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Research Article

On the matrix representation of Bezier curves and derivatives in E^3

Şeyda KILIÇOĞLU[1,*], Süleyman ŞENYURT[2]

[1]Department of Mathematics, Başkent University, Ankara, 06790, Türkiye
[2]Department of Mathematics, Ordu University, Ordu, 52200, Türkiye

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ABSTRACT

In this study we have examined, the coefficient matrix of a cubic, 4^{th} order and n^{th} order Bezier curves using combinations as the elements to get a pattern. Also their first, second, third derivativies are examined based on the control points, in matrix represation in E^3. Further as a simple way has been given to find the equation of a Bezier curves and its derivatives using matrix product, based on the control points.

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INTRODUCTION

A Bézier curve is frequently used in computer graphics and related fields, in vector graphics, used in animation as a tool to control motion see generally [5] and [3]. We have been motivated by the following studies generally [7, 8, 11-13] wich are related in many properties of curves, Bezier curves and surfaces. In [4] there is a new approach to design the ruled surface. Cubic bezier-like transition curves are examined in [1], also we have examined Frenet apparatus and involute of the cubic Bézier curves in [9, 10] respectively.

A Bézier curve is defined by a set of control points P_0 through P_n, where n is called its order. If $n = 1$ for linear, if $n = 2$ for quadratic, if $n = 3$ for cubic Bézier curve, etc. The first and last control points are always the end points of the curve; however, the intermediate control points (if any) generally do not lie on the curve. Generaly Béziers curve can be defined by $n + 1$ control points $P_0, P_1,..., P_n$ and has the following form:

Definition 1.1 *The points P_I are called control points for the Bézier curve. The polygon formed by connecting the Bézier points with lines, starting with P_0 and finishing with P_n, is called the Bézier polygon (or control polygon). The convex hull of the Bézier polygon contains the Bézier curve. Bézier curve with $n + 1$ control points $P_0, P_1,..., P_n$ has the following equation [6], [11], [2]:*

$$\mathbf{B}(t)=\sum_{I=0}^{n}\binom{n}{I}t^{I}(1-t)^{n-I}(t)[P_I],\quad t\in[0,1]$$

$$\mathbf{B}(t)=\sum_{I=0}^{n}B_{n,I}(t)[P_I]$$

*Corresponding author.
*E-mail address: seyda@baskent.edu.tr

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where $\mathbf{B}_{n,I}(t)=\binom{n}{I}t^I(1-t)^{n-I}$ and $\binom{n}{I}$ are the binomial coefficients, also expressed as ${}^{n}C_I$ or C_I^n is $\binom{n}{I}=\frac{n!}{I!(n-I)!}$.

Theorem 1.1 *The derivatives of the any Bézier curve* $\mathbf{B}(t)$ *is*

$$\mathbf{B}'(t)=\sum_{i=0}^{n-1}\binom{n-1}{i}t^i(1-t)^{n-i-1}Q_i$$

where $Q_0=n(P_1-P_0), Q_1=n(P_2-P_1), Q_2=n(P_3-P_2),\ldots,Q_n=n(P_{i+1}-P_i)$.

Proof. Given points P_0 and P_1, a linear Bézier curve is simply represents a straight line between those two points. It is also known as a homotopy equation with homotopy parameter t, and it is widely used in the homotopy perturbation method. The curve is defined by

$$\alpha(t)=(1-t)P_0+tP_1$$

and also it has the matrix form with control points P_0 and P_1

$$\alpha(t)=\begin{bmatrix}t & 1\end{bmatrix}\begin{bmatrix}-1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}P_0\\ P_1\end{bmatrix}.$$

A quadratic Bézier curve is the path traced by the function $\alpha(t)$, given points P_0, P_1 and P_2, which can be interpreted as the linear interpolant of corresponding points on the linear Bézier curves from P_0 to P_1 and from P_1 to P_2 respectively.and also a quadratic Bézier curve has the matrix form with control points P_0, P_1 and P_2

$$\alpha(t)=\begin{bmatrix}t^2 & t & 1\end{bmatrix}\begin{bmatrix}\binom{2}{0}\binom{2}{2} & -\binom{2}{1}\binom{1}{1} & \binom{2}{2}\\ -\binom{2}{0}\binom{2}{1} & \binom{2}{1}\binom{1}{0} & 0\\ \binom{2}{0}\binom{2}{0} & 0 & 0\end{bmatrix}\begin{bmatrix}P_0\\ P_1\\ P_2\end{bmatrix}.$$

Four points P_0, P_1, P_2, P_3, and P_4 in the plane or in higher-dimensional space define a cubic Bézier curve with the following equation

$$\alpha(t)=(1-t)^3P_0+3t(1-t)^2P_1+3t^2(1-t)P_2+t^3P_3.$$

Those curve starts at P_0 going toward P_1 and arrives at P_3 coming from the direction of P_2. Usually, it will not pass through P_1 or P_2; these points are only there to provide directional information. The distance between P_0 and P_1 determines "how long" the curve moves into direction P_2 before turning towards P_3.

MATRIX REPRESENTATION OF BEZIER CURVES AND DERIVATIVE

We have already examine in cubic Bézier curves and involutes in [9] and [10], respectively. In this section we will use the coefficients expressed as combination $C_I^n=\binom{n}{I}$, to get a pattern.

