Sigma J Eng Nat Sci, Vol. 41, No. 5, pp. 916–925, October, 2023

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2023.00111

Research Article

On orthogonal fuzzy b-metric-like spaces and their fixed point application

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ARTICLE INFO

Article history
Received: 19 July 2021
Revised: 10 November 2021
Accepted: 20 November 2021

Keywords:
Orthogonal b-metric-like Space;
Fuzzy B-Metric-Like Space;
Fractional Differential Equation

ABSTRACT

This manuscript aims to introduce the concept of orthogonal fuzzy b-metric-like spaces and discuss some fixed point results. A non-trivial example is imparted to illustrate the feasibility of the proposed methods. Finally, to validate the superiority of the obtained results, we provide an application to fractional differential equations.

Cite this article as: Uddin F, Ishtiaq U, Aydi H, Javed K, Arshad M. On orthogonal fuzzy b-metric-like spaces and their fixed point application. Sigma J Eng Nat Sci 2023;41(5):916–925.

INTRODUCTION

The concept of fuzzy sets was initiated by Zadeh [13], which gave a new aspect to research activity leading to the improvement of fuzzy systems. Afterward, several researchers contributed towards some basic significant results in fuzzy sets.

Kramosil and Michalek [11] introduced the concept of fuzzy metric spaces by generalizing the concepts of probabilistic metric spaces to fuzzy metric spaces. George and Veeramani [15] derived a Hausdorff topology initiated by a fuzzy metric to modify the concept of fuzzy metric spaces. Later on, fixed point theory via a fuzzy metric has been enriched with several different generalizations. Garbiec [26] displayed the fuzzy version of the Banach contraction principle in fuzzy metric spaces. For some necessary definitions, examples, and basic results, we refer to [13, 14] and the references therein.

As we know, fixed point theory plays a crucial role in proving the existence of solutions of different mathematical models and has a wide range of applications in different fields related to mathematics. This theory has intrigued many researchers. Recently, Harandi [17] initiated the concept of metric-like spaces, which generalizes the notion of metric spaces in a nice way. Alghamdi et al. [2] used the concept metric-like spaces to introduce the notion of b-metric-like spaces. Since then, several authors have

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This Paper was recommended for Publication in revised form by Editor Sania Qureshi worked on metric-like spaces and b-metric-like spaces. For more details, refer to [5-8]. In this sequel, Shukla and Abbas [2] generalized the concept of metric-like spaces and introduced fuzzy metric-like spaces. Recently, Javed et al. [27] introduced the concept of fuzzy b-metric-like spaces and they proves several fixed point results. For more details on this topic, please see [1, 3, 4, 10, 11, 17, 20, 28-34]. Eshaghi Gordji et al. [21] introduced the concept of orthogonal sets. More details can be found in [22-25].

In this article, we aim to generalize the concept of fuzzy b-metric-like spaces by introducing orthogonal fuzzy b-metric-like spaces. We also prove some related fixed point results in the setting of orthogonal fuzzy b-metric-like spaces. Moreover, we give examples and an application of fractional differential equations to support our obtained results that shows the superiority of present notions in the existing literature.

First, we write some shortcut notations used throughout this paper: CTM for continuous triangular norm, BML for b-metric-like, FML for fuzzy metric-like, FBM for fuzzy b-metric, FBML for fuzzy b-metric-like, and s.t. for such that.

Definition 1.1. [1] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a CTM if it satisfies the following assertions:

1. $a * b = b * a$, $(\forall)$ $a, b \in [0, 1]$;
2. $a * 1 = a$, $(\forall)$ $a \in [0, 1]$;
3. $(a * b) * c = a * (b * c)$, $(\forall)$ $a, b, c \in [0, 1]$;
4. If $a \le c$ and $b \le d$, with $a, b, c, d \in [0, 1]$, then $a * b \le c * d$.

Some fundamental examples of a t-norm are $a * b = a \cdot b$, $a * b = \min\{a, b\}$ and $a * b = \max\{a + b - 1, 0\}$.

Definition 1.2. [2] A BML on a set $K \neq \Phi$ is a function $\mu: K \times K \to [0, +\infty)$ so that for all $e, k, z \in K$ and $u \ge 1$, it satisfies the following conditions:

1. If $\mu(e, k) = 0 \Rightarrow e = k$;
2. $\mu(e, k) = \mu(k, e)$;
3. $\mu(e, k) \le u[\mu(e, z) + \mu(z, k)]$.

The pair (K, μ) is called a BML space.

Example 1.1. [2] Let $K = [0, \infty)$. Define $\mu: K \times K \to [0, +\infty)$ by

$$\mu(e, k) = (e + k)^2.$$

Then (K, μ) is called a BML space with $u = 2$.

Example 1.2. [2] Let $K = [0, \infty)$. Define $\mu: K \times K \to [0, +\infty)$ by

$$\mu(e, k) = (\max\{e, k\})^2.$$

Then (K, μ) is a BML space with $u = 2$.

Definition 1.3. [3] A 3-tuple $(K, \Delta, *)$ is said to be an FML space if $K \neq \Phi$ is a random set, $*$ is a CTM and Δ is a fuzzy set on $K \times K \times (0, \infty)$ meeting the points below for all $e, k, z \in K$, $r, s > 0$:

FL1) $\Delta(e, k, r) > 0$;
FL2) If $\Delta(e, k, r) = 1$, then $e = k$;
FL3) $\Delta(e, k, r) = \Delta(k, e, r)$;
FL4) $\Delta(e, z, r + s) \ge \Delta(e, k, r) * \Delta(k, z, s)$;
FL5) $\Delta(e, k, \cdot): (0, \infty) \to [0,1]$ is continuous.

Example 1.3. [3] Let $K = \mathbb{R}^+$, $p \in \mathbb{R}^+$ and $m > 0$. Define a t-norm by $a * b = ab$ and the fuzzy set Δ on $K \times K \times (0, \infty)$ by

$$\Delta(e, k, r) = \frac{pr}{pr + m(\max\{e, k\})}, \forall\, e, k \in K, r > 0.$$

Then $(K, \Delta, *)$ is an FML space.

