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Research Article

Spinor representation of directional q -frame

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ABSTRACT

Spinors play a fundamental role in geometry and physics. The Clifford algebra is the natural linear algebraic setting where spinors and spin groups are formulated. In this study, spinors, which have many uses in Clifford algebras, have been given with a different representation in E^3. Firstly, the directional q -frame has been considered, and this frame is represented with a spinor in spinor space. Later, considering the relations between the directional q -frame and Frenet frame, the relations between spinors corresponding to these frames have been obtained. In this way, a different construction of the spinors by using in Clifford algebras has been shown.

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INTRODUCTION

Spinors are used in a wide field of study, from geometry and Clifford algebra to quantum mechanics and general relativity in physics. The most important characteristic feature of spinors is their behavior under rotations. In other words, spinors are characterized by the specific way in which they behave under rotations. Namely, if a vector or tensorial object rotates with θ, then a spinor which represents this rotates by $\theta / 2$. So, the spinor takes two complete cycles to return to its original configuration. Spin groups, including rotation groups, are frequently defined in geometry and Clifford algebras [18, 19]. Hestenes first described spinors with real even multi-vectors in geometric algebra [12, 15]. Spinors are defined as components of left minimal ideals of the Clifford algebra. Moreover, spinors in the left ideal approach lie in geometric algebra rather than its even subset. Spinors in geometry and physics are elements of a complex vector space that can be associated with Euclidean space. According to the definition of spinors, they are equipped with a complex linear representation of the spin group. Namely, the elements of the spin group act as linear transformations on the spinor space. Both the spin group and its Lie algebra are naturally embedded in Clifford algebra, and in practice, Clifford algebra is usually the easiest method to work together.

Cartan expressed the most general mathematical form of spinors in 1913 [5]. Based on Cartan's study, Torres del Castillo and Barrales expressed the relationship between

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spinor and curve theory [7]. That study is a fundamental work for scientists studying curve theory. With the help of that study, the expression of the curve theory with spinor has been understandably expressed. After that, Kişi and Tosun gave the spinor representation of the Darboux frame on a directed surface in E^3 [17]. Moreover, Ünal et al. described the spinor formulation of Bishop frame of curves in E^3 in [20]. Then, Ketenci et al. investigated the spinor corresponding to a mutually orthogonal vector triad in three-dimensional Minkowski space E_1^3. Thus, they introduced hyperbolic spinors. Based on that study, they gave the hyperbolic spinor formulation of the Frenet curve in Minkowski space E_1^3 [16]. After that, Erişir et al. expressed the spinor equations of an alternative frame of a curve in Minkowski space and the spinor formulation of the relationship between the Frenet and Bishop frames [10]. In addition, the hyperbolic spinor representation of the Darboux frame was obtained in E_1^3 [1]. After that, Erişir and Kardağ expressed a new representation of the involute evolute curves in E^3 with the help of spinors [13]. Then, the spinor formulation of Bertrand curves in E^3 was given in [11].

In this paper, firstly, spinors have been introduced algebraically. Then, the spinor equations of directional q -frame of curves have been defined. Later, the spinor relations between the Frenet frame and q -frame of any curve in E^3 have been established. Finally, the angle notation for these spinors has been given. In this way, a different geometric construction of spinors has been established. The aim of this study is to research the spinor structure lying on the basis of the differential geometry.

PRELIMINARIES

The Frenet-Serret formulas in differential geometry describe the geometric properties of the curve itself irrespective of any motion and the kinematic properties of a particle moving along a continuous, differentiable curve. More specifically, these formulas consist of equations written in terms of each other of the derivatives of vectors called tangent, normal, and binormal of the differentiable curve. The formulas are named after the two French mathematicians who independently discovered them; Frenet in 1847 and Serret in 1851. The tangent, normal, and binormal unit vectors, often denoted by t, n and b, or collectively the Frenet-Serret frame, in E^3 and are defined as: "A curve is considered as differentiable at each point of an open interval. Thus, one can construct a set of mutually orthogonal unit vectors on this curve. "So, let us consider that a regular curve (α) is given by the differentiable function $\alpha : I \rightarrow E^3$ where $s \in I \subseteq R$ is the arc-length parameter. If $\|\alpha'(s)\| = 1$ for all $s \in I$, the curve (α) is called unit speed curve. Then the Frenet vectors of a unit speed curve (α) can be obtained by $\boldsymbol{t}(s) = \alpha'(s)$, $\boldsymbol{n}(s) = \frac{1}{\|\alpha''(s)\|}\alpha''(s)$ and $\boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{n}(s)$ where "'" is the derivative with respect to the arc-length parameter s. Moreover, the Frenet formulas of this curve are $\boldsymbol{t}' = \kappa \boldsymbol{n}$, $\boldsymbol{n}' = \kappa \boldsymbol{t} + \tau \boldsymbol{b}$ and $\boldsymbol{b}' = -\tau \boldsymbol{n}$ [14].