Theorem 2.1 *The matrix form of the cubic Bezier curve with control points* P_0, P_1, P_2, *and* P_3 *is*

$$\alpha(t)=\begin{bmatrix}t^3 & t^2 & t & 1\end{bmatrix}\begin{bmatrix}-\binom{3}{0}\binom{3}{3} & \binom{3}{1}\binom{2}{2} & -\binom{3}{2}\binom{1}{1} & \binom{3}{3}\\ \binom{3}{0}\binom{3}{2} & -\binom{3}{1}\binom{2}{1} & \binom{3}{2}\binom{1}{0} & 0\\ -\binom{3}{0}\binom{3}{1} & \binom{3}{1}\binom{2}{0} & 0 & 0\\ \binom{3}{0}\binom{3}{0} & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}P_0\\ P_1\\ P_2\\ P_3\end{bmatrix}.$$

Proof. Since the cubic Bezier curve has the following equation

$$\alpha(t)=\binom{3}{0}(1-t)^3P_0+\binom{3}{1}t(1-t)^2P_1+\binom{3}{2}t^2(1-t)P_2+\binom{3}{3}t^3P_3$$

we have the proof.

Theorem 2.2 *The matrix form of the first derivative of a cubic Bezier curve based on the control points* P_0, P_1, P_2, *and* P_3 *is*

$$\alpha'(t)=\begin{bmatrix}t^2 & t & 1\end{bmatrix}\begin{bmatrix}-3\binom{3}{0}\binom{3}{3} & 3\binom{3}{1}\binom{2}{2} & -3\binom{3}{2}\binom{1}{1} & 3\binom{3}{3}\\ 2\binom{3}{0}\binom{3}{2} & -2\binom{3}{1}\binom{2}{1} & 2\binom{3}{2}\binom{1}{0} & 0\\ -\binom{3}{0}\binom{3}{1} & \binom{3}{1}\binom{2}{0} & 0 & 0\end{bmatrix}\begin{bmatrix}P_0\\ P_1\\ P_2\\ P_3\end{bmatrix}.$$

The first derivative of a cubic Bezier curve is a quadratic Bezier curve with control points $Q_0=3(P_1-P_0)$, $Q_1=3(P_2-P_1)$, and $Q_2=3(P_3-P_2)$ and

$$\alpha'(t)=\begin{bmatrix}t^2 & t & 1\end{bmatrix}\begin{bmatrix}\binom{2}{0}\binom{2}{2} & -\binom{2}{1}\binom{1}{1} & \binom{2}{2}\\ -\binom{2}{0}\binom{2}{1} & \binom{2}{1}\binom{1}{0} & 0\\ \binom{2}{0}\binom{2}{0} & 0 & 0\end{bmatrix}\begin{bmatrix}3(P_1-P_0)\\ 3(P_2-P_1)\\ 3(P_3-P_2)\end{bmatrix}.$$

Proof. Since the first derivative of the cubic Bezier curve has the following equation

$$\alpha'(t) = -3\binom{3}{0}\binom{3}{3}t^2[P_0] + 2\binom{3}{0}\binom{3}{2}t[P_0] - \binom{3}{0}\binom{3}{1}[P_0] + 3\binom{3}{1}\binom{2}{2}t^2[P_1]$$

$$-2\binom{3}{1}\binom{2}{1}t[P_1] + \binom{3}{1}\binom{2}{0}[P_1] + \left(-3\binom{3}{2}\binom{1}{1}t^2[P_2] + 2\binom{3}{2}\binom{1}{0}t[P_2]\right)$$

$$+3\binom{3}{3}t^2[P_3]$$

we get the first part of the proof. Also, we can write

$$\alpha'(t) = \begin{bmatrix}\binom{2}{0}(1-t)^2 & \binom{2}{1}t(1-t) & \binom{2}{2}t^2\end{bmatrix}\begin{bmatrix}3(P_1-P_0)\\ 3(P_2-P_1)\\ 3(P_3-P_2)\end{bmatrix},$$

$$\alpha'(t) = \begin{bmatrix}t^2 & t & 1\end{bmatrix}\begin{bmatrix}\binom{2}{0}\binom{2}{2} & -\binom{2}{1}\binom{1}{1} & \binom{2}{2}\\ -\binom{2}{0}\binom{2}{1} & \binom{2}{1}\binom{1}{0} & 0\\ \binom{2}{0}\binom{2}{0} & 0 & 0\end{bmatrix}\begin{bmatrix}Q_0\\ Q_1\\ Q_2\end{bmatrix}$$

with control points $Q_0 = 3(P_1 - P_0)$, $Q_1 = 3(P_2 - P_1)$, and $Q_2 = 3(P_3 - P_2)$.

