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Research Article

Wong-Zakai approximation for stochastic models of smoking

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ABSTRACT

In this study, Wong-Zakai approximation method has been used to obtain approximate solutions for two compartmental models of smoking dynamics. Stratonovich stochastic differential equation systems are obtained for these two stochastic models for the application of Wong-Zakai method. Wong-Zakai method is used together with the predictor-corrector deterministic approximation method where Adams-Bashforth method is used as the predictor pair and Adams-Moulton method is used as the corrector pair. Stochastic Runge-Kutta IV, Euler-Maruyama and stochastic Runge-Kutta strong order 1.0 schemes are also used to investigate the models and the results are compared to the results from Wong-Zakai approximation. The comparison shows that Wong-Zakai method is a reliable tool for the analysis of stochastic models and can be considered as an alternative investigation method for modeling studies. Solution graphs, error graphs and numerical results have been given as evidence to show that Wong-Zakai method can also be a reliable method for analyzing various models. An alternate technique for parallelizing the algorithm has also been given to decrease CPU times for Wong-Zakai method. This technique is suggested to overcome the extra calculation load that comes with Wong-Zakai method.

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INTRODUCTION

Mathematical modeling studies concentrate on an accurate description of real life events through the use of mathematical equations. Models are used in a wide range of applications from metastasis evolution to voter modeling [14, 16]. However, in literature, it is seen that most of the mathematical modeling studies contain deterministic analyses, meaning that the random nature of the event under consideration is neglected. Deterministic events produce the same results for every trial under the same conditions but it is known that some events in nature do not show deterministic behavior. These types of events, random events in particular, should not be modeled with deterministic equation systems if they are to be described accurately. The use of random and stochastic equation systems is more appropriate in such cases.

Smoking models in literature are often comprised of compartmental models. The SIR (susceptible-infected-recovered) model and its versions form the basis for many

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mathematical models just like the smoking models. Most of these smoking models are formed of deterministic compartmental models analyzing certain aspects of the transmission of this habit. Some of the recent studies include models with relapse classes [22], models for analyzing the effect of electronic cigarettes on smoking cessation [6], giving up smoking models with harmonic type incidence rate [13] and models on smoking dynamics with health education effect [20]. There are also many recent studies on the fractional modeling of smoking dynamics such as the analysis of a model with modified homotopy analysis transform method [18], a model with Atangana-Baleanu derivative [17], a fractional model with local and nonlocal kernel [10] and the analysis of a model with Laplace Adomian decomposition method [5]. It is seen that there are not so many random modeling studies for smoking models, with a study by Lahrouz et al. being the only prominent example for this case [8]. Although there are some stochastic studies, it is known that most of the recent works in the literature are concentrated on fractional calculus [24-35].

In this study, Wong-Zakai approximation will be used for analyzing mathematical models of smoking consisting of stochastic differential equations. Wong-Zakai method is based on Stratonovich stochastic integration which preserves some of the properties of deterministic integration. Most of the stochastic studies in literature use the popular stochastic approximation methods Euler-Maruyama and Runge-Kutta [7, 9, 11]. Hence, we propose this method as a new alternative for the analysis of smoking models. Wong-Zakai method has been recently used for the analysis of stochastic Landau-Lifschitz-Gilbert equations [2], stochastic reaction-diffusion equations [23], random Navier-Stokes equations [3] and stochastic heat equation [4]. The next section contains the basic formulas for the methods used for the analysis. The models and results for the models are given in sections 3 and 4. Finally the conclusion is given for the study.

MATERIALS AND METHODS

Wong-Zakai Approximation

Wong-Zakai method is based on obtaining deterministic approximations for each discretized time subinterval. A deterministic approximation method is used together with Wong-Zakai method to obtain the approximate solution of a stochastic differential equation (SDE).

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t \tag{1}$$

on the time interval $[t_0, T]$ with the initial condition $X_{t_0} = X_0$. The corresponding Stratonovich SDE can be given as:

$$dX_t = \underline{a}(t, X_t)dt + b(t, X_t) \circ dW_t, \tag{2}$$

where the modified drift coefficient $\underline{a}(t, X_t)dt$ can be obtained as [7]:

$$\underline{a}(t, X_t) = a(t, X_t) - \frac{1}{2} b(t, X_t) \frac{\partial b}{\partial X_t}(t, X_t). \tag{3}$$

The Ito SDE (1) is transformed into the Stratonovich SDE (2) to implement Wong-Zakai method. Let the interval $[t_0, T]$ be divided into k subintervals such as $t_0 < t_1 < \cdots < t_{k-1} < t_k = T$. Using this discretization, an ordinary differential equation is examined for each interval $[t_j, t_{j+1}]$ for $j = 0,1, \dots, k-1$. Here, the approximations $X_{t_{j+1}}$ are obtained from

$$\frac{d\hat{X}_t}{dt} = \underline{a}(t, \hat{X}_t) + \frac{1}{\Delta t_n} b(t, \hat{X}_t)\Delta W_n, \tag{4}$$

where $\hat{X}_{tn} = \hat{X}_j$, $j = 0,1, \dots, k-1$ and $\Delta W_n = W_{tj+1} - W_{tj}$ is the approximation for dW_t. The ordinary differential equations will be solved by predictor corrector method with Adams-Bashford method as the predictor method

$$X_{i+1}^* = X_i + \frac{h}{2}[3f(t_i, X_i) - f(t_{i-1}, X_{i-1})], \tag{5}$$

and Adams-Moulton method is used as the corrector method [12]:

$$X_{i+1} = X_i + \frac{h}{2}[f(t_i, X_i) + f(t_{i+1}, X_{i+1}^*)], i = 2,3, \dots, m. \tag{6}$$

where m is the number of subintervals and h is the step size.

Smoking Models

Two compartmental models will be used as numerical examples. First example is a stochastic model with three compartments, whereas the second model is a deterministic model containing three compartments with additional stochastic noise.