The most known frame of a curve is the Frenet frame. Moreover, the Frenet frame has an important place in curve theory. Because many known types of curves such as Bertrand curves, involute evolute curves, Mannheim curves, spherical curves are defined with the help of these frame vectors. However, there are some disadvantages of the Frenet frame. The main disadvantage of the Frenet frame is that it has undesirable rotation around the tangent vector [2]. Alternatively, it is possible to define this with different frames at the points of space curves. The first study that was emphasized a different frame given by Bishop [3] and introduced a new frame along a space curve which is more suitable for applications. The Bishop frame is widely used in computer graphics; however, it is not easy to compute [22]. So, as an alternative to the Frenet or Bishop frames, a new adapted frame along a space curve, the directional q -frame, has been defined [8, 9]. The directional q -frame is defined with the help of the quasi-normal vector introduced by Coquillart [6]. The q -frame has many advantages compared to other frames (Frenet, Bishop). For instance, the directional q -frame can be defined even along a line ($\kappa = 0$). Moreover, the directional q -frame can be determined easily. So, let (α) be a curve with arc-length parameter s, the directional q -frame $\{t, n_q, b_q, k\}$ along the curve is given by

$$\boldsymbol{t} = \alpha', \quad \boldsymbol{n}_q = \frac{\boldsymbol{t} \wedge \boldsymbol{k}}{\|\boldsymbol{t} \wedge \boldsymbol{k}\|}, \quad \boldsymbol{b}_q = \boldsymbol{t} \wedge \boldsymbol{n}_q \tag{1}$$

where t is the unit tangent vector, n_q is the quasi-normal vector, b_q is the quasi-binormal vector and k is the projection vector which can be chosen $k_x = (1,0,0)$, $k_y = (0,1,0)$ or $k_z = (0,0,1)$. However, the q -frame is singular in all cases where t and k are parallel [8, 9]. The variation equations of the q -frame are given by

$$\begin{bmatrix} \boldsymbol{t}' \\ \boldsymbol{n}_q' \\ \boldsymbol{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n}_q \\ \boldsymbol{b}_q \end{bmatrix} \tag{2}$$

where the q -curvatures $\{k_1, k_2, k_3\}$ are expressed as follows

$$k_1 = \kappa cos\theta, \quad k_2 = -\kappa sin\theta, \quad k_3 = \theta' + \tau \tag{3}$$

and θ is the Euclidean angle between the principal normal vector n and the quasi-normal vector n_q. Moreover, one can see the relation between Frenet frame and directional q -frame

$$\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} \tag{4}$$

[8, 9].

Spinors

Geometrically, spinors are introduced by Cartan [4] as follows. One considers the vector $x = (x_1,x_2,x_3) \in C^3$ is an isotropic vector. So, $x_1^2 + x_2^2 + x_3^2 = 0$. The set of isotropic vectors in the vector space C^3 forms a two-dimensional surface in the space C^2. Let this two-dimensional surface be parameterized by the coordinates ξ_1 and ξ_2 where $x_1 = \xi_1^2 - \xi_2^2, x_2 = i\left(\xi_1^2 + \xi_2^2\right)$, $x_3 = -2\xi_1,\xi_2$. The two vectors (ξ_1,ξ_2) and $(-\xi_1, -\xi_2)$ in the space C^2 represent an isotropic vector in the complex vector space C^3. On the contrary, the same isotropic vector x corresponds to both of these vectors in space C^2. Thus, the two-dimensional complex vector $\xi = (\xi_1,\xi_2)$ described in this way is called spinor and can be shown by the column matrix

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}.$$

Moreover, Cartan expressed that the spinors are two-dimensional complex vectors and represent three-dimensional complex isotropic vectors [4]. Afterwards, Torres del Castillo and Barrales [7] said that the isotropic vector $a + ib$ can be represented by the spinor $\xi = (\xi_1,\xi_2)$ where $a, b \in R^3$. Moreover, if one considers the matrices