Theorem 2.3 *The matrix form of the second derivative of a cubic Bezier curve based on the control points P_0, P_1, P_2, and P_3 is*

$$\alpha''(t) = \begin{bmatrix}t & 1\end{bmatrix}\begin{bmatrix}-6\binom{3}{0}\binom{3}{3} & 6\binom{3}{1}\binom{2}{2} & -6\binom{3}{2}\binom{1}{1} & 6\binom{3}{3}\\ 2\binom{3}{0}\binom{3}{2} & -2\binom{3}{1}\binom{2}{1} & 2\binom{3}{2}\binom{1}{0} & 0\end{bmatrix}\begin{bmatrix}P_0\\ P_1\\ P_2\\ P_3\end{bmatrix}$$

The second derivative of a cubic Bezier curve is a linear Bezier curve with control points $6(P_2 - 2P_1 + P_0)$, and $6(P_3 - 2P_2 + P_1)$.

$$\alpha''(t) = \begin{bmatrix}t & 1\end{bmatrix}\begin{bmatrix}-1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}6(P_2 - 2P_1 + P_0)\\ 6(P_3 - 2P_2 + P_1)\end{bmatrix}.$$

4TH ORDER BEZIER CURVE AND ITS DERIVATIVES

Definition 3.1 *Five points P_0, P_1, P_2, P_3, and P_4 in the plane or in higher-dimensional space define a 4th order Bézier curve with the following equation*

$$\alpha(t) = \sum_{I=0}^{4}\binom{4}{I}t^I(1-t)^{4-I}(t)[P_I], \quad t \in [0,1].$$

Theorem 3.1 *Let α be a 4th order Bézier curve with control points P_0, P_1, P_2, P_3, and P_4. The matrix form of the 4th order Bezier curve based on the control points is*

$$\alpha(t) = \begin{bmatrix}t^4 & t^3 & t^2 & t & 1\end{bmatrix}\begin{bmatrix}\binom{4}{0}\binom{4}{4} & -\binom{4}{1}\binom{3}{3} & \binom{4}{2}\binom{2}{2} & -\binom{4}{3}\binom{1}{1} & \binom{4}{4}\binom{0}{0}\\ -\binom{4}{0}\binom{4}{3} & \binom{4}{1}\binom{3}{2} & -\binom{4}{2}\binom{2}{1} & \binom{4}{3}\binom{1}{0} & 0\\ \binom{4}{0}\binom{4}{2} & -\binom{4}{1}\binom{3}{1} & \binom{4}{2}\binom{2}{0} & 0 & 0\\ -\binom{4}{0}\binom{4}{1} & \binom{4}{1}\binom{3}{0} & 0 & 0 & 0\\ \binom{4}{0}\binom{4}{0} & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}P_0\\ P_1\\ P_2\\ P_3\\ P_4\end{bmatrix}.$$

Proof. 4th order Bezier curve has the following equation

$$\alpha(t) = \binom{4}{0}\binom{4}{4}t^4[P_0] - \binom{4}{0}\binom{4}{3}t^3[P_0] + \binom{4}{0}\binom{4}{2}t^2[P_0] - \binom{4}{0}\binom{4}{1}t[P_0]$$

$$+\binom{4}{0}\binom{4}{0}[P_0] - \binom{4}{1}\binom{3}{3}t^4[P_1] + \binom{4}{1}\binom{3}{2}t^3[P_1] - \binom{4}{1}\binom{3}{1}t^2[P_1]$$

$$+\binom{4}{1}\binom{3}{0}t[P_1] + \binom{4}{2}\binom{2}{2}t^4[P_2] - \binom{4}{2}\binom{2}{1}t^3[P_2] + \binom{4}{2}\binom{2}{0}t^2[P_2]$$

$$-\binom{4}{3}\binom{1}{1}t^4[P_3] + \binom{4}{3}\binom{1}{0}t^3[P_3] + \binom{4}{4}t^4[P_4].$$

Hence it is easy to compute that

$$\alpha(t) = \begin{bmatrix}t^4 & t^3 & t^2 & t & 1\end{bmatrix}\begin{bmatrix}\binom{4}{0}\binom{4}{4} & -\binom{4}{1}\binom{3}{3} & \binom{4}{2}\binom{2}{2} & -\binom{4}{3}\binom{1}{1} & \binom{4}{4}\\ -\binom{4}{0}\binom{4}{3} & \binom{4}{1}\binom{3}{2} & -\binom{4}{2}\binom{2}{1} & \binom{4}{3}\binom{1}{0} & 0\\ \binom{4}{0}\binom{4}{2} & -\binom{4}{1}\binom{3}{1} & \binom{4}{2}\binom{2}{0} & 0 & 0\\ -\binom{4}{0}\binom{4}{1} & \binom{4}{1}\binom{3}{0} & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}P_0\\ P_1\\ P_2\\ P_3\\ P_4\end{bmatrix}. \quad (1)$$

Corollary 3.1 *The 4th order Bezier curve has the following matrix representation*

$$\alpha(t) = \begin{bmatrix}t^4 & t^3 & t^2 & t & 1\end{bmatrix}\begin{bmatrix}1 & -4 & 6 & -4 & 1\\ -4 & 12 & -12 & 4 & 0\\ 6 & -12 & 6 & 0 & 0\\ -4 & 4 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}P_0\\ P_1\\ P_2\\ P_3\\ P_4\end{bmatrix}$$

and it's inverse matrix is

$$\begin{bmatrix}1 & -4 & 6 & -4 & 1\\ -4 & 12 & -12 & 4 & 0\\ 6 & -12 & 6 & 0 & 0\\ -4 & 4 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\end{bmatrix}^{-1} = \begin{bmatrix}0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & \frac{1}{4} & 1\\ 0 & 0 & \frac{1}{6} & \frac{1}{2} & 1\\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1\\ 1 & 1 & 1 & 1 & 1\end{bmatrix}. \quad (2)$$