The stochastic model of smoking

The stochastic differential equation system is given as [8]:

$$\begin{aligned} dP &= [\mu - \mu P - \beta PS]dt + \sigma_1 P(P - P^*)dB_1, \\ dS &= [-(\mu + \gamma)S + \beta PS + \alpha Q_T]dt + \sigma_2 S(S - S^*)dB_2, \\ dQ_T &= [-(\mu + \alpha)Q_T + \gamma(1 - \sigma)S]dt + \sigma_3 Q_T(Q_T - Q_T^*)dB_3, \end{aligned} \tag{7}$$

where B_i, $i = 1, 2, 3$ are independent Brownian motions and σ_i, $i = 1, 2, 3$ are constants. Here, $P^* = \frac{1}{\Re_s}$, $S^* = \frac{\mu}{\beta}(\Re_s - 1)$ and $Q_T^* = \frac{\gamma(1-\sigma)}{\mu+\alpha} S^*$ with $\Re_s = \frac{\beta(\mu+\alpha)}{\mu(\mu+\alpha)+\gamma(\sigma\alpha+\mu)}$. Here, P denotes the fraction of the potential smokers who have a possibility of becoming smokers in the future, S denotes the fraction of smokers and Q_T denotes the fraction of smokers who temporarily quit smoking. Also, μ denotes the average duration of smoking activity $(1/\mu)$, γ denotes the rate of quitting smoking, α denotes the rate at which temporary quitters revert back to smoking, σ denotes the fraction of smokers who permanently quit smoking and β denotes contact rate [15].

The Stratonovich form of problem is given as

$$\begin{aligned} dP &= \left[\mu - \mu P - \beta PS - \frac{1}{2}\sigma_1^2 P(P - P^*)(2P - P^*)\right]dt + \sigma_1 P(P - P^*) \circ dB_1, \\ dS &= \left[-(\mu + \gamma)S + \beta PS + \alpha Q_T - \frac{1}{2}\sigma_2^2 S(S - S^*)(2S - S^*)\right]dt + \sigma_2 S(S - S^*) \circ dB_2, \\ dQ_T &= \left[-(\mu + \alpha)Q_T + \gamma(1 - \sigma)S - \frac{1}{2}\sigma_3^2 Q_T(Q_T - Q_T^*)(2Q_T - Q_T^*)\right]dt + \sigma_3 Q_T(Q_T - Q_T^*) \circ dB_3. \end{aligned} \tag{8}$$

The values for the parameters are obtained from literature [1] as: $\mu = 0.04$, $\gamma = 0.3$, $\alpha = 0.25$, $\sigma = 0.4$, $\beta = 0.5$. The diffuson coefficients are assumed as: $\sigma_1 = \sigma_2 = \sigma_3 = 0.5$, whereas the initial conditions are: $P(0) = 0.634$, $S(0) = 0.288$, $Q_T(0) = 0.048$. t denotes the number of months.

Theorem 1. Let $m_i(X, t)$, $i = 1,2,3$ for $X \in \{P, S, Q_T\}$ be the drift coefficients of the stochastic differential equation system (7) and $\sigma_i^*(X, t)$, $i = 1,2,3$ be the diffusion coefficients of the system (7), satisfying the following conditions

H_1: $\sigma_i^{*'}(X, t) = \frac{\partial \sigma_i^*(X,t)}{\partial X}$ is continuous in both $X \in \{P, S, Q_T\}$ and t.

H_2: $f(X, t)$ is continuous in t.

H_3: $|f(X, t) - f(X_0, t)| \leq K|X - X_0|$

where $f(X, t)$ stands for any of the quantities $\sigma_i^*(X, t)$, $\sigma_i^{*\prime}(X, t)\sigma_i^*(X, t)$ and $m_i(X, t)$, and K is a constant independent of t, X and X_0.

Furthermore, assume that the initial conditions $P(0)$, $S(0)$, $Q_T(0)$ satisfy $E[X^4] < \infty$. Then the sequence of solutions $X_t^{(n)}$ converges in mean to X_t as $n \to \infty$, where X_t is the unique solution of the stochastic differential equation

$$dX_t = m_i(X_t, t)dt + \frac{1}{2}\sigma_i^*(X_t, t)\sigma_i^{*\prime}(X_t, t)dt + \sigma_i^*(X_t, t)dY_t$$

[19].

The mathematical model of smoking with determination and education

The second mathematical model is a deterministic model for analyzing smoking dynamics with determination and education [21]. The system is added stochastic noise to obtain a stochastic model which can be given as:

$$\begin{aligned} dP &= [\mu - \beta PS + \alpha(1-\epsilon)S - \mu P - \delta P]dt + P(\sigma_4 P - \sigma_5)dB_1, \\ dS &= [\beta PS - \mu S - \alpha S]dt + S(\sigma_6 S - \sigma_7)dB_2, \\ dQ &= [\alpha\epsilon S - \mu Q + \delta P]dt + Q(\sigma_8 Q - \sigma_9)dB_3, \end{aligned} \tag{9}$$

where B_i, $i = 1, 2, 3$ are independent Brownian motions and $\sigma_i = 0.3$, $i = \overline{4,9}$ are constants. Here, $P(t)$ denotes the fraction of potential smokers, $S(t)$ denotes the fraction of smokers and $Q(t)$ denotes the fraction of quitters. The parameters used in this model are given as follows: μ denotes the inflow rate of individuals into compartment P and the natural death rate in each compartment, β denotes the rate of transmission of smoking habit, α denotes the quitting rate from smoking, ϵ denotes the measure of determination and δ denotes the rate of individuals in compartment P moving to compartment Q due to education.

The Stratonovich form of problem is given as

$$\begin{aligned} dP &= \left[\mu - \beta PS + \alpha(1-\epsilon)S - \mu P - \delta P - \frac{1}{2}P(\sigma_4 P - \sigma_5)(2\sigma_4 P - \sigma_5)\right]dt + P(\sigma_4 P - \sigma_5) \circ dB_1, \\ dS &= \left[\beta PS - \mu S - \alpha S - \frac{1}{2}S(\sigma_6 S - \sigma_7)(2\sigma_6 S - \sigma_7)\right]dt + S(\sigma_6 S - \sigma_7) \circ dB_2, \\ dQ &= \left[\alpha\epsilon S - \mu Q + \delta P - \frac{1}{2}Q(\sigma_8 Q - \sigma_9)(2\sigma_8 Q - \sigma_9)\right]dt + Q(\sigma_8 Q - \sigma_9) \circ dB_3. \end{aligned} \tag{10}$$

Parameters are used with the following values (obtained from the referred study [21]: $\mu = 0.02$, $\epsilon = 0.2$, $\alpha = 0.05$, $\sigma = 0.4$, $\beta = 0.4$, $\delta = 0.06$. Initial conditions are given as $P(0) = 0.6$, $S(0) = 0.3$, $Q(0) = 0.1$. It should be noted that the diffusion coefficients are assumed to be $\sigma_i = 0.3$, $i = \overline{4,9}$.