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

by the help of Pauli matrices, then the spinor equations of the vector triad a, b, c can be written $a + ib = \xi^t\sigma\xi$, $c = -\xi^t\sigma\xi$ where $a + ib$ is the isotropic vector in the space C^3 and $c \in R^3$. Here, the mate ξ of the spinor ξ is $\hat{\xi} = \begin{pmatrix} -\overline{\xi}_2 \\ \overline{\xi}_1 \end{pmatrix}$ [7]. So, one can see that the vectors a, b and c have the same length $\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \|\boldsymbol{c}\| = \overline{\xi}^t\xi$ and these vectors are mutually orthogonal. Spinors are needed if it is desired to code basic information about the topology of the rotations group since this rotation group is not simply connected. Thus, the spin group has two elements that representing it for each rotation. The relation between orthogonal basis and spinors mentioned above is two to one. That is to say that the spinors ξ and $-\xi$ represent the same orthogonal basis $\{a,b,c\}$ with $\langle a x b,c\rangle$. It should also be emphasized here that the triads $\{a,b,c\}$, $\{b,c,a\}$ and $\{c,a,b\}$ correspond to different spinors [7]. So, the following proposition can be given.

Proposition 2.1 Let ξ and ϕ be two arbitrary spinors. So, the following statements are held;

$$\begin{aligned} &i) \ \ \overline{\phi^t\sigma\xi} = -\hat{\phi}^t\sigma\hat{\xi}, \\ &ii) \ \ \lambda\phi + \mu\xi = \overline{\lambda}\hat{\phi} + \overline{\mu}\hat{\xi}, \\ &iii) \ \ \hat{\hat{\xi}} = -\xi, \\ &iv) \ \ \phi^t\sigma\xi = -\hat{\xi}\sigma\phi, \end{aligned} \tag{5}$$

where $\lambda, \mu \in C$ "¯" denotes the complex conjugate, "^{t}" denotes the transpose, and "ˆ" denotes the mate [7].

Moreover, in [7], the spinor equation of a unit speed curve was given as follows. Let $\alpha : I \to E^3$, $(I \subseteq R)$ be a curve parameterized by arc-length. Namely, $\|\alpha'(s)\| = 1$ and s is the arc-length parameter of the curve (α). In addition, the triad $\{n,b,t\}$ is considered as the Frenet frame of this curve and the spinor ξ corresponds to this Frenet frame of (α). Similar to the orthogonal triad $\{a,b,c\}$, the Frenet frame $\{n,b,t\}$ can be represented by a spinor ϕ as follows;

$$\begin{aligned} \boldsymbol{n} + i\boldsymbol{b} &= \phi^t\sigma\xi = \left(\phi_1^2 - \phi_2^2, i\left(\phi_1^2 + \phi_2^2\right), -2\phi_1\phi_2\right), \\ \boldsymbol{t} &= -\hat{\phi}^t\sigma\phi = \left(\phi_1\overline{\phi}_2 + \overline{\phi}_1\phi_2, i\left(\phi_1\overline{\phi}_2 - \overline{\phi}_1\phi_2\right), |\phi_1|^2 - |\phi_2|^2\right), \end{aligned} \tag{6}$$

where $\overline{\phi}^t\phi = 1$ [7]. Thus, one can give the following theorem.

Theorem 2.2 Let the spinor ϕ be represented the Frenet frame $\{n, b, t\}$ of the unit speed curve (α). So, the Frenet frame of this curve is characterized by a single spinor equation as

$$\frac{d\phi}{ds} = \frac{1}{2}\left(-i\tau\phi + \kappa\hat{\phi}\right),$$

where κ and τ are the curvature and torsion of the curve (α), respectively [7].

MAIN THEOREMS AND PROOFS

In this section, the spinor representation of the directional q -frame is given. Let $\alpha : I \subseteq R \to E^3$ be a regular and unit speed curve and the directional q -frame of this curve be $\{n_q, b_q, t\}$. Moreover, we consider that the spinor γ corresponds to this directional q -frame. So, similar to the equation (2.6), we can write the spinor equation of this frame

$$\begin{aligned} \boldsymbol{n_q} + i\boldsymbol{b_q} &= \gamma^t\sigma\gamma, \\ \boldsymbol{t} &= -\hat{\gamma}^t\sigma\gamma, \end{aligned} \tag{7}$$

where $n_q + ib_q \in C^3$ is an isotropic vector, $t \in R^3$, "^{t}" is the transpose, and "ˆ" is the mate. In addition to that, since the directional q -frame is an orthonormal frame, $\overline{\gamma}^t\gamma = 1$. Now, we write the directional q -curvatures in terms of a single spinor equation as follows.