Proof. Since

$$\begin{bmatrix} \binom{4}{0}\binom{4}{4} & -\binom{4}{1}\binom{3}{3} & \binom{4}{2}\binom{2}{2} & -\binom{4}{3}\binom{1}{1} & \binom{4}{4} \\ -\binom{4}{0}\binom{4}{3} & \binom{4}{1}\binom{3}{2} & -\binom{4}{2}\binom{2}{1} & \binom{4}{3}\binom{1}{0} & 0 \\ \binom{4}{0}\binom{4}{2} & -\binom{4}{1}\binom{3}{1} & \binom{4}{2}\binom{2}{0} & 0 & 0 \\ -\binom{4}{0}\binom{4}{1} & \binom{4}{1}\binom{3}{0} & 0 & 0 & 0 \\ \binom{4}{0}\binom{4}{0} & 0 & 0 & 0 & 0 \end{bmatrix}_{5x5} = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

if find the inverse this complete the proof.

Theorem 3.2 *The matrix form of the first derivative of 4^{th} order Bezier curve based on the control points P_0, P_1, P_2, P_3, and P_4 is*

$$\alpha'(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 4\binom{4}{0}\binom{4}{4} & -4\binom{4}{1}\binom{3}{3} & 4\binom{4}{2}\binom{2}{2} & -4\binom{4}{3}\binom{1}{1} & 4\binom{4}{4}\binom{0}{0} \\ -3\binom{4}{0}\binom{4}{3} & 3\binom{4}{1}\binom{3}{2} & -3\binom{4}{2}\binom{2}{1} & 3\binom{4}{3}\binom{1}{0} & 0 \\ 2\binom{4}{0}\binom{4}{2} & -2\binom{4}{1}\binom{3}{1} & 2\binom{4}{2}\binom{2}{0} & 0 & 0 \\ -\binom{4}{0}\binom{4}{1} & \binom{4}{1}\binom{3}{0} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}.$$

The first derivative of 4^{th} *order* Bezier curve is a cubic Bezier curve with control points $R_0 = 4(P_1 - P_0)$, $R_1 = 4(P_2 - P_1)$, $R_2 = 4(P_3 - P_2)$, $R_3 = 4(P_3 - P_2)$. It has the following representation

$$\alpha'(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -\binom{3}{0}\binom{3}{3} & \binom{3}{1}\binom{2}{2} & -\binom{3}{2}\binom{1}{1} & \binom{3}{3} \\ \binom{3}{0}\binom{3}{2} & -\binom{3}{1}\binom{2}{1} & \binom{3}{2}\binom{1}{0} & 0 \\ -\binom{3}{0}\binom{3}{1} & \binom{3}{1}\binom{2}{0} & 0 & 0 \\ \binom{3}{0}\binom{3}{0} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4(P_1 - P_0) \\ 4(P_2 - P_1) \\ 4(P_3 - P_2) \\ 4(P_4 - P_3) \end{bmatrix}.$$

Proof. Since the first derivative of a 4^{th} *order* Bezier curve has the following equation is

$$\alpha'(t) = 4\binom{4}{0}\binom{4}{4}t^3[P_0] - 3\binom{4}{0}\binom{4}{3}t^2[P_0] + 2\binom{4}{0}\binom{4}{2}t[P_0] - \binom{4}{0}\binom{4}{1}[P_0]$$

$$-4\binom{4}{1}\binom{3}{3}t^3[P_1] + 3\binom{4}{1}\binom{3}{2}t^2[P_1] - 2\binom{4}{1}\binom{3}{1}t[P_1] + \binom{4}{1}\binom{3}{0}[P_1]$$

$$+\left(4\binom{4}{2}\binom{2}{2}t^3[P_2] - 3\binom{4}{2}\binom{2}{1}t^2[P_2] + 2\binom{4}{2}\binom{2}{0}t[P_2]\right)$$

$$-4\binom{4}{3}\binom{1}{1}t^3[P_3] + 3\binom{4}{3}\binom{1}{0}t^2[P_3] + 4\binom{4}{4}t^3[P_4]$$

it is easy to compute in matrix form

$$\alpha'(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 4\binom{4}{0}\binom{4}{4} & -4\binom{4}{1}\binom{3}{3} & 4\binom{4}{2}\binom{2}{2} & -4\binom{4}{3}\binom{1}{1} & 4\binom{4}{4} \\ -3\binom{4}{0}\binom{4}{3} & 3\binom{4}{1}\binom{3}{2} & -3\binom{4}{2}\binom{2}{1} & 3\binom{4}{3}\binom{1}{0} & 0 \\ 2\binom{4}{0}\binom{4}{2} & -2\binom{4}{1}\binom{3}{1} & 2\binom{4}{2}\binom{2}{0} & 0 & 0 \\ -\binom{4}{0}\binom{4}{1} & \binom{4}{1}\binom{3}{0} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}. \quad (3)$$