Similar to Theorem 1, the numerical solution of (10) is convergent to the unique solution of the stochastic differential equation system.

RESULTS

Wong-Zakai method is compared with the other stochastic methods Euler-Maruyama, Runge-Kutta 1-0 strong order and stochastic Runge-Kutta IV methods as follows

Results for the Stochastic Model (8)

The stochastic results for model (8) are given for the parameters in the figures below (Figures 1,2,3).

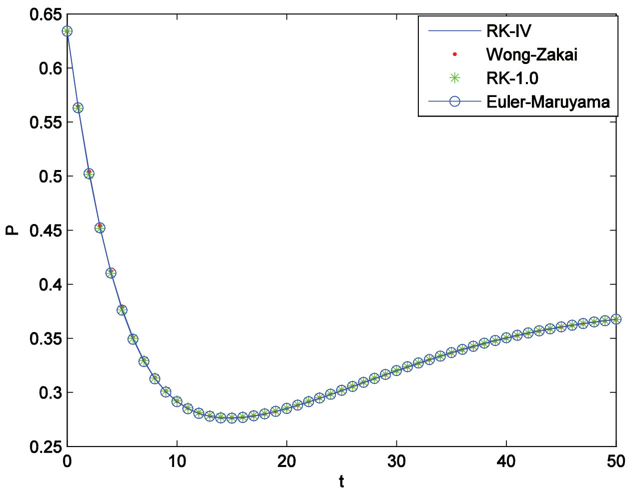

Figure 1. Results for the compartment $P(t)$ in model (8).

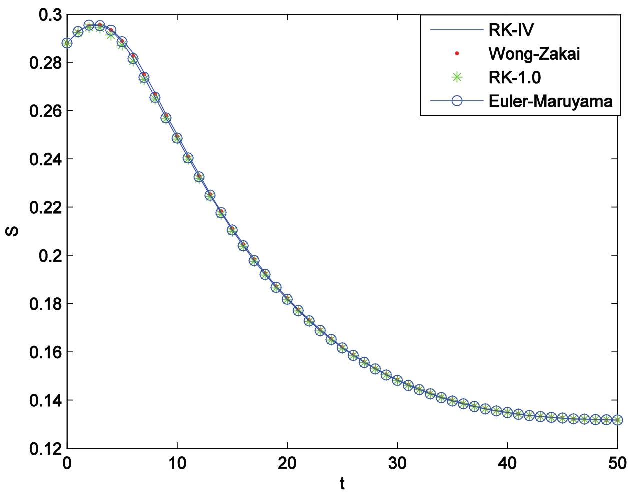

Figure 2. Results for the compartment $S(t)$ in model (8).

The results for all of the methods are given in the table below (Table 1).

The relative errors (relative to the solutions of stochastic Runge-Kutta IV) at several points of the process can be shown as below (Table 2).

The relative errors throughout the process are shown for the compartments below Figures (4,5,6).

The relative errors have been computed in relevance to the stochastic Runge-Kutta method of order IV since it is the highest ordered method within the scope of the study.

Table 1. Results from the methods for model (8)

t	RK-IV			Euler-Maruyama			RK-1.0			WZ		
	$E(P(t))$	$E(S(t))$	$E(Q_T(t))$	$E(P(t))$	$E(S(t))$	$E(Q_T(t))$	$E(P(t))$	$E(S(t))$	$E(Q_T(t))$	$E(P(t))$	$E(S(t))$	$E(Q_T(t))$
0.0000	0.6340	0.2880	0.0480	0.6340	0.2880	0.0480	0.6340	0.2880	0.0480	0.6340	0.2880	0.0480
1.0000	0.5642	0.2928	0.0813	0.5631	0.2927	0.0815	0.5627	0.2923	0.0813	0.5644	0.2926	0.0813
2.0000	0.5048	0.2955	0.1068	0.5023	0.2954	0.1072	0.5013	0.2945	0.1067	0.5043	0.2952	0.1068
3.0000	0.4531	0.2959	0.1262	0.4522	0.2952	0.1265	0.4513	0.2943	0.1259	0.4543	0.2956	0.1261
4.0000	0.4111	0.2939	0.1405	0.4103	0.2933	0.1407	0.4099	0.2913	0.1399	0.4122	0.2934	0.1404
5.0000	0.3775	0.2893	0.1507	0.3760	0.2885	0.1508	0.3764	0.2871	0.1499	0.3785	0.2888	0.1505
6.0000	0.3502	0.2839	0.1575	0.3491	0.2816	0.1574	0.3499	0.2805	0.1566	0.3509	0.2826	0.1573
7.0000	0.3292	0.2757	0.1616	0.3285	0.2738	0.1612	0.3293	0.2729	0.1603	0.3298	0.2750	0.1613
8.0000	0.3129	0.2674	0.1633	0.3125	0.2655	0.1627	0.3133	0.2648	0.1619	0.3135	0.2669	0.1630
9.0000	0.3006	0.2584	0.1632	0.3003	0.2570	0.1625	0.3012	0.2566	0.1618	0.3011	0.2581	0.1630
10.0000	0.2916	0.2501	0.1618	0.2914	0.2486	0.1611	0.2922	0.2482	0.1604	0.2919	0.2494	0.1617
11.0000	0.2850	0.2416	0.1594	0.2849	0.2404	0.1588	0.2858	0.2399	0.1582	0.2854	0.2409	0.1591
12.0000	0.2805	0.2336	0.1562	0.2805	0.2325	0.1557	0.2814	0.2318	0.1552	0.2809	0.2329	0.1561
13.0000	0.2777	0.2255	0.1526	0.2778	0.2249	0.1522	0.2786	0.2243	0.1517	0.2781	0.2254	0.1526
14.0000	0.2763	0.2181	0.1488	0.2763	0.2177	0.1484	0.2772	0.2170	0.1480	0.2768	0.2183	0.1489
15.0000	0.2761	0.2112	0.1448	0.2761	0.2105	0.1445	0.2768	0.2100	0.1441	0.2764	0.2112	0.1449
16.0000	0.2766	0.2047	0.1409	0.2768	0.2040	0.1405	0.2774	0.2036	0.1402	0.2769	0.2046	0.1408
17.0000	0.2780	0.1986	0.1369	0.2782	0.1978	0.1365	0.2787	0.1975	0.1361	0.2782	0.1984	0.1368
18.0000	0.2798	0.1927	0.1330	0.2801	0.1921	0.1325	0.2806	0.1918	0.1323	0.2800	0.1926	0.1329
19.0000	0.2823	0.1873	0.1292	0.2824	0.1867	0.1288	0.2829	0.1865	0.1285	0.2823	0.1872	0.1290
20.0000	0.2849	0.1823	0.1255	0.2851	0.1818	0.1252	0.2855	0.1816	0.1249	0.2850	0.1822	0.1253
21.0000	0.2879	0.1776	0.1220	0.2882	0.1771	0.1217	0.2886	0.1769	0.1214	0.2880	0.1775	0.1219
22.0000	0.2911	0.1733	0.1187	0.2913	0.1728	0.1183	0.2918	0.1726	0.1181	0.2913	0.1731	0.1186
23.0000	0.2945	0.1693	0.1156	0.2947	0.1688	0.1152	0.2952	0.1686	0.1150	0.2947	0.1691	0.1154
24.0000	0.2979	0.1655	0.1126	0.2982	0.1651	0.1123	0.2986	0.1649	0.1121	0.2982	0.1654	0.1125
25.0000	0.3015	0.1620	0.1098	0.3018	0.1616	0.1095	0.3023	0.1615	0.1094	0.3018	0.1619	0.1097
26.0000	0.3051	0.1588	0.1072	0.3055	0.1585	0.1069	0.3059	0.1584	0.1068	0.3055	0.1588	0.1071
27.0000	0.3088	0.1559	0.1048	0.3092	0.1556	0.1045	0.3095	0.1555	0.1044	0.3091	0.1558	0.1047
28.0000	0.3125	0.1532	0.1026	0.3129	0.1529	0.1023	0.3132	0.1529	0.1022	0.3127	0.1531	0.1025
29.0000	0.3161	0.1507	0.1005	0.3165	0.1504	0.1002	0.3168	0.1504	0.1002	0.3164	0.1507	0.1004
30.0000	0.3197	0.1484	0.0985	0.3201	0.1482	0.0983	0.3203	0.1482	0.0983	0.3199	0.1484	0.0985
31.0000	0.3232	0.1464	0.0968	0.3236	0.1461	0.0965	0.3238	0.1462	0.0965	0.3235	0.1464	0.0967
32.0000	0.3267	0.1445	0.0951	0.3271	0.1443	0.0949	0.3272	0.1443	0.0949	0.3269	0.1445	0.0951
33.0000	0.3300	0.1428	0.0936	0.3303	0.1426	0.0934	0.3305	0.1426	0.0934	0.3302	0.1428	0.0936
34.0000	0.3333	0.1412	0.0922	0.3335	0.1410	0.0921	0.3337	0.1411	0.0920	0.3334	0.1412	0.0922
35.0000	0.3363	0.1398	0.0910	0.3367	0.1396	0.0908	0.3368	0.1397	0.0908	0.3365	0.1399	0.0909
40.0000	0.3501	0.1348	0.0862	0.3504	0.1348	0.0861	0.3504	0.1349	0.0861	0.3501	0.1349	0.0862
45.0000	0.3602	0.1324	0.0835	0.3605	0.1325	0.0834	0.3607	0.1324	0.0835	0.3604	0.1325	0.0835
50.0000	0.3674	0.1317	0.0823	0.3676	0.1317	0.0823	0.3674	0.1317	0.0822	0.3673	0.1317	0.0822