Theorem 3.1 Let us consider that the spinor γ corresponds to the directional q -frame of the unit speed curve (α). So, the derivative of the spinor γ can be written in terms of the q -curvatures as follows

$$\frac{d\gamma}{ds}=\frac{-ik_3}{2}\gamma+\frac{k_1+ik_2}{2}\hat{\gamma},$$

where $\{k_1, k_2, k_3\}$ corresponds to the q -curvatures of the curve (α) and $\hat{\gamma}$ is the mate of spinor γ, respectively.

Proof. Let γ be a spinor corresponding to the directional q -frame of the unit speed curve (α). So, if the derivative of the second equation of (7) with respect to the arc-length parameter s is considered, then one can obtain

$$\frac{d\boldsymbol{n}_q}{ds}+\frac{d\boldsymbol{b}_q}{ds}=\frac{d\gamma^t}{ds}\sigma\gamma+\gamma^t\sigma\frac{d\gamma}{ds}.$$

On the other hand, if the reference [7] is taken into consideration, we can say that $\{\gamma,\hat{\gamma}\}$ is a basis for space of spinors with two complex components. In this case, there are two complex functions f and g that provide the equation $\frac{d\gamma}{ds}=f\gamma+g\hat{\gamma}$. So, if these last two expressions are used, we can write

$$\left(-k_1-ik_2\right)\boldsymbol{t}-ik_3\left(\boldsymbol{n}_q+i\boldsymbol{b}_q\right)=2f\left(\boldsymbol{n}_q+i\boldsymbol{b}_q\right)-2g\left(\boldsymbol{t}\right)$$

with the help of the equations (2) and (5). Thus, we obtain $f=\frac{-ik_3}{2}$, $g=\frac{k_1+ik_2}{2}$ and then

$$\frac{d\gamma}{ds}=\frac{-ik_3}{2}\gamma+\frac{k_1+ik_2}{2}\hat{\gamma}.$$

Now, we consider the Frenet curvatures $\{\kappa,\tau\}$ of the same curve α. Thus, with the help of equation (3), the following conclusion can be obtained.

Conclusion 3.2 Let γ be a spinor corresponding to the directional q -frame $\{n_q, b_q, t\}$ of the unit speed curve (α). So, the q -frame of this curve is characterized by a single spinor equation with the help of the Frenet curvatures $\{\kappa,\tau\}$ of the same curve as

$$\frac{d\gamma}{ds}=\frac{-i}{2}\left(\frac{d\theta}{ds}+\tau\right)\gamma+\frac{\kappa}{2}e^{-i\theta}\hat{\gamma}.$$

Proof. Consider that the spinor γ corresponds to the directional q -frame $\{n_q, b_q, t\}$, $\{k_1, k_2, k_3\}$ and $\{\kappa,\tau\}$ are the directional q -curvatures and the Frenet curvatures of the curve (α), respectively. So, we know that the relationship between these curvatures is given by equation (3). Thus, if we use the equation (3), we have

$$\frac{d\gamma}{ds}=\frac{-i}{2}\left(\frac{d\theta}{ds}+\tau\right)\gamma+\frac{\kappa}{2}\left(cos\theta-isin\theta\right)\hat{\gamma}$$

and finally

$$\frac{d\gamma}{ds}=\frac{-i}{2}\left(\frac{d\theta}{ds}+\tau\right)\gamma+\frac{\kappa}{2}e^{-i\theta}\hat{\gamma}.$$

Now, we write the spinor equations of the quasi-normal vector n_q and quasi-binormal vector b_q of the isotropic vector $n_q + ib_q$. So, the following conclusion can be given as a result of equation (7).