Since

$$\begin{bmatrix} 4\binom{4}{0}\binom{4}{4} & -4\binom{4}{1}\binom{3}{3} & 4\binom{4}{2}\binom{2}{2} & -4\binom{4}{3}\binom{1}{1} & 4\binom{4}{4} \\ -3\binom{4}{0}\binom{4}{3} & 3\binom{4}{1}\binom{3}{2} & -3\binom{4}{2}\binom{2}{1} & 3\binom{4}{3}\binom{1}{0} & 0 \\ 2\binom{4}{0}\binom{4}{2} & -2\binom{4}{1}\binom{3}{1} & 2\binom{4}{2}\binom{2}{0} & 0 & 0 \\ -\binom{4}{0}\binom{4}{1} & \binom{4}{1}\binom{3}{0} & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -16 & 14 & -16 & 4 \\ -12 & 36 & -36 & 12 & 0 \\ 12 & -24 & 12 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \end{bmatrix}$$

we have another represation as in the following way

$$\alpha'(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 4 & -16 & 24 & -16 & 4 \\ -12 & 36 & -36 & 12 & 0 \\ 12 & -24 & 12 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}. \quad (4)$$

Also since the derivative of 4^{th} *order* Bezier curve is

$$\alpha'(t) = n\left(\sum_{i=0}^{n-1}\binom{n-1}{i}t^i(1-t)^{n-1-i}[P_{i+1}] - \sum_{i=0}^{n-1}\binom{n-1}{i}t^i(1-t)^{n-1-i}[P_i]\right),$$

$$\alpha'(t) = \begin{bmatrix} \binom{3}{0}(1-t)^3 & \binom{3}{1}t(1-t)^2 & \binom{3}{2}t^2(1-t) & \binom{3}{3}t^3 \end{bmatrix} \begin{bmatrix} 4(P_1 - P_0) \\ 4(P_2 - P_1) \\ 4(P_3 - P_2) \\ 4(P_4 - P_3) \end{bmatrix}$$

with control points $Q_0 = 4(P_1 - P_0)$, $Q_1 = 4(P_2 - P_1)$, $Q_2 = 4(P_3 - P_2)$ and $Q_3 = 4(P_3 - P_2)$ we have the result;

$$\alpha'(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -\binom{3}{0}\binom{3}{3} & \binom{3}{1}\binom{2}{2} & -\binom{3}{2}\binom{1}{1} & \binom{3}{3} \\ \binom{3}{0}\binom{3}{2} & -\binom{3}{1}\binom{2}{1} & \binom{3}{2}\binom{1}{0} & 0 \\ -\binom{3}{0}\binom{3}{1} & \binom{3}{1}\binom{2}{0} & 0 & 0 \\ \binom{3}{0}\binom{3}{0} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4(P_1 - P_0) \\ 4(P_2 - P_1) \\ 4(P_3 - P_2) \\ 4(P_4 - P_3) \end{bmatrix}.$$

Corollary 3.2 *The first derivative of 4^{th} order Bezier curve with the control points P_0, P_1, P_2, P_3, and P_4, has* (4+1)−1 *times control points as* $n(P_i - P_{i-1})$, *for* i = 1,2,3,4.

Corollary 3.3 *The first derivative of n^{th} orderBezier curve with the control points P_0, P_1,..., P_n, has* $(n+1)-1$ *times control points as* $n(Pi - P_{i-1})$, *for* $i = 1,2,..., n$.

Theorem 3.3 *The matrix form of the second derivative of 4^{th} order Bezier curve based on the control points P_0, P_1, P_2, P_3, and P_4 is*

$$\alpha''(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 12\binom{4}{0}\binom{4}{4} & -12\binom{4}{1}\binom{3}{3} & 12\binom{4}{2}\binom{2}{2} & -12\binom{4}{3}\binom{1}{1} & 12\binom{4}{4}\binom{0}{0} \\ -6\binom{4}{0}\binom{4}{3} & 6\binom{4}{1}\binom{3}{2} & -6\binom{4}{2}\binom{2}{1} & 6\binom{4}{3}\binom{1}{0} & 0 \\ 2\binom{4}{0}\binom{4}{2} & -2\binom{4}{1}\binom{3}{1} & 2\binom{4}{2}\binom{2}{0} & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}.$$

The second derivative of *4th order* Bezier curve is a quadratic curve with control points S_0, S_1, and S_2, it has the following representation.

$$\alpha''(t)=\begin{bmatrix} t^2 & t & 1\end{bmatrix}\begin{bmatrix} 1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix} S_0\\ S_1\\ S_2\end{bmatrix}$$

where $S_0=12P_0-24P_1+12P_2$, $S_1=12P_1-24P_2+12P_3$, $S_2=12P_2-24P_3+12P_4$.