Table 2. Relative errors at selected points within the interval for model (8)

t	Relative Error (Euler-Mar.)			Relative Error (RK 1.0)			Relative Error (Wong-Zakai)		
	$e_r(P)$	$e_r(S)$	$e_r(Q_T)$	$e_r(P)$	$e_r(S)$	$e_r(Q_T)$	$e_r(P)$	$e_r(S)$	$e_r(Q_T)$
10.00	0.0007	0.0060	0.0043	0.0021	0.0076	0.0087	0.0010	0.0028	0.0006
20.00	0.0007	0.0027	0.0024	0.0021	0.0038	0.0048	0.0004	0.0005	0.0016
30.00	0.0013	0.0013	0.0020	0.0019	0.0013	0.0020	0.0006	0.0000	0.0000
40.00	0.0009	0.0000	0.0012	0.0009	0.0007	0.0012	0.0000	0.0007	0.0000
45.00	0.0008	0.0008	0.0012	0.0014	0.0000	0.0000	0.0006	0.0008	0.0000
50.00	0.0005	0.0000	0.0000	0.0000	0.0000	0.0012	0.0003	0.0008	0.0012

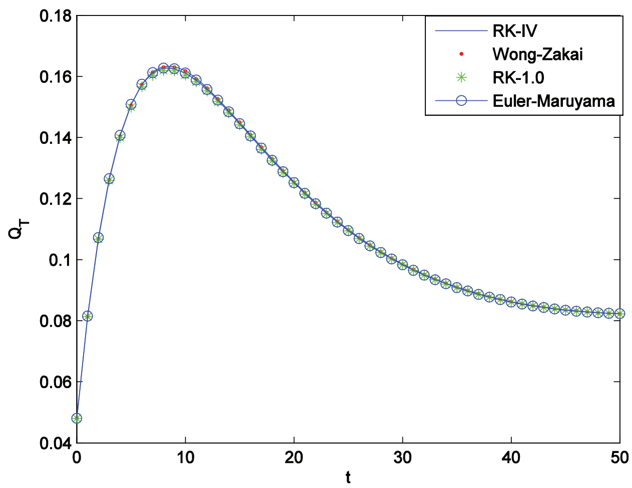

Figure 3. Results for the compartment $Q_T(t)$ in model (8).

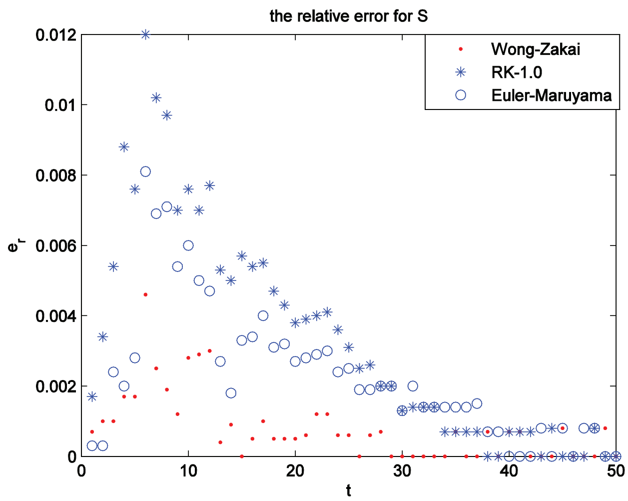

Figure 5. Relative errors of Wong-Zakai, Runge Kutta 1.0 and Euler Maruyama for S.