Conclusion 3.3 Let γ be a spinor corresponding to the directional q -frame of a unit speed curve (α). So, the quasi-normal vector n_q, quasi-binormal vector b_q in the isotropic vector $n_q + ib_q$, and the tangent vector t are written by

$$\begin{aligned}\boldsymbol{t}&=-\hat{\gamma}^t\sigma\gamma\\&=\left(\gamma_1\overline{\gamma}_2+\overline{\gamma}_1\gamma_2,i\left(\gamma_1\overline{\gamma}_2-\overline{\gamma}_1\gamma_2\right),|\gamma_1|^2-|\gamma_2|^2\right),\\ \boldsymbol{n}_q&=\frac{1}{2}\left(\gamma^t\sigma\gamma-\hat{\gamma}^t\sigma\hat{\gamma}\right)\\&=\frac{1}{2}\left(\gamma_1^2-\gamma_2^2+\overline{\gamma}_1^2-\overline{\gamma}_2^2,i\left(\gamma_1^2+\gamma_2^2-\overline{\gamma}_1^2-\overline{\gamma}_2^2\right),-2\left(\gamma_1\gamma_2+\overline{\gamma_1\gamma_2}\right)\right),\\ \boldsymbol{b}_q&=-\frac{i}{2}\left(\gamma^t\sigma\gamma+\hat{\gamma}^t\sigma\hat{\gamma}\right)\\&=-\frac{i}{2}\left(\gamma_1^2-\gamma_2^2-\overline{\gamma}_1^2+\overline{\gamma}_2^2,i\left(\gamma_1^2+\gamma_2^2+\overline{\xi}_1^2+\overline{\xi}_2^2\right),-2\left(\gamma_1\gamma_2-\overline{\gamma_1\gamma_2}\right)\right),\end{aligned}$$

where $\overline{\gamma^t\sigma\gamma}=-\hat{\gamma}^t\sigma\hat{\gamma}$, "$-$" is the complex conjugate, and "t" denotes the transpose.

Theorem 3.4 (Main Theorem) Let $\alpha : I \subseteq \mathbb{R}\rightarrow E^3$ be a unit speed curve and the Frenet frame and directional q -frame of this curve be $\{n, b, t\}$ and $\{n_q, b_q, t\}$. Moreover, the spinors ϕ and γ correspond to the Frenet frame and directional q -frame of this curve, respectively. So, the relation between these spinors is

$$\gamma^t\sigma\gamma=e^{-i\theta}\phi^t\sigma\phi \tag{8}$$

where θ is the Euclidean angle between the principal normal vector n and the quasi-normal vector n_q, and "t" denotes the transpose.

Proof. Suppose that the spinors γ and ϕ corresponds to the frames $\{n_q, b_q, t\}$ and $\{n, b, t\}$. So, from the equation (4), we know that

$$\begin{aligned}\boldsymbol{n}_q+i\boldsymbol{b}_q&=cos\theta\boldsymbol{n}+sin\theta\boldsymbol{b}+i\left(-sin\theta\boldsymbol{n}+cos\theta\boldsymbol{b}\right)\\&=\left(cos\theta-isin\theta\right)\boldsymbol{n}+i\left(sin\theta+icos\theta\right)\boldsymbol{b}\\&=e^{-i\theta}\left(\boldsymbol{n}+i\boldsymbol{b}\right).\end{aligned}$$

Then, if we use the equations (6) and (7), we have

$$\gamma^t\sigma\gamma=e^{-i\theta}\phi^t\sigma\phi.$$

In the light of Theorem 3.4, the following theorem related to the relation between the spinors γ and ϕ can be given.

Theorem 3.5 Let $\{n, b, t\}$ and $\{n_q, b_q, t\}$ be the Frenet frame and directional q -frame of a unit speed curve $\alpha : I \subseteq \mathbb{R} \to E^3$, ϕ and γ be the spinors corresponding to these frames, respectively. Then, the spinor γ is written in terms of the spinor ϕ as

$$\gamma = \pm e^{-i\theta/2}\phi,$$

where θ is the Euclidean angle between the principal normal vector n and the quasi-normal vector n_q.