Definition 1.4. [14] A 3-tuple $(K, \Delta, *)$ is said an FBM space if K is a random (non-empty) set, $*$ is a CTM and Δ is a fuzzy set on $K \times K \times (0, \infty)$ meeting the points below for all $e, k, z \in K$, $r, s > 0$ and a provided real number $u \ge 1$;

FB1) $\Delta(e, k, r) > 0$;
FB2) $\Delta(e, k, r) = 1$ iff $e = k$;
FB3) $\Delta(e, k, r) = \Delta(k, e, r)$;
FB4) $\Delta(e, z, r + s) \ge \Delta\left(e, k, \frac{r}{u}\right) * \Delta\left(k, z, \frac{s}{u}\right)$;
FB5) $\Delta(e, k, \cdot): (0, \infty) \to [0,1]$ is continuous.

Example 1.4. [5] Let $\Delta(e, k, r) = e^{\frac{-|e-k|^p}{r}}$, where $p > 1$ is a real number. It is then simple to show that Δ is an FBM with $b = 2^{p-1}$.

Definition 1.5. [27] A 4-tuple $(K, \Delta, *, u)$ is named as an FBML space if $K \neq \Phi$ is a random set, $*$ is a CTM and Δ is a fuzzy set on $K \times K \times (0, \infty)$ meeting the following points below for all $e, k, z \in K$, $r, s > 0$:

B1) $\Delta(e, k, r) > 0$;
B2) If $\Delta(e, k, r) = 1$, then $e = k$;
B3) $\Delta(e, k, r) = \Delta(k, e, r)$;
B4) $\Delta(e, z, u(r + s)) \ge \Delta(e, k, r) * \Delta(k, z, s)$, for $u \in \mathbb{N}$;
B5) $\Omega(e, k, \cdot): (0, \infty) \to [0,1]$ is continuous.

Example 1.5 [27] Take $K = (0, \infty)$. Given a t-norm as $a * b = ab$, then

$$\Delta(e, k, r) = \left[e^{\frac{(e+k)^2}{r}}\right]^{-1}, \forall\, e, k \in K, r > 0$$

is a FBML. But, it is not a FBM.

Lemma 1.1. [9] If for some $v \in (0,1)$ and $e, k \in K$,

$$\Delta(e, k, r) \ge \Delta\left(e, k, \frac{r}{v}\right), \qquad r > 0,$$

then $e = k$.

Definition 1.6. [21] Assume that $K \neq \Phi$ and $\perp \in K \times K$ is a binary relation. Suppose there exists $e_0 \in K$ such that $e_0 \perp e$ or $e \perp e_0$ for all $e \in K$. Thus, we say that K is an orthogonal set (O-set). Further, we denote an orthogonal set by $(K, \perp)$.

Example 1.6.

i. Let $K = [0, \infty)$ and define $e \perp k$ if $ek = \min\{e, k\}$, then by putting $e_0 = 1$, $(K, \perp)$ is an O-set.

ii. Suppose Δ is the set of scalar matrices of order 2×2 with entries from natural numbers (i.e. $Q = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, for all $a \in N$). Define the relation $\perp$ by

$$A \perp B \text{ if } \det(A) \leq \det(B).$$

Then by taking $A = I$, $(Q, \perp)$ is an O-set.

Definition 1.7. [21] Suppose that (К, $\perp$) is an O-set. A sequence $\{e_n\}$ for all $n \in \mathbb{N}$ is called an O-sequence if ($\forall n$, $e_n \perp e_{n+1}$) or ($\forall n$, $e_{n+1} \perp e_n$).

Definition 1.8. [21] A metric space (К, d) is an orthogonal metric space if (К, $\perp$) is an O-set. Further, ζ: К → К is $\perp$-continuous at $e \in$ К if for each O-sequence $\{e_n\}$ for all $n \in \mathbb{N}$ in К so that $\lim_{n\to\infty} d(e_n, e) = 0$ then $\lim_{n\to\infty} d(\zeta e_n, \zeta e) = 0$ Furthermore, ζ is $\perp$-continuous if ζ is $\perp$-continuous at each $e \in$ К. Also, ζ is $\perp$-preserving if $\zeta e \perp \zeta k$, whence $e \perp k$. Finally, К is orthogonally complete (O-complete) if every Cauchy O-sequence is convergent.

RESULTS AND DISCUSSIONS

In this section, we introduce orthogonal BML spaces and orthogonal FBML spaces. We will prove some fixed point results in the class of orthogonal FBML spaces.

Definition 2.1. Let К ≠ Φ be an orthogonal set and $u \geq 1$. A function d: К × К → $\mathbb{R}^+$ is called orthogonal BML if it meets the below points: ∀ $e, k, z \in$ К,

OB1) If $d(e, k) = 0 \Rightarrow e = k$ such that $e \perp k$ and $k \perp e$;

OB2) $d(e, k) = d(k, e)$ such that $e \perp k$ and $k \perp e$;

OB3) $d(e, k) \leq u[d(e, z) + d(z, k)]$ such that $e \perp z$, $z \perp k$ and $e \perp k$.

Then the set К is named an orthogonal BML space and is denoted by (К, d, u, $\perp$).

Example 2.1. Let К = $\mathbb{R}$. The set К is orthogonal if $e \perp k$ iff e, k∈ {|e|, |k|} . Define $d(e, k) = (e + k)^p$ for all e, k ∈ К, where p belongs to the set of natural numbers. Clearly, d is an orthogonal BMLike space. But, it is not a metric-like space. It suffices to take $e, k \in \mathbb{R}^-$ and an odd p, then clearly $d(e, k)$ is not in $\mathbb{R}^+$.

Remark 2.1. Every BML space is an orthogonal BML space, but the converse is not true.