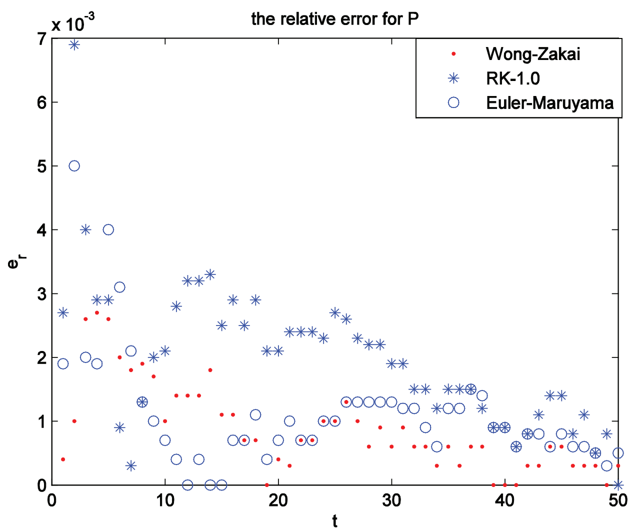

Figure 4. Relative errors of Wong-Zakai, Runge Kutta 1.0 and Euler Maruyama for P.

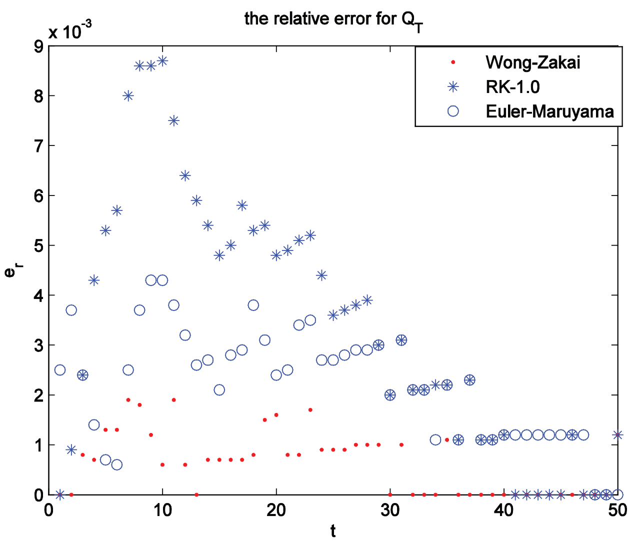

Figure 6. Relative errors of Wong-Zakai, Runge Kutta 1.0 and Euler Maruyama for Q_T.

The relative error plots (Figures 4-6), error comparisons (Table 2) and the results (Table 1) show that Wong-Zakai method performs similarly compared to the other stochastic methods. For instance, at $t = 30$, Wong-Zakai method produces the relative errors 0.0006, 0.0000 and 0.0000 for the compartments P, S and Q_T respectively whereas the relative errors for Euler-Maruyama and Runge-Kutta strong order 1.0 are found as 0.0013, 0.0013, 0.0020 and 0.0019, 0.0013, 0.0020, respectively. Although Wong-Zakai method gives the best results at this point in comparison to stochastic Runge-Kutta IV, it can be seen that it is also outperformed by the other methods in several points of the time interval. Hence, an overall interpretation can be made that the methods perform similarly for model (8).

Results for the Stochastic Model (10)

The stochastic results for model (10) are given for the parameters in the figures below (Figures 7,8,9).

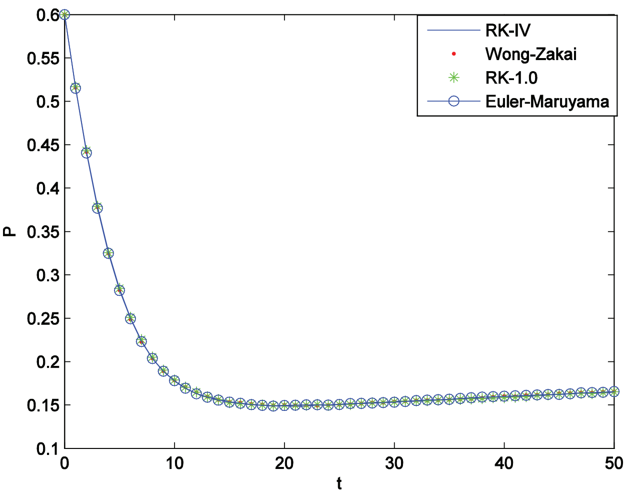

Figure 7. Results for the compartment $P(t)$ in model (10).

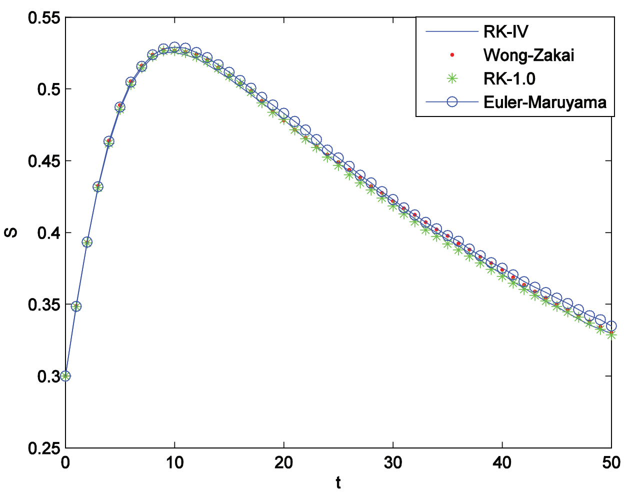

Figure 8. Results for the compartment $S(t)$ in model (10).

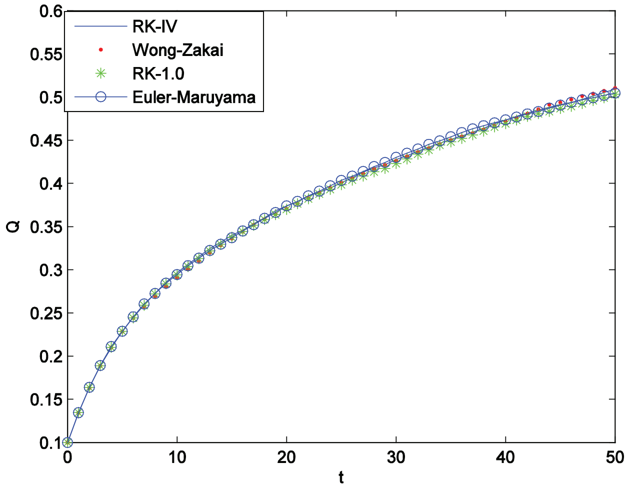

Figure 9. Results for the compartment $Q(t)$ in model (10).