Proof. Assume that θ and γ be denote spinors corresponding to the Frenet frame and directional q -frame. So, from the equations (6), (7), and (8), we obtain

$$\gamma_1^2 - \gamma_2^2 = e^{-i\theta}\left(\phi_1^2 - \phi_2^2\right),$$
$$\gamma_1^2 + \gamma_2^2 = e^{-i\theta}\left(\phi_1^2 + \phi_2^2\right),$$
$$\gamma_1\gamma_2 = e^{-i\theta}\phi_1\phi_2.$$

It is known from Section 2.1 that the spinors γ (or ϕ) and $-\gamma$ (or $-\phi$) correspond to the same ordered orthonormal basis. So, we can write

$$\gamma_1 = \pm e^{-i\theta/2}\phi_1, \quad \gamma_2 = \pm e^{-i\theta/2}\phi_2.$$

So, we obtain

$$\gamma = \pm e^{-i\theta/2}\phi.$$

Thus, the proof is completed.

Thus, the following geometric interpretation can be given as a result of Theorem 3.5.

Conclusion 3.6 Let the spinors ϕ and γ correspond to the Frenet frame and directional q -frame of the curve $\alpha : I \to E^3$, respectively. If θ is the angle between the normal vector n and the quasi-normal vector n_q, then the angle between the spinors ϕ and γ is $\frac{\theta}{2}$.

We know that the vector k is a projection vector that can be chosen as $k_x = (1,0,0)$, $k_y = (0,1,0)$ or $k_z = (0,0,1)$. If we consider Vivarelli's study [21], we can write the spinor representation of the projection vector k. We consider the spinors ε_x, ε_y, ε_z corresponding to the projection vector k (k_x, k_y, k_z) as follows

$$i)\quad \varepsilon_x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{for} \quad \boldsymbol{k_x} = (1,0,0),$$

$$ii)\quad \varepsilon_y = \begin{bmatrix} 0 \\ i \end{bmatrix} \quad \text{for} \quad \boldsymbol{k_y} = (0,1,0),$$

$$iii)\quad \varepsilon_z = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{for} \quad \boldsymbol{k_z} = (0,0,1).$$

Moreover, these spinors ε_x, ε_y, ε_z correspond to the first columns of the Pauli matrices

$$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ P_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let us examine these three situations separately. From equation (2.1), we know that $\boldsymbol{n_q} = \frac{\boldsymbol{t} \wedge \boldsymbol{k}}{\|\boldsymbol{t} \wedge \boldsymbol{k}\|}$.

Conclusion 3.7 Let $\alpha : I \subseteq \mathbb{R} \to E^3$ be a unit speed curve and the directional q -frame of this curve be $\{n_q, b_q, t\}$. Moreover, the spinor $\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$ corresponds to the q -frame of this curve where $\gamma_1, \gamma_2 \notin \mathbb{C}$, and "$^{-}$" is the complex conjugate. So, the following situations are valid.

i) For $\varepsilon_x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\gamma_1^2 - \gamma_2^2 + \overline{\gamma}_1^2 - \overline{\gamma}_2^2 = 0$. So, the real part of the spinor equation $\gamma_1^2 - \gamma_2^2$ is zero.

ii) For $\varepsilon_y = \begin{bmatrix} 0 \\ i \end{bmatrix}$, $\gamma_1^2 + \gamma_2^2 - \overline{\gamma}_1^2 - \overline{\gamma}_2^2 = 0$. Thus, the imaginary part of the spinor equation $\gamma_1^2 + \gamma_2^2$ is zero. Namely, $\gamma_1^2 + \gamma_2^2 \in \mathbb{R}$

iii) For $\varepsilon_z = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\gamma_1\gamma_2 + \overline{\gamma}_1\overline{\gamma}_2 = 0$. Thus, the real part of the spinor equation $\gamma_1\gamma_2$ is zero.

CONCLUSION

Spinors are frequently used in physics, quantum physics of fermions, general relativity, and abstract areas of algebra and geometry. According to physicists, spinors are very linear transformations. Thanks to these properties, spinors are somehow mathematical entities like tensors, allowing the concept of invariance under rotation and Lorentz boosts to be considered in a more general way. In this paper, we have approached spinors from a geometric perspective and considered spinors as two-dimensional vectors in the complex plane. Regardless of a specific application in geometry, the most crucial feature of spinors is their behavior under rotations. That is, when a vector or tensor object rotates by a certain angle, a spinor corresponding to this object rotates half that angle. Therefore, for this object to return to its original position, the spinor must rotate two full turns. Starting from this, geometric interpretations of the angle between these spinors have been made by corresponding one spinor each to the Frenet frame and the directional q-frame in three-dimensional Euclidean space. Therefore, in this study, it is thought that by using spinors, a new perspective will be gained in the fields of physics, algebra, and geometry with this study.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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