Proof. Since

$$\alpha'(t)=4\begin{pmatrix}\binom{3}{0}t^0(1-t)^3([P_1]-[P_0])+\binom{3}{1}t^1(1-t)^2([P_2]-[P_1])\\ +\binom{3}{2}t^2(1-t)([P_3]-[P_2])+\binom{3}{3}t^3([P_4]-[P_3])\end{pmatrix}$$

the second derivative can be calculated easily

$$\alpha''(t)=4\begin{pmatrix}-\binom{3}{0}3(1-t)^2(P_1-P_0)+\binom{3}{1}\left((1-t)^2-2t(1-t)\right)(P_2-P_1)\\ +\binom{3}{2}\left(2t(1-t)-t^2\right)(P_3-P_2)+\binom{3}{3}3t^2(P_4-P_3)\end{pmatrix}.$$

Hence it is easy to write its matrix form as in the following way

$$\alpha''(t)=\begin{bmatrix} t^2 & t & 1\end{bmatrix}\begin{bmatrix} 12 & -48 & 72 & -48 & 12\\ -24 & 72 & -72 & 24 & 0\\ 12 & -24 & 12 & 0 & 0\end{bmatrix}\begin{bmatrix} P_0\\ P_1\\ P_2\\ P_3\\ P_4\end{bmatrix}$$

$$=\begin{bmatrix} t^2 & t & 1\end{bmatrix}\begin{bmatrix} 1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 0 & 0\end{bmatrix}12\begin{bmatrix} 1 & -2 & 1 & 0 & 0\\ 0 & 1 & -2 & 1 & 0\\ 0 & 0 & 1 & -2 & 1\end{bmatrix}\begin{bmatrix} P_0\\ P_1\\ P_2\\ P_3\\ P_4\end{bmatrix}.$$

To find the control points S_0, S_1, S_2 of the second derivative of *4th order* Bezier curve we use the inerse matrix as in the following way

$$\begin{bmatrix} 0 & 0 & 1\\ 0 & \frac{1}{2} & 1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix} 1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 0 & 0\end{bmatrix}=\begin{bmatrix} 0 & 0 & 1\\ 0 & \frac{1}{2} & 1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix} 12 & -48 & 72 & -48 & 12\\ -24 & 72 & -72 & 24 & 0\\ 12 & -24 & 12 & 0 & 0\end{bmatrix}\begin{bmatrix} P_0\\ P_1\\ P_2\\ P_3\\ P_4\end{bmatrix}$$

hence

$$\alpha''(t)=\begin{bmatrix} t^2 & t & 1\end{bmatrix}\begin{bmatrix} 1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix} 12P_0-24P_1+12P_2\\ 12P_1-24P_2+12P_3\\ 12P_2-24P_3+12P_4\end{bmatrix}.$$

also this complete the proof that; $S_0=12P_0-24P_1+12P_2$, $S_1=12P_1-24P_2+12P_3$, $S_2=12P_2-24P_3+12P_4$ are the control points of the second derivative of *4th order*Bezier curve.

Corollary 3.4 *The second derivative of 4th order curve with the control points P_0, P_1, P_2, P_3, and P_4, has* (4+1)−2 *times control points as $n(n-1)(P_{i-1}-2P_i+P_{i+1})$ for i =* 1,2,3.

Corollary 3.5 *The second derivative of nth order curve with the control points P_0, P_1,..., P_n has* (n+1)−2 *times control points as $n(n-1)(P_{i-1}-2P_i+P_{i+1})$ for i =* 1,2,..., *$n-1$.*

Theorem 3.4 *The matrix form of the third derivative of 4th order Bezier curve based on the control points P_0, P_1, P_2, P_3, and P_4 is*

$$\alpha'''(t)=\begin{bmatrix} t & 1\end{bmatrix}\begin{bmatrix} 24\binom{4}{0}\binom{4}{4} & -24\binom{4}{1}\binom{3}{3} & 24\binom{4}{2}\binom{2}{2} & -24\binom{4}{3}\binom{1}{1} & 24\binom{4}{4}\binom{0}{0}\\ -6\binom{4}{0}\binom{4}{3} & 6\binom{4}{1}\binom{3}{2} & -6\binom{4}{2}\binom{2}{1} & 6\binom{4}{3}\binom{1}{0} & 0\end{bmatrix}\begin{bmatrix} P_0\\ P_1\\ P_2\\ P_3\\ P_4\end{bmatrix}.$$

The third derivative of *4th order* Bezier curve is a linear Bezier curve with control points T_0 and T_1. The matrix form of is

$$\alpha'''(t)=\begin{bmatrix} t & 1\end{bmatrix}\begin{bmatrix} -1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} T_0\\ T_1\end{bmatrix} \tag{5}$$

where $T_0=24(-P_0+3P_1-3P_2+P_3)$, $T_1=24(3P_2-P_1-3P_3+P_4)$ are the control points of the third derivative of *4th order* Bezier curve.

Proof. Since

$$\alpha''(t)=4\begin{pmatrix}-\binom{3}{0}3(1-t)^2(P_1-P_0)+\binom{3}{1}\left((1-t)^2-2t(1-t)\right)(P_2-P_1)\\ +\binom{3}{2}\left(2t(1-t)-t^2\right)(P_3-P_2)+\binom{3}{3}3t^2(P_4-P_3)\end{pmatrix}.$$

the third derivative can be calculated easily. Hence

$$\alpha'''(t)=\begin{bmatrix} t & 1\end{bmatrix}\begin{bmatrix} 24P_0-96P_1+144P_2-96P_3+24P_4\\ 72P_1-24P_0-72P_2+24P_3\end{bmatrix},$$

$$\alpha'''(t)=\begin{bmatrix} t & 1\end{bmatrix}\begin{bmatrix} -1 & 1\\ 1 & 0\end{bmatrix}24\begin{bmatrix} 3P_1-P_0-3P_2+P_3\\ 3P_2-P_1-3P_3+P_4\end{bmatrix},$$

$$\alpha'''(t)=\begin{bmatrix} t & 1\end{bmatrix}\begin{bmatrix} -1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} T_0\\ T_1\end{bmatrix}$$

this complete the proof.