The relative errors throughout the process are shown for the compartments below (Figures 10,11,12)

The relative errors (relative to the solutions of stochastic Runge-Kutta IV) at several points of the process can be shown as below (Table 4):

It should be noted that the diffusion coefficients for model (8) have been used as $\sigma_i = 0.5, i = \overline{1,3}$ whereas $\sigma_i = 0.3, i = \overline{4,9}$ have been used for model (10). This is due to the fact that the methods Euler-Maruyama and Runge-Kutta produced instable results with $\sigma_i = 0.5, i = \overline{4,9}$ in the simulations for model (10). However, no instability in the results was seen with $\sigma_i = 0.5, i = \overline{4,9}$ for Wong-Zakai method. This is seen as an advantage for Wong-Zakai, showing that it gives more stable results numerically.

The relative error plots (Figures 10-12), error comparisons (Table 4) and the results (Table 3) show that Wong-Zakai method performs similarly compared to the other

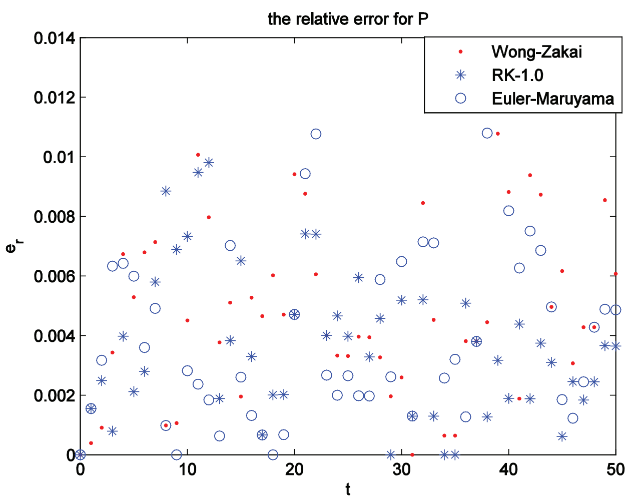

Figure 10. Relative errors of Wong-Zakai, Runge Kutta 1.0 and Euler Maruyama for P.

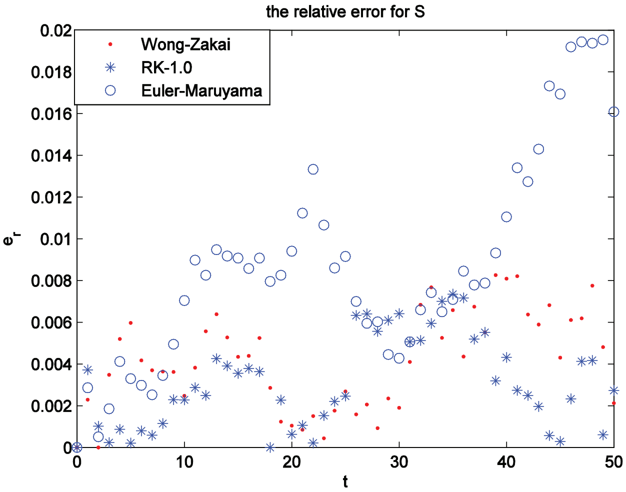

Figure 11. Relative errors of Wong-Zakai, Runge Kutta 1.0 and Euler Maruyama for *S*.

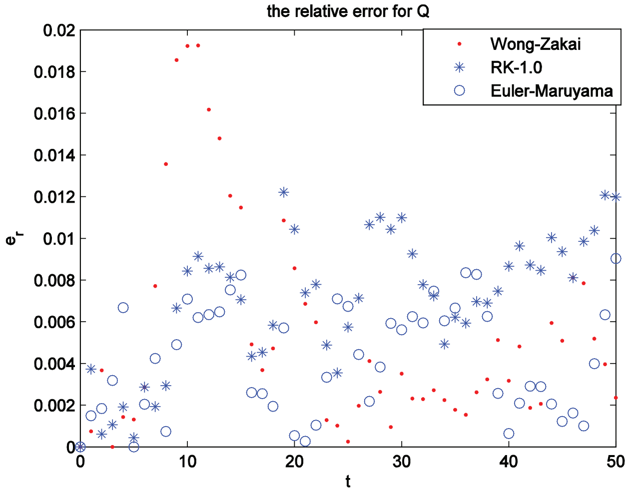

Figure 12. Relative errors of Wong-Zakai, Runge Kutta 1.0 and Euler Maruyama for *Q*.

stochastic methods for the model (10) too. At $t = 30$, Wong-Zakai method gives the relative errors 0.0026, 0.0019 and 0.0035 for the compartments P, S and Q respectively; whereas the relative errors for Euler-Maruyama and Runge-Kutta strong order 1.0 are found as 0.0065, 0.0043, 0.0056 and 0.0052, 0.0064, 0.0110, respectively. Although Wong-Zakai method gives the best results at this point in comparison to stochastic Runge-Kutta IV, it can be seen that it is also outperformed by the other methods in several points of the time interval. Hence, an overall interpretation can be made that the methods perform similarly for model (10) too.

It is seen that for model (8), the fraction of potential smokers decreases to about 27.6% around $t = 15$ reaching its minimum value, after which it starts increasing. The fraction of smokers increases in the beginning reaching its maximum point about 29.5% around $t = 3$ and then starts decreasing. The fraction of quitters increases in the beginning, obtaining the maximal value about 16.3% around $t = 8$ and then starts decreasing. For model (10) the fraction of potential smokers decreases in the beginning and reaches its minimum about 14.8% around $t = 21$ and then starts increasing. The fraction of smokers increases to about 52.5% around $t = 10$ and starts decreasing. The fraction of temporary quitters increases throughout the process reaching its maximum at $t = 50$ about 50.92%. The analysis could be performed on a longer time scale, however, $[0,50]$ is the region where the results are most nonlinear. After $t = 50$ the results become more stable, meaning similar results should be expected for all of the methods. Since the main point of the study is the comparison of the numerical performance of the methods, we have used this time interval where nonlinear solutions are seen for the compartments.