Definition 2.2. A 5-tuple (К, Δ,∗, u, $\perp$) is called an orthogonal FBML space if К ≠ Φ is a random orthogonal set (К, $\perp$), ∗ is a CTM and Δ is a fuzzy set on К × К × (0, ∞) meeting the following points below (for a given real number $u \geq 1$);

$B_\perp$L1) $\Delta(e, k, r) > 0$, ∀ $e, k \in$ К, $r > 0$ such that $e \perp k$ and $k \perp e$;

$B_\perp$L2) $\Delta(e, k, r) = 1 \Rightarrow e = k$, ∀ $e, k \in$ К, $r > 0$ such that $e \perp k$ and $k \perp e$;

$B_\perp$L3) $\Delta(e, k, r) = \Delta(k, e, r)$, ∀ $e, k \in$ К, $r > 0$ such that $e \perp k$ and $k \perp e$;

$B_\perp$L4) $\Delta(e, z, u(r + s)) \geq \Delta(e, k, r) * \Delta(k, z, s)$, ∀ e, k, $z \in$ К, $r, s > 0$ such that $e \perp k$, $k \perp z$ and $e \perp z$;

$B_\perp$L5) $\Delta(e, k, \cdot)$: (0, ∞) → [0,1] is continuous, ∀ $e, k \in$ К such that $e \perp k$ and $k \perp e$.

Example 2.2. Let К = $\mathbb{R}$ and define a t-norm as $a * b = a.b$. Given a binary relation $\perp$ as: $e \perp k$ iff $e, k \in \{|e|, |k|\}$, then for all $e, k \in$ К, $r > 0$ and p belongs to odd positive integer,

$$\Delta(e, k, r) = \frac{r}{r + (e + k)^p},$$

is an orthogonal FBML. But, clearly it is not an FBML.

Proof. ($B_\perp$L1), ($B_\perp$L2), ($B_\perp$L3) and ($B_\perp$L5) are obvious. Here, we prove ($B_\perp$L4). For an arbitrary integer u, we know that

$$(e + z)^p \leq u[(e + k)^p + (k + z)^p]$$

$$\Rightarrow rs(e + z)^p \leq u(r + s)s(e + k)^p + u(r + s)r(k + z)^p$$

$$\Rightarrow rs(e + z)^p \leq u(r + s)[s(e + k)^p + r(k + z)^p + (e + k)^p(k + z)^p]$$

$$\Rightarrow rs[u(r + s) + (e + z)^p] \leq u(r + s)[r + (e + k)^p][s + (k + z)^p]$$

$$\Rightarrow \frac{u(r + s)}{u(r + s) + (e + z)^p} \geq \frac{r}{r + (e + k)^p} \cdot \frac{s}{s + (k + z)^p}$$

$$\Rightarrow \Delta(e, z, u(r + s)) \geq \Delta(e, k, r) * \Delta(k, z, s).$$

Now, we shall show that (К, Δ,∗, u) is not an FBML space. For $e, k \in$ К, from (B4),

$$\Delta(e, z, u(r + s)) \geq \Delta(e, k, r) * \Delta(k, z, s), \forall\, e, k, z \in К, r, s > 0.$$

We have

$$\frac{u(r+s)}{u(r+s)+(e+z)^p} \geq \frac{r}{r+(e+k)^p} \cdot \frac{s}{s+(k+z)^p}.$$

In particular, assume that $p = 3 = u,\ r = s = \frac{1}{2}$ and $e = k = z = -1$, then

$$\frac{3\left(\frac{1}{2} + \frac{1}{2}\right)}{3\left(\frac{1}{2} + \frac{1}{2}\right) + (-1 - 1)^3} \geq \frac{\frac{1}{2}}{\frac{1}{2} + (-1 - 1)^3} \cdot \frac{\frac{1}{2}}{\frac{1}{2} + (-1 - 1)^3}$$

$$\Rightarrow -\frac{3}{5} \geq \frac{1}{225},$$

which is a contradiction.

Example 2.3. Let К = $\mathbb{R}$ and define a t-norm as $a * b = ab$. Given the binary relation $\perp$ as $e \perp k$ iff $e, k \in \{|e|, |k|\}$. Then for all $e, k \in$ К, $r > 0$,

$\Delta(e, k, r) = e^{-\frac{(e+k)^p}{r}}$ belongs to odd positive integer), is an orthogonal FBML. But, clearly it not an FBML.

Proof. ($B_\perp$L1), ($B_\perp$L3) and ($B_\perp$L5) are obvious. Here, we prove ($B_\perp$L2) and ($B_\perp$L4). We have

$$\Delta(e, k, r) = 1$$

$$\Rightarrow e^{-\frac{(e+k)^p}{r}} = e^0$$

$$\Rightarrow \frac{(e + k)^p}{r} = 0$$

$$\Rightarrow (e + k)^p = 0.$$

Then $\varDelta$ is an orthogonal BML. This implies that $e = k$. Now, assume that $\mu(e,k) = (e+k)^p$. Then

$$\mu(e,z) \le u[\mu(e,k) + \mu(k,z)]$$

$$\frac{\mu(e,z)}{r+s} \le \frac{u[\mu(e,k) + \mu(k,z)]}{r+s}$$

$$\frac{\mu(e,z)}{u(r+s)} \le \frac{\mu(e,k)}{r} + \frac{\mu(k,z)}{s}$$

$$e^{\frac{\mu(e,z)}{u(r+s)}} \ge e^{\frac{\mu(e,k)}{r}} + e^{\frac{\mu(k,z)}{s}}$$

$$\Rightarrow \varDelta(e,z,u(r+s)) \ge \varDelta(e,k,r) * \varDelta(k,z,s).$$

Hence, ($B_\perp$L2) and ($B_\perp$L4) are satisfied and (К, $\varDelta$,∗, u, ⊥) is an orthogonal FBML space . Now, we prove that (К, $\varDelta$,∗, u) is not an FBML space. From (B4),

$$\varDelta(e,z,u(r+s)) \ge \varDelta(e,k,r) * \varDelta(k,z,s), \forall\, e,k,z \in К, r,s > 0.$$

We have

$$e^{-\frac{(e+z)^p}{u(r+s)}} \ge e^{-\frac{(e+k)^p}{r}} \cdot e^{-\frac{(k+z)^p}{s}}.$$

In particular, assume that $p = 3 = u,\ r = s = \frac{1}{2}$ and $e = k = z = -1$. Then

$$e^{-\frac{(-1-1)^3}{3\left(\frac{1}{2}+\frac{1}{2}\right)}} \ge e^{-\frac{(-1-1)^3}{\frac{1}{2}}} \cdot e^{-\frac{(-1-1)^3}{\frac{1}{2}}}$$

$$\Rightarrow e^{\frac{8}{3}} \ge e^{32}.$$

This is wrong.