nth order BEZIER CURVE AND ITS DERIVATIVE

Theorem 4.1 *The matrix representation of nth order Bézier curve with control points P_0, P_1, P_2, ..., and P_n is*

$$\alpha(t)=\begin{bmatrix} t^{n} & t^{n-1} & t^{n-2} & .... & t & 1\end{bmatrix}\left[noBc\right]_{(n+1)\times(n+1)}\begin{bmatrix} P_0 \\ P_1 \\ . \\ . \\ . \\ P_n \end{bmatrix}$$

where $\left[noBc\right]_{(n+1)\times(n+1)}$ is the coefficient matrix of n^{th} *order* of Bézier curve. "$\left[noBc\right]_{(n+1)\times(n+1)}$" is obtained by the initial letters of "Coefficient Matrix of n^{th} *order* of Bézier curve" , and it is

$$(-1)^{n}\begin{bmatrix}
\binom{n}{0}\binom{n}{n} & -\binom{n}{1}\binom{n-1}{n-1} & \binom{n}{2}\binom{n-2}{n-2} & . & . & . & \binom{n}{n-1}\binom{1}{1} & \binom{n}{n}\binom{0}{0} \\
-\binom{n}{0}\binom{n}{n-1} & \binom{n}{1}\binom{n-1}{n-2} & -\binom{n}{2}\binom{n-2}{n-3} & . & . & . & \binom{n}{n-1}\binom{1}{0} & 0 \\
\binom{n}{0}\binom{n}{n-2} & -\binom{n}{1}\binom{n-1}{n-3} & . & . & . & . & 0 & 0 \\
-\binom{n}{0}\binom{n}{n-3} & . & . & . & . & . & 0 & 0 \\
. & . & . & . & . & . & 0 & 0 \\
. & . & \binom{n}{2}\binom{n-2}{1} & \binom{n}{3}\binom{n-3}{0} & 0 & 0 & 0 & 0 \\
. & \binom{n}{1}\binom{n-1}{n-(n-1)} & \binom{n}{2}\binom{n-2}{0} & 0 & 0 & 0 & 0 & 0 \\
-\binom{n}{0}\binom{n}{1} & \binom{n}{1}\binom{n-1}{0} & 0 & 0 & 0 & 0 & 0 & 0 \\
\binom{n}{0}\binom{n}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Proof. If n is an even number, using combinations $C_I^n=\binom{n}{I}$, we get a pattern. Hence

$$\begin{bmatrix}
\binom{n}{0}\binom{n}{n} & -\binom{n}{1}\binom{n-1}{n-1} & \binom{n}{2}\binom{n-2}{n-2} & -\binom{n}{3}\binom{n-3}{n-3} & . & . & \binom{n}{n-1}\binom{1}{1} & \binom{n}{n}\binom{0}{0} \\
-\binom{n}{0}\binom{n}{n-1} & \binom{n}{1}\binom{n-1}{n-2} & -\binom{n}{2}\binom{n-2}{n-3} & \binom{n}{3}\binom{n-3}{n-4} & . & . & \binom{n}{n-1}\binom{1}{0} & 0 \\
\binom{n}{0}\binom{n}{n-2} & -\binom{n}{1}\binom{n-1}{n-3} & . & . & . & . & 0 & 0 \\
-\binom{n}{0}\binom{n}{n-3} & . & . & . & . & . & 0 & 0 \\
. & . & . & . & . & . & 0 & 0 \\
. & . & \binom{n}{2}\binom{n-2}{1} & \binom{n}{3}\binom{n-3}{0}. & 0 & 0 & 0 & 0 \\
. & \binom{n}{1}\binom{n-1}{1} & \binom{n}{2}\binom{n-2}{0} & 0 & 0 & 0 & 0 & 0 \\
-\binom{n}{0}\binom{n}{1} & \binom{n}{1}\binom{n-1}{0} & 0 & 0 & 0 & 0 & 0 & 0 \\
\binom{n}{0}\binom{n}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}_{(n+1)\times(n+1)}$$

If n is an odd number, using combinations $C_I^n=\binom{n}{I}$, we get a pattern. Hence

$$\begin{bmatrix}
-\binom{n}{0}\binom{n}{n} & \binom{n}{1}\binom{n-1}{n-1} & -\binom{n}{2}\binom{n-2}{n-2} & \binom{n}{3}\binom{n-3}{n-3} & . & . & \binom{n}{(n-1)}\binom{1}{1} & -\binom{n}{n}\binom{0}{0} \\
\binom{n}{0}\binom{n}{n-1} & -\binom{n}{1}\binom{n-1}{n-2} & \binom{n}{2}\binom{n-2}{n-3} & -\binom{n}{3}\binom{n-3}{n-4} & . & . & \binom{n}{(n-1)}\binom{1}{0} & 0 \\
-\binom{n}{0}\binom{n}{n-2} & \binom{n}{1}\binom{n-1}{n-3} & . & . & . & . & 0 & 0 \\
\binom{n}{0}\binom{n}{n-3} & . & . & . & . & . & 0 & 0 \\
. & . & . & . & . & . & 0 & 0 \\
. & . & \binom{n}{2}\binom{n-2}{1} & \binom{n}{3}\binom{n-3}{0}. & 0 & 0 & 0 & 0 \\
. & \binom{n}{1}\binom{n-1}{1} & -\binom{n}{2}\binom{n-2}{0} & 0 & 0 & 0 & 0 & 0 \\
\binom{n}{0}\binom{n}{n-(n-1)} & -\binom{n}{1}\binom{n-1}{0} & 0 & 0 & 0 & 0 & 0 & 0 \\
\binom{n}{0}\binom{n}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}_{(n+1)\times(n+1)}$$