DISCUSSION

It is known that an important fraction of the numerical studies on stochastic differential models are based on the use of the popular stochastic schemes Euler-Maruyama and Milstein. Our findings show that once the corresponding Stratonovich stochastic differential equations are obtained for the stochastic models under consideration, Wong-Zakai method gives equally reliable numerical results. Figures 1, 2 and 3 have been given to show the results for the stochastic model (8) using all of these stochastic schemes. The figures show that all of the methods provide similar results for the model. Table 1 contains the numerical results for model (8) and Table 2 contains errors relative to Runge-Kutta scheme. Figures 4, 5 and 6 show these relative errors and using the results from the Tables and the Figures, it is once again seen that all of the models give similar results. The situation is the same for model (10) too. Figures 7, 8 and 9 contain the results for the compartments of model (10) obtained with all of the stochastic models considered. Relative errors shown in Figures 10, 11 and 12, numerical results given in Table 3 and relative errors at selected points given in Table 4 underline the identical performance of the stochastic methods for the system (10). Note that Figures 1, 2, 3 and 7, 8, 9 have matching solution curves for all of the compartments obtained from all of the stochastic schemes. Relative error figures 4, 5, 6 and 10, 11, 12 show the distribution of errors at various time points. The figures show that all of the methods have time points in the focused interval where they are better than the others and other time points where they are outperformed by the rest of the methods.

Improving the Algorithm Performance

The calculations have been performed on a computer with Intel® Core™ i7-6700HQ @ 2.60GHz 2.59 GHz processor and a 16 GB RAM in a 64-bit OS. The same step size

Table 3. Results from the methods for model (10)

t	RK-IV			Euler-Maruyama			RK-1.0			WZ		
	$E(P(t))$	$E(S(t))$	$E(Q(t))$	$E(P(t))$	$E(S(t))$	$E(Q(t))$	$E(P(t))$	$E(S(t))$	$E(Q(t))$	$E(P(t))$	$E(S(t))$	$E(Q(t))$
0	0.6000	0.3000	0.1000	0.6000	0.3000	0.1000	0.6000	0.3000	0.1000	0.6000	0.3000	0.1000
1.0000	0.5159	0.3496	0.1342	0.5151	0.3486	0.1344	0.5167	0.3483	0.1347	0.5157	0.3488	0.1343
2.0000	0.4419	0.3931	0.1636	0.4405	0.3933	0.1639	0.4430	0.3927	0.1637	0.4415	0.3931	0.1630
3.0000	0.3791	0.4311	0.1886	0.3767	0.4319	0.1892	0.3788	0.4312	0.1888	0.3778	0.4326	0.1886
4.0000	0.3268	0.4616	0.2096	0.3247	0.4635	0.2110	0.3255	0.4620	0.2100	0.3246	0.4640	0.2099
5.0000	0.2837	0.4858	0.2287	0.2820	0.4874	0.2287	0.2843	0.4857	0.2288	0.2822	0.4887	0.2284
6.0000	0.2503	0.5033	0.2449	0.2494	0.5048	0.2454	0.2510	0.5029	0.2442	0.2486	0.5054	0.2442
7.0000	0.2241	0.5144	0.2593	0.2230	0.5157	0.2604	0.2254	0.5147	0.2588	0.2225	0.5163	0.2573
8.0000	0.2034	0.5220	0.2728	0.2036	0.5238	0.2726	0.2052	0.5226	0.2720	0.2036	0.5239	0.2691
9.0000	0.1888	0.5253	0.2857	0.1888	0.5279	0.2843	0.1901	0.5265	0.2838	0.1890	0.5272	0.2804
10.0000	0.1774	0.5254	0.2965	0.1779	0.5291	0.2944	0.1787	0.5266	0.2940	0.1782	0.5267	0.2908
11.0000	0.1689	0.5237	0.3065	0.1693	0.5284	0.3046	0.1705	0.5252	0.3037	0.1706	0.5257	0.3006
12.0000	0.1632	0.5211	0.3154	0.1629	0.5254	0.3134	0.1648	0.5224	0.3127	0.1645	0.5240	0.3103
13.0000	0.1591	0.5169	0.3245	0.1590	0.5218	0.3224	0.1594	0.5191	0.3217	0.1597	0.5202	0.3197
14.0000	0.1567	0.5122	0.3321	0.1556	0.5169	0.3296	0.1561	0.5142	0.3294	0.1559	0.5149	0.3281
15.0000	0.1537	0.5070	0.3399	0.1533	0.5116	0.3371	0.1527	0.5088	0.3375	0.1534	0.5092	0.3360
16.0000	0.1518	0.5016	0.3458	0.1520	0.5059	0.3449	0.1513	0.5035	0.3443	0.1526	0.5038	0.3441
17.0000	0.1505	0.4960	0.3531	0.1504	0.5005	0.3522	0.1506	0.4978	0.3515	0.1512	0.4986	0.3518
18.0000	0.1496	0.4903	0.3602	0.1496	0.4942	0.3595	0.1499	0.4903	0.3581	0.1505	0.4917	0.3585
19.0000	0.1489	0.4848	0.3684	0.1488	0.4888	0.3663	0.1492	0.4837	0.3639	0.1496	0.4842	0.3644
20.0000	0.1487	0.4785	0.3737	0.1494	0.4830	0.3739	0.1494	0.4788	0.3698	0.1501	0.4780	0.3705
21.0000	0.1484	0.4720	0.3792	0.1498	0.4773	0.3791	0.1495	0.4715	0.3764	0.1497	0.4716	0.3766
22.0000	0.1486	0.4652	0.3852	0.1502	0.4714	0.3856	0.1497	0.4653	0.3822	0.1495	0.4659	0.3829
23.0000	0.1498	0.4599	0.3896	0.1502	0.4648	0.3909	0.1504	0.4592	0.3877	0.1492	0.4597	0.3891
24.0000	0.1502	0.4535	0.3944	0.1499	0.4574	0.3972	0.1495	0.4525	0.3930	0.1497	0.4543	0.3948
25.0000	0.1509	0.4478	0.4008	0.1505	0.4519	0.4035	0.1503	0.4467	0.3985	0.1504	0.4490	0.4007
26.0000	0.1514	0.4430	0.4066	0.1517	0.4461	0.4084	0.1505	0.4402	0.4037	0.1508	0.4437	0.4058
27.0000	0.1522	0.4373	0.4129	0.1519	0.4399	0.4138	0.1517	0.4345	0.4085	0.1516	0.4382	0.4112
28.0000	0.1531	0.4319	0.4179	0.1522	0.4345	0.4195	0.1524	0.4295	0.4133	0.1526	0.4323	0.4168
29.0000	0.1530	0.4266	0.4217	0.1526	0.4285	0.4242	0.1530	0.4240	0.4173	0.1527	0.4276	0.4213
30.0000	0.1542	0.4211	0.4276	0.1532	0.4229	0.4300	0.1534	0.4184	0.4229	0.1538	0.4219	0.4261
31.0000	0.1542	0.4151	0.4322	0.1540	0.4172	0.4349	0.1544	0.4130	0.4282	0.1542	0.4168	0.4312
32.0000	0.1539	0.4095	0.4372	0.1550	0.4122	0.4398	0.1547	0.4074	0.4338	0.1552	0.4123	0.4362
33.0000	0.1547	0.4041	0.4419	0.1558	0.4071	0.4452	0.1549	0.4017	0.4387	0.1554	0.4072	0.4407
34.0000	0.1556	0.4000	0.4465	0.1560	0.4026	0.4492	0.1556	0.3972	0.4443	0.1557	0.4021	0.4455
35.0000	0.1561	0.3950	0.4509	0.1566	0.3978	0.4539	0.1561	0.3921	0.4481	0.1562	0.3976	0.4501
40.0000	0.1588	0.3710	0.4732	0.1601	0.3751	0.4735	0.1585	0.3694	0.4691	0.1602	0.3740	0.4717
45.0000	0.1623	0.3484	0.4912	0.1620	0.3543	0.4906	0.1624	0.3485	0.4866	0.1633	0.3499	0.4937
50.0000	0.1646	0.3295	0.5092	0.1654	0.3348	0.5046	0.1652	0.3286	0.5031	0.1656	0.3302	0.5104