Remark 2.2. Every FBML space is an orthogonal FBML space, but the converse is not true.

Definition 2.3. A sequence $\{e_n\}$ in an orthogonal FBML space (К, $\varDelta$,∗, u, ⊥) is named to be convergent to $e \in$ К, if

$$\lim_{n\to\infty} \varDelta(e_n, e, r) = \varDelta(e,e,r), \forall\, r > 0.$$

Definition 2.4. A sequence $\{e_n\}$ in an orthogonal FBML space (К, $\varDelta$,∗, u, ⊥) is named to be Cauchy if

$$\lim_{n\to\infty} \varDelta\left(e_n, e_{n+p}, r\right), \forall\, r > 0, p \ge 1$$

exists and is finite.

Definition 2.5. ζ: К → К is ⊥-continuous at $e \in$ К in an orthogonal FBML space (К, $\varDelta$,∗, u, ⊥) if for each O-sequence $\{e_n\}$ for all $n \in \mathbb{N}$ in К, $\lim_{n\to\infty} \varDelta(e_n, e, r)$ exists and is finite for all $r > 0$, then $\lim_{n\to\infty} \varDelta(\zeta e_n, \zeta e, r)$ exists and is finite for all $r > 0$. Furthermore, ζ is ⊥-continuous if ζ is ⊥-continuous at each $e \in$ К. Also, ζ is ⊥-preserving if $\zeta e \perp \zeta k$, whence $e \perp k$. Finally, К is orthogonally complete (O-complete) if every Cauchy O-sequence is convergent.

Definition 2.6. An orthogonal FBML space (К, $\varDelta$,∗, u, ⊥) is said to be complete if every Cauchy sequence $\{e_n\}$ in К, converges to some $e \in$ К such that

$$\lim_{n\to\infty} \varDelta(e_n, e, r) = \varDelta(e,e,r) = \lim_{n\to\infty} \varDelta\left(e_n, e_{n+p}, r\right), \forall\, r > 0, p \ge 1.$$

Definition 2.7. Let (К, $\varDelta$,∗, u, ⊥) be an orthogonal FBML space. A map ζ: К → К is an orthogonal contraction if ∃ $q \in (0,1)$ such that for every $r > 0$ and $e, k \in$ К with $e \perp k$, we have

$$\varDelta(\zeta e, \zeta k, qr) \ge \varDelta(e,k,r). \tag{1}$$

Theorem:2.1. Assume (К, $\varDelta$,∗, u, ⊥) is an orthogonal complete FBML space such that

$$\lim_{r\to\infty} \varDelta(e,k,r) = 1, \forall\, e,k \in К.$$

Let ζ: К → К be ⊥-continuous, ⊥-contraction and ⊥-preserving. Then, ζ has a unique fixed point $e_* \in$ К. Furthermore,

$$\lim_{n\to\infty} \varDelta(\zeta^n e, e_*, r) = \varDelta(e_*, e_*, r), \forall\, e \in К \text{ and } r > 0.$$

Proof: Since (К, $\varDelta$,∗, u, ⊥) is an Orthogonal complete FBML space, there exists $e_0 \in$ К such that

$$e_0 \perp k, \forall\, k \in К. \tag{2}$$

That is, $e_0 \perp \zeta e_0$. Assume that

$$e_1 = \zeta e_0\ , e_2 = \zeta^2 e_0 = \zeta e_1, \ldots, e_n = \zeta^n e_0 = \zeta e_{n-1}, \forall\, n \in N.$$

Since ζ is ⊥-preserving, $\{e_n\}$ is an O-sequence. Now, since ζ is an ⊥-contraction, we can get

$$\varDelta(e_{n+1}, e_n, qr) = \varDelta(\zeta e_n, \zeta e_{n-1}, qr) \ge \varDelta(e_n, e_{n-1}, r)$$

for all $n \in N$ and $r > 0$. Note that $\varDelta$ is b-nondecreasing on $(0, \infty)$. Therefore, by applying the above expression, we can deduce

$$\begin{aligned} &\varDelta(e_{n+1}, e_n, r) \ge \varDelta(e_{n+1}, e_n, qr) = \varDelta(\zeta e_n, \zeta e_{n-1}, qr) \ge \varDelta(e_n, e_{n-1}, r). \\ &= \varDelta(\zeta e_{n-1}, \zeta e_{n-2}, r) \ge \varDelta\left(e_{n-1}, e_{n-2}, \frac{r}{q}\right) \ge \cdots \ge \varDelta\left(e_1, e_0, \frac{r}{q^n}\right) \end{aligned} \tag{3}$$

for all $r > 0$. Thus, from (3), we have

$$\begin{aligned} &\varDelta(e_n, e_{n+p}, r) \ge \varDelta\left(e_n, e_{n+1}, \frac{r}{u}\right) * \varDelta\left(e_{n+1}, e_{n+p}, \frac{r}{u}\right) \\ &\ge \varDelta\left(e_n, e_{n+1}, \frac{r}{u}\right) * \varDelta\left(e_{n+1}, e_{n+2}, \frac{r}{u^2}\right) * \varDelta\left(e_{n+2}, e_{n+3}, \frac{r}{u^3}\right) * \ldots * \varDelta\left(e_{n+p-1}, e_{n+p}, \frac{r}{u^{p-1}}\right) \\ &\ge \varDelta\left(e_1, e_0, \frac{r}{uq^n}\right) * \varDelta\left(e_1, e_0, \frac{r}{u^2 q^{n+1}}\right) * \ldots * \varDelta\left(e_1, e_0, \frac{r}{u^{p-1} q^{n+p-1}}\right). \end{aligned} \tag{4}$$

Here, u is an arbitrary positive integer. We know that $\lim_{n\to\infty} \Delta(e,k,r) = 1$, $\forall\ e,\ k \in$ К and $r > 0$. Thus, from (4), we get

$$\lim_{n\to\infty} \Delta\left(e_n, e_{n+p}, r\right) \geq 1 * 1 * \ldots * 1 = 1.$$

So, $\{e_n\}$ is a Cauchy O-sequence. The hypothesis of O-completeness of the FBML space (К, Δ,$*$, u, $\perp$) ensures that there exists $e_* \in$ К such that $\Delta(e_n, e_*, r) \to 1$ as $n \to +\infty$, $\forall\ r > 0$. Now, since ζ is an $\perp$-continuous mapping, one writes $\Delta(e_{n+1}, \zeta e_*, r) = \Delta(\zeta e_n, \zeta e_*, r) \to 1$ as $n \to +\infty$. Now, we have

$$\Delta(e_*, \zeta e_*, r) \geq \Delta\left(e_*, e_{n+1}, \frac{r}{2u}\right) * \Delta\left(e_{n+1}, \zeta e_*, \frac{r}{2u}\right).$$

Taking limit as $n \to +\infty$, we get $\Delta(e_*, \zeta e_*, r) = 1 * 1 = 1$, and hence $\zeta e_* = e_*$. Therefore, e_* is a fixed point of ζ and $\Delta(e_*, e_*, r) = 1, \forall\, r > 0$.

Now, we show the uniqueness of the fixed point of the mapping ζ. Assume that e_* and k_* are two fixed points of ζ such that $e_* \neq k_*$. One writes

$$e_0 \perp e_* \text{ and } e_0 \perp k_*.$$

Since ζ is $\perp$-preserving, we get

$$\zeta^n e_0 \perp \zeta^n e_* \text{ and } \zeta^n e_0 \perp \zeta^n k_*$$

for all $n \in N$. So from (1), we can derive

$$\Delta(\zeta^n e_0, \zeta^n e_*, r) \geq \Delta(\zeta^n e_0, \zeta^n e_*, qr) \geq \Delta\left(e_0, e_*, \frac{r}{q^n}\right)$$

and

$$\Delta(\zeta^n e_0, \zeta^n k_*, r) \geq \Delta(\zeta^n e_0, \zeta^n k_*, qr) \geq \Delta\left(e_0, k_*, \frac{r}{q^n}\right).$$

Consequently,

$$\Delta(e_*, k_*, r) = \Delta(\zeta^n e_*, \zeta^n k_*, r)$$
$$\geq \Delta\left(\zeta^n e_0, \zeta^n e_*, \frac{r}{2u}\right) * \Delta\left(\zeta^n e_0, \zeta^n k_*, \frac{r}{2u}\right)$$
$$\geq \Delta\left(e_0, e_*, \frac{r}{2uq^n}\right) * \Delta\left(e_0, k_*, \frac{r}{2uq^n}\right) \to 1 \text{ as } n \to \infty.$$

So, $e_* = k_*$, hence e_* is the unique fixed point.

Corollary 2.1. Let (К, Δ, $*$, u, $\perp$) be an O-complete fuzzy b-metric space. Let ζ: К → К be an $\perp$- contraction and $\perp$-preserving. Also, assume that if $\{e_n\}$ is an O-sequence with $e_n \to e \in$ К, then $e \perp e_n$ for all $n \in \mathbb{N}$. Then, ζ has a unique fixed point $e_* \in$ К. Furthermore, $\lim_{n\to\infty} \Delta(\zeta^n e, e_*, r) = \Delta(e_*, e_*, r)$, for all $e \in$ К and $r > 0$.

Proof: We can similarly derive as in the proof of Theorem 2.1 that $\{e_n\}$ is a Cauchy sequence and converges to $e_* \in$ К. Hence, $e_* \perp e_n$ for all $n \in \mathbb{N}$. From (1), we can get

$$\Delta(\zeta e_*, e_{n+1}, r) = \Delta(\zeta e_*, \zeta e_n, r) \geq \Delta(\zeta e_*, \zeta e_n, rq) \geq \Delta(e_*, e_n, r)$$

and

$$\lim_{n\to\infty} \Delta(\zeta e_*, e_{n+1}, r) = 1.$$

Thus,

$$\Delta(e_*, \zeta e_*, r) \geq \Delta\left(e_*, e_{n+1}, \frac{r}{2u}\right) * \Delta\left(e_{n+1}, \zeta e_*, \frac{r}{2u}\right).$$

Taking limit as $n \to +\infty$, we get $\Delta(e_*, \zeta e_*, r) = 1 * 1 = 1$, and hence $\zeta e_* = e_*$. The rest of proof is similarly as in Theorem 2.1.

Example 2.4. Let К = $[-2, 2]$ and define a binary relation $\perp$ by

$$e \perp k \iff e, k \in \{|e|, |k|\}.$$

Define Δ by

$$\Delta(e,k,r) = e^{\frac{-(e+k)^3}{r}}, \text{ for all } e,k \in \text{К and } r > 0$$

Take the t-norm: $a * b = a.\, b$. Then Δ is an orthogonal complete FBML space, but it is not a FBML space. Also, observe that $\lim_{r\to\infty} \Delta(e,k,r) = 1, \forall\ e, k \in$ К.

Define ζ: К → К by

$$\zeta(e) = \begin{cases} \frac{e}{4}, & e \in [-2,0] \\ 0, & e \in (0,2]. \end{cases}$$

Then, it satisfies the following:

1. If $e \in [-2,0]$ and $k \in (0,2]$, then $\zeta(e) = \frac{e}{4}$ and $\zeta(k) = 0$.
2. If $e, k \in [-2,0]$, then $\zeta(e) = \frac{e}{4}$ and $\zeta(k) = \frac{k}{4}$.
3. If $e, k \in (0,2]$, then $\zeta(e) = \zeta(k) = 0$.
4. If $e \in (0,2]$ and $k \in [-2,0]$, then $\zeta(e) = 0$ and $\zeta(k) = \frac{k}{4}$.

We have $e \perp k \iff e,\ k \in \{|e|,\ |k|\}$. This implies that $\zeta(e), \zeta(k) \in \{|\zeta(e)|, |\zeta(k)|\}$. Hence ζ is $\perp$-preserving. Let $\{e_n\}$ be an arbitrary o-sequence in К that $\{e_n\}$ converges to $e \in$ К. We have

$$\lim_{n\to\infty} \Delta(e_n, e, r) = \lim_{n\to\infty} e^{-\frac{(e_n+e)^3}{r}} = \Delta(e,e,r)$$

as $\{e_n\}$ converges to e. We can easily see that if $\lim_{n\to\infty} \Delta(e_n, e, r)$ exists and is finite, then $\lim_{n\to\infty} \Delta(\zeta e_n, \zeta e, r)$ exists and is finite for all $e \in$ К and $r > 0$. Hence, ζ is orthogonally continuous. But, ζ is not continuous. For this, take $e_n, e \in [-2, 0]$, so

$$\lim_{n\to\infty} \Delta(\zeta e_n, \zeta e, r) = \lim_{n\to\infty} \Delta\left(\frac{e_n}{4}, \frac{e}{4}, r\right) = \lim_{n\to\infty} e^{-\frac{(e_n+e)^3}{64r}}.$$

As $e_n \to e$ as $n \to \infty$ and taking $e = -2$, we have $\lim_{n\to\infty} \Delta(\zeta e_n, \zeta e, r) = e^{\frac{1}{r}} > 1$, which is wrong. We have 4 cases for $q \in \left[\frac{1}{2}, 1\right)$,

Case 1) If $e \in [-2, 0]$ and $k \in (0, 2]$. Then $\zeta e = \frac{e}{4}$ and $\zeta k = 0$. We have

$$\Delta(\zeta e, \zeta k, qr) = \Delta\left(\frac{e}{4}, 0, qr\right) = e^{-\frac{e^3}{64qr}} \geq e^{-\frac{(e+k)^3}{r}} = \Delta(e, k, r).$$

Case 2) If $e, k \in [-2, 0]$, then $\zeta e = \frac{e}{4}$ and $\zeta k = \frac{k}{4}$. We have

$$\Delta(\zeta e, \zeta k, qr) = \Delta\left(\frac{e}{4}, \frac{k}{4}, qr\right) = e^{-\frac{\left(\frac{e}{4}+\frac{k}{4}\right)^3}{qr}} = e^{-\frac{(e+k)^3}{64qr}} \geq e^{-\frac{(e+k)^3}{r}} = \Delta(e, k, r).$$

Case 3) If $e, k \in (0, 2]$, then $\zeta e = 0$ and $\zeta k = 0$. We have

$$\Delta(\zeta e, \zeta k, qr) = \Delta(0, 0, qr) = e^0 \geq e^{-\frac{(e+k)^3}{r}} = \Delta(e, k, r).$$

Case 4) If $e \in (0, 2]$ and $k \in [-2, 0]$, then $\zeta e = 0$ and $\zeta k = \frac{k}{4}$. Here,

$$\Delta(\zeta e, \zeta k, qr) = \Delta\left(0, \frac{k}{4}, qr\right) = e^{-\frac{k^3}{64qr}} \geq e^{\frac{-(e+k)^3}{r}} = \Delta(e, k, r).$$

From all 4 cases, we obtain that

$$\Delta(\zeta e, \zeta k, qr) \geq \Delta(e, k, r).$$

Hence, ζ is an orthogonal contraction. But, ζ is not a contraction. Indeed, taking $e = -2$ and $k = 1$, one gets

$$\Delta(\zeta e, \zeta k, qr) = e^{-\frac{\left(-\frac{1}{2}+0\right)^3}{qr}} = e^{\frac{1}{8qr}} \geq 1.$$

This is wrong.

All the conditions of Theorem 2.1 are satisfied and ζ has a unique fixed point, which is 0.

Theorem 2.2. Assume that (К, Δ,$*$, u, $\perp$) is an orthogonal complete FBML space such that

$$\lim_{r\to\infty} \Delta(e, k, r) = 1, \forall\, e, k \in К \text{ and } r > 0.$$

Let ζ: К $\to$ К be an $\perp$-continuous, $\perp$-contraction and $\perp$-preserving. Suppose that there exist $q \in \left(0, \frac{1}{u}\right)$ and $r > 0$ such that

$$\Delta(\zeta e, \zeta k, qr) \geq \min\{\Delta(\zeta e, e, r), \Delta(\zeta k, k, r)\}$$

ffor all $e, k \in$ К, $r > 0$. Then ζ has a unique fixed point $e_* \in$ К.

Proof. Since (К, Δ,$*$, u, $\perp$) is an orthogonal complete FBML space, there exists $e_0 \in$ К such that

$$e_0 \perp k, \forall\, k \in К. \tag{2}$$

Thus, $e_0 \perp \zeta e_0$. Consider,

$$e_1 = \zeta e_0\,,\, e_2 = \zeta^2 e_0 = \zeta e_1, \ldots, e_n = \zeta^n e_0 = \zeta e_{n-1}, \forall\, n \in N.$$

Since ζ is $\perp$-preserving, $\{e_n\}$ is an O-sequence. Note that Δ is b-nondecreasing on $(0, \infty)$, so

$$\Delta(e_{n+1}, e_n, r) \geq \Delta(e_{n+1}, e_n, qr) = \Delta(\zeta e_n, \zeta e_{n-1}, qr)$$
$$\geq \min\{\Delta(\zeta e_n, e_n, r), \Delta(\zeta e_{n-1}, e_{n-1}, r)\}.$$

Two cases occur:

Case 1. If $\Delta(e_{n+1}, e_n, r) \geq \Delta(\zeta e_n, e_n, r)$, then

$$\Delta(e_{n+1}, e_n, r) \geq \Delta(e_{n+1}, e_n, qr) \geq \Delta(\zeta e_n, e_n, r) = \Delta(e_{n+1}, e_n, r).$$

Then by Lemma 1.1, we get $e_n = e_{n+1}$ for all $n \in \mathbb{N}$ and $r > 0$.

Case 2. If $\Delta(e_{n+1}, e_n, r) \geq \Delta(\zeta e_{n-1}, e_{n-1}, r)$, then

$$\Delta(e_{n+1}, e_n, r) \geq \Delta(e_{n+1}, e_n, qr) \geq \Delta(\zeta e_{n-1}, e_{n-1}, r) \geq \Delta(e_n, e_{n-1}, r)$$

for all $n \in \mathbb{N}$ and $r > 0$. Then by Theorem 2.1, $\{e_n\}$ is a Cauchy orthogonal sequence. By completeness of (К, Δ,$*$, u, $\perp$), there exists $e_* \in$ К such that

$$\lim_{n\to\infty} \Delta(e_n, e_*, r) = 1, \text{ for all } r > 0.$$

We know that ζ is an $\perp$-continuous mapping, then

$$\lim_{n\to\infty} \Delta(e_{n+1}, \zeta e_*\,, r) = \lim_{n\to\infty} \Delta(\zeta e_n, \zeta e_*\,, r) \to 1.$$

Now, we prove that e_* is a fixed point for ζ. Let $r_1 \in (qu, 1)$ and $r_2 = 1 - r_1$. Then

$$\Delta(\zeta e_*, e_*, r) \geq \Delta\left(\zeta e_*, e_{n+1}, \frac{rr_1}{u}\right) * \Delta\left(e_{n+1}, e_*, \frac{rr_2}{u}\right)$$
$$= \Delta\left(\zeta e_*, \zeta e_n, \frac{rr_1}{u}\right) * \Delta\left(e_{n+1}, e_*, \frac{rr_2}{u}\right)$$
$$\geq \min\left\{\Delta\left(\zeta e_*, e_*, \frac{rr_1}{uq}\right), \Delta\left(\zeta e_n, e_n, \frac{rr_1}{uq}\right)\right\} * \Delta\left(e_{n+1}, e_*, \frac{rr_2}{u}\right)$$
$$= \min\left\{\Delta\left(\zeta e_*, e_*, \frac{rr_1}{uq}\right), \Delta\left(e_{n+1}, e_n, \frac{rr_1}{uq}\right)\right\} * \Delta\left(e_{n+1}, e_*, \frac{rr_2}{u}\right).$$

Taking $n \to \infty$, we get

$$\varDelta(\zeta e_*, e_*, r) \geq \min\left\{\varDelta\left(\zeta e_*, e_*, \frac{rr_1}{uq}\right), 1\right\} * 1$$

$$\varDelta(\zeta e_*, e_*, r) \geq \varDelta\left(\zeta e_*, e_*, \frac{r}{v}\right), r > 0.$$

Here, $v = \frac{uq}{r_1} \in (0,1)$, and from Lemma 1.1, we have $\zeta e_* = e_*$.

Let e_* and k_* be two different fixed points of ζ. We have

$$e_0 \perp e_* \text{ and } e_0 \perp k_*.$$

Since, ζ is $\perp$-preserving, we can write

$\zeta^n e_0 \perp \zeta^n e_*$ and $\zeta^n e_0 \perp \zeta^n k_*$ for all $n \in \mathbb{N}$.

We have

$$\varDelta(\zeta^n e_0, \zeta^n e_*, r) \geq \varDelta(\zeta^n e_0, \zeta^n e_*, qr) \geq \min\{\varDelta(\zeta^n e_0, e_0, r), \varDelta(\zeta^n e_*, e_*, r)\}$$

and

$$\varDelta(\zeta^n e_0, \zeta^n k_*, r) \geq \varDelta(\zeta^n e_0, \zeta^n k_*, qr) \geq \min\{\varDelta(\zeta^n e_0, e_0, r), \varDelta(\zeta^n k_*, k_*, r)\}.$$

Hence,

$\varDelta(e_*, k_*, r) = \varDelta(\zeta^n e_*, \zeta^n k_*, r) \geq \min\left\{\varDelta\left(\zeta^n e_*, e_*, \frac{r}{q}\right), \varDelta\left(\zeta^n k_*, k_*, \frac{r}{q}\right)\right\} = \min\{1, 1\} = 1$, for all $r > 0$. That is, $e_* = k_*$.

Corollary 2.2. Let $(\text{К}, \varDelta, *, u, \perp)$ be an orthogonal complete fuzzy b-metric space and $\zeta: \text{К} \to \text{К}$ be an $\perp$-continuous and $\perp$-preserving mapping. Assume that there exist $q \in \left(0, \frac{1}{u}\right)$ and $r > 0$ such that

$$\varDelta(\zeta e, \zeta g, qr) \geq \min\{\varDelta(\zeta e, e, r), \varDelta(\zeta g, g, r), \varDelta(e, g, r)\}.$$

Then ζ has s a unique fixed point.

Proof. We can easily prove this result by the help of Theorem 2.1 and Theorem 2.2.

Example 2.5. Let $\text{К} = [-2, 2]$ and define a binary relation $\perp$ by

$$e \perp k \Leftrightarrow e, k \in \{|e|, |k|\}.$$

Define $\varDelta$ by

$$\varDelta(e, k, r) = \frac{r}{r + (e+k)^3}, \text{ for all } e, k \in \text{К and } r > 0$$

Consider the t-norm: $a * b = a.b$, then $\varDelta$ is an orthogonal complete FBML, but it is not an FBML. Observe that $\lim_{r\to\infty} \varDelta(e, k, r) = 1, \forall\, e, k \in \text{К}$.

Define $\zeta: \text{К} \to \text{К}$ by

$$\zeta(e) = \begin{cases} \frac{e}{4}, & e \in \left[-2, \frac{2}{3}\right] \\ 1 - e, & e \in \left(\frac{2}{3}, 1\right] \\ e - \frac{1}{2}, & e \in (1, 2]. \end{cases}$$

Then, it satisfies the following:

1. If $e, k \in \left[-2, \frac{2}{3}\right]$, then $\zeta(e) = \frac{e}{4}$ and $\zeta(k) = \frac{k}{4}$.
2. If $e, k \in \left(\frac{2}{3}, 1\right]$, then $\zeta(e) = 1 - e$ and $\zeta(k) = 1 - k$.
3. If $e, k \in (1, 2]$, then $\zeta(e) = e - \frac{1}{2}$ and $\zeta(k) = k - \frac{1}{2}$.
4. If $e \in \left[-2, \frac{2}{3}\right]$ and $k \in \left(\frac{2}{3}, 1\right]$, then $\zeta(e) = \frac{e}{4}$ and $\zeta(k) = 1 - k$.
5. If $e \in \left[-2, \frac{2}{3}\right]$ and $k \in (1, 2]$, then $\zeta(e) = \frac{e}{4}$ and $\zeta(k) = k - \frac{1}{2}$.
6. If $e \in \left(\frac{2}{3}, 1\right]$ and $k \in (1, 2]$, then $\zeta(e) = 1 - e$ and $\zeta(k) = k - \frac{1}{2}$.
7. If $e \in (1, 2]$ and $k \in \left(\frac{2}{3}, 1\right]$, then $\zeta(e) = e - \frac{1}{2}$ and $\zeta(k) = 1 - k$.
8. If $e \in (1, 2]$ and $k \in \left[-2, \frac{2}{3}\right]$, then $\zeta(e) = e - \frac{1}{2}$ and $\zeta(k) = \frac{k}{4}$.
9. If $e \in \left(\frac{2}{3}, 1\right]$ and $k \in \left[-2, \frac{2}{3}\right]$, then $\zeta(e) = 1 - e$ and $\zeta(k) = \frac{k}{4}$.

We have $e \perp k \Leftrightarrow e, k \in \{|e|, |k|\}$. This implies that $\zeta(e), \zeta(k) \in \{|\zeta(e)|, |\zeta(k)|\}$. Hence, ζ is $\perp$-preserving. Let $\{e_n\}$ be an arbitrary o-sequence in К so that $\{e_n\}$ converges to $e \in \text{К}$. We have

$$\lim_{n\to\infty} \varDelta(e_n, e, r) = \lim_{n\to\infty} \frac{r}{r + (e_n + e)^3} = \varDelta(e, e, r).$$

We can easily see that if $\lim_{n\to\infty} \varDelta(e_n, e, r)$ exists and is finite, $\lim_{n\to\infty} \varDelta(\zeta e_n, \zeta e, r)$ also exists and is finite for all $e \in \text{К}$ and $r > 0$. Hence, ζ is orthogonal continuous. But, ζ is not continuous. For this, take $e_n, e \in [-2, \frac{2}{3}]$. Here,

$$\lim_{n\to\infty} \varDelta(\zeta e_n, \zeta e, r) = \lim_{n\to\infty} \varDelta\left(\frac{e_n}{4}, \frac{e}{4}, r\right) = \lim_{n\to\infty} \frac{r}{r + \left(\frac{e_n}{4} + \frac{e}{4}\right)^3}.$$

As $e_n \to e$ as $n \to \infty$ and taking $e = -2$ and $r = \frac{1}{8}$, we have $\lim_{n\to\infty} \varDelta(\zeta e_n, \zeta e, r) = -\frac{1}{7} < 0$, which is wrong. Also, all above cases satisfy the orthogonal contraction:

$$\varDelta(\zeta e, \zeta k, qr) \geq \min\{\varDelta(\zeta e, e, r), \varDelta(\zeta k, k, r)\}.$$

But, it is not a contraction. Assume $\min\{\varDelta(\zeta e, e, r), \varDelta(\zeta k, k, r)\} = \varDelta(\zeta e, e, r)$, then for $e = k = -2$, we have

$$\varDelta(\zeta e, \zeta k, qr) = \frac{qr}{qr + \left(\frac{e}{4} + \frac{k}{4}\right)^3} = \frac{qr}{qr - 1} \geq 1.$$

This is wrong. Hence, all the conditions of Theorem 2.2 are satisfied and 0 is the unique fixed point of ζ.

APPLICATION

Within this part, we apply Theorem 2.1 to investigate the existence and uniqueness of a solution of a nonlinear fractional differential equation (see [18]) given by

$$D_c^\alpha e(t) = f\big(t, e(t)\big) \quad (t \in (0,1), \alpha \in (1,2]) \tag{3}$$

with boundary conditions

$$e(0) = 0, e'(0) = Ie(t)\ t \in (0,1),$$

where D_c^α is the Caputo fractional derivative of order α defined by

$$D_c^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s) ds$$

$$(n-1 < \alpha < n, \;\; n = [\alpha]+1),$$

and f: $[0,1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function. Let $К = C([0,1], \mathbb{R})$ be endowed with the supremum $|e| = \operatorname{Sup}_{t\in[0,1]} |e(t)|$.

The Riemann-Liouville fractional integral of order α (see [19]) is given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \quad (\alpha > 0).$$

Theorem 3.1. Assume that
i. f: $[0,1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function,
ii. $e(t)$: $[0,1] \to \mathbb{R}$ is continuous,
So that

$$|f(t,e) + f(t,k)| \le L|e+k|$$

for all $t \in [0,1]$ and for all $e, k \in К$ such that $e(t) + k(t) \ge 0$. L is a constant with $LЛ < 1$ where

$$Л = \frac{1}{\Gamma(\alpha+1)} + \frac{2k^{\alpha+1}\Gamma(\alpha)}{(2-k^2)\Gamma(\alpha+1)}.$$

Then the differential equation (3) has a unique solution.
Proof. We take the following orthogonal relation on К:

$e \perp k$ iff $e(t) + k(t) \ge 0$ for all $t \in [0,1]$

Also, we take

$$\Delta(e,k,r) = e^{-\frac{(e(t)+k(t))^3}{r}}.$$

For all $e, k \in К$, we consider $|e+k| = \operatorname{Sup}_{t\in[0,1]} |e(t)+k(t)|$. $(К, \Delta, *, u, \perp)$ is a complete orthogonal fuzzy BML space. Observe that it is not a fuzzy BML space. We define a mapping ζ: $К \to К$ by

$$\zeta e(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,e(s)) ds + \frac{2t}{(2-k^2)\Gamma(\alpha)} \int_0^k \left(\int_0^s (s-m)^{\alpha-1} f(m,e(m)) dm \right) ds$$

for all $t \in [0,1]$. Note that the equation (3.1) has a solution $e \in К$ iff $e(t) = \zeta e(t)$ for all $t \in [0,1]$. To check the existence of a fixed point of ζ, we are going to show that ζ is $\perp$-preserving, $\perp$-contraction and $\perp$-continuous.

For all $t \in [0,1]$, $e(t) \perp k(t)$ means that $e(t) + k(t) \ge 0$. Clearly, from