Theorem 4.2 *The matrix representation of the first derivative of n^{th} order of a Bézier curve with control points P_0 , P_1, P_2, ..., and P_n is*

$$\alpha^{'}(t)=\begin{bmatrix} t^{n-1} & t^{n-2} & ... & t & 1\end{bmatrix}\left[noBc\right]^{'}_{n\times(n+1)}\begin{bmatrix} P_0 \\ P_1 \\ . \\ . \\ . \\ P_n \end{bmatrix}$$

where the coefficient matrix of the first derivative of n^{th} *order* of a Bézier curve $\left[noBc\right]^{'}_{n\times(n+1)}$ is

$$(-1)^{n}\begin{bmatrix}
n\binom{n}{0}\binom{n}{n} & -n\binom{n}{1}\binom{n-1}{n-1} & . & . & . & . & n\binom{n}{n}\binom{0}{0} \\
-(n-1)\binom{n}{0}\binom{n}{n-1} & (n-1)\binom{n}{1}\binom{n-1}{n-2} & . & . & . & . & 0 \\
(n-2)\binom{n}{0}\binom{n}{n-2} & -(n-2)\binom{n}{1}\binom{n-1}{n-3} & . & . & . & 0 & 0 \\
-(n-3)\binom{n}{0}\binom{n}{n-3} & . & . & . & . & 0 & 0 \\
. & . & . & . & . & 0 & 0 \\
. & . & . & \binom{n}{3} & 0 & 0 & 0 \\
2\binom{n}{0}\binom{n}{2}. & 2\binom{n}{1}\binom{n-1}{1} & 2\binom{n}{2} & 0 & 0 & 0 & 0 \\
-\binom{n}{0}\binom{n}{1} & \binom{n}{1}\binom{n-1}{0} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}_{n\times(n+1)}$$

Proof. Using combinations $C_I^n=\binom{n}{I}$, we get a pattern. Hence the proof is completed.

Theorem 4.3 *The matrix representation of the second derivative of n^{th} order of a Bézier curve with control points P_0 , P_1, P_2, ..., and P_n is*

$$\alpha^{''}(t)=\begin{bmatrix} t^{n-2} & t^{n-3} & ... & t & 1\end{bmatrix}\left[noBc\right]^{''}_{(n-1)\times(n+1)}\begin{bmatrix} P_0 \\ P_1 \\ . \\ . \\ . \\ P_n \end{bmatrix}_{n\times(n+1)}$$

where the coefficient matrix of the second derivative of n^{th} *order* of a Bézier curve $\left[noBc\right]^{''}_{(n-1)\times(n+1)}$ is

$$(-1)^{n}\begin{bmatrix}
n(n-1)\binom{n}{0}\binom{n}{n} & -n(n-1)\binom{n}{1}\binom{n-1}{n-1} & . & ... & n(n-1)\binom{n}{n-1} & n(n-1)\binom{n}{n}\binom{0}{0} \\
-(n-1)(n-2)\binom{n}{0}\binom{n}{n-1} & (n-1)(n-2)\binom{n}{1}\binom{n-1}{n-2} & . & ... & (n-1)(n-2)\binom{n}{n-1} & 0 \\
(n-2)(n-3)\binom{n}{0}\binom{n}{n-2} & -(n-2)(n-3)\binom{n}{1}\binom{n-1}{n-3} & . & ... & 0 & 0 \\
-(n-3)(n-4)\binom{n}{0}\binom{n}{n-3} & . & . & ... & 0 & 0 \\
. & . & . & ... & 0 & 0 \\
. & . & 2\binom{n}{2}\binom{n-2}{1} & ... & 0 & 0 \\
2\binom{n}{0}\binom{n}{2}. & 2\binom{n}{1}\binom{n-1}{1} & 2\binom{n}{2}\binom{n-2}{0} & 0 & 0 & 0
\end{bmatrix}$$

Proof. Using combinations $C_I^n = \binom{n}{I}$, we get a pattern. Hence the proof is completed.

CONCLUSION

In this study, we focus on the matrix representation of Bezier curves of any nth order. Because matrix representation makes it easy to write the equation of any order Bezier curve with given control points. For this reason, in this matrix representation, firstly the coefficients matrix corresponding to the 3rd and 4th order Bezier curves was examined. Further, the generalization of the matrix of coefficients corresponding to nth order Bezier curve was also calculated. Knowing this coefficients matrix has been made it easier for us to find the control points of a Bezier curve whose equation is given.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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