Table 4. Relative errors at selected points within the interval for model (10)

t	Relative Error (Euler-Mar.)			Relative Error (RK 1.0)			Relative Error (Wong-Zakai)		
	$e_r(P)$	$e_r(S)$	$e_r(Q)$	$e_r(P)$	$e_r(S)$	$e_r(Q)$	$e_r(P)$	$e_r(S)$	$e_r(Q)$
10.00	0.0028	0.0070	0.0071	0.0073	0.0023	0.0084	0.0045	0.0025	0.0192
20.00	0.0047	0.0094	0.0005	0.0047	0.0006	0.0104	0.0094	0.0010	0.0086
30.00	0.0065	0.0043	0.0056	0.0052	0.0064	0.0110	0.0026	0.0019	0.0035
40.00	0.0082	0.0111	0.0006	0.0019	0.0043	0.0087	0.0088	0.0081	0.0032
45.00	0.0018	0.0169	0.0012	0.0006	0.0003	0.0094	0.0062	0.0043	0.0051
50.00	0.0049	0.0161	0.0090	0.0036	0.0027	0.0120	0.0061	0.0021	0.0024

Table 5. CPU times for Wong-Zakai method (using 10^4 iterations)

	$N_1 = 50 \times N_2 = 10$ subintervals	$N_1 = 100 \times N_2 = 10$ subintervals	$N_1 = 100 \times N_2 = 20$ subintervals
With parfor	94.977495	201.633941	390.185824
Without parfor	227.234240	475.945581	914.946211

has been used within the methods for a fair comparison. Models (8) and (10) have been simulated in MATLAB 10^4 times with a time step of $\Delta t_n = 0.05$ within the interval [0,50]. However, Wong-Zakai method requires the solution of an ordinary differential equation in each time sub-interval $[t_j, t_{j+1}]$ and hence, a new time step size h is used within this subinterval for the deterministic method. This results in a higher calculation load for Wong-Zakai method. The most prominent disadvantage of Wong-Zakai method is this calculation load. To tackle this disadvantage, we have parallelized the loops for decreasing the calculation time. "parfor" command has been used in a computer with 4 workers to obtain better CPU times, which have been shown in the table below (Table 5).

Here, N_1 denotes the number of intervals for the total time [0,50], whereas N_2 denotes the number of subintervals used within each $[t_j, t_{j+1}]$ for Wong-Zakai method. "parfor" command has resulted in a 2-3 times decrease in the total CPU times for the computations. Hence, a significant amount of decrease has been obtained using the new "parfor" command in the algorithm. This result could be obtained at a better level if more computers or processors are used for the algorithms.

Hence, it is seen that "parfor" command could be coupled with Wong-Zakai method to swiftly obtain accurate approximate solutions to mathematical models of smoking consisting of stochastic differential equation systems. "parfor" is an important tool for the algorithm as it enables a swift approximate analysis of the stochastic model. This algorithm could be generalized to analyze any stochastic mathematical model in various research fields such as engineering, biology or medicine.

CONCLUSION

In this study, Wong-Zakai approximation method has been applied to two compartmental stochastic models of smoking. Wong-Zakai method relies upon the deterministic properties of Stratonovich stochastic integration and the performance of such an approximation method has been compared to the more popular stochastic methods such as Euler-Maruyama and stochastic Rung-Kutta. Both of the models have been transformed to systems of Stratonovich stochastic differential equations and are investigated numerically. Additionally, "parfor" command has been embedded into the algorithm to deal with the calculation load that comes with Wong-Zakai stochastic approximation. The results show that Wong-Zakai method performs similarly to the other stochastic schemes. The solution graphs, numerical values for solutions and relative errors for the methods are all similar to each other meaning Wong-Zakai method performs just as good as the other popular stochastic methods. However, it is seen that Wong-Zakai is not used in the literature as much as other schemes that work with Ito SDEs. Considering that "parfor" can be used to decrease the calculation time needed for the algorithm and the fact that Wong-Zakai method can be used with a selection of deterministic approximation techniques, it is obvious that Wong-Zakai acts as a valuable alternative. The results show that the method is an accurate and valuable approximation technique for the analysis of stochastic equations and systems. The motivation of this study was to underline the matching performance of Wong-Zakai scheme in comparison to the other popular stochastic methods in the literature. Two stochastic models with three compartments have been used as examples and have been investigated numerically using all of the aforementioned methods. The findings of the study show that the relative error distributions of the methods and the solution graphs of the compartments are obtained similarly for Wong-Zakai method and the rest of the methods. Considering that "parfor" command can be used to overcome the extra computation load that comes with the use of Wong-Zakai method, it is obvious that this method is a valuable alternative for the investigation of many stochastic models in the literature.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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