the relations between rough Cauchy sequences and rough convergence of functions defined on discrete countable amenable semigroups. These concepts can also be studied for statistical convergence and ideal convergence of functions defined on discrete countable amenable semigroups in the future. Also, these convergence types of of functions defined on discrete countable amenable semigroups be studied for double sequences.

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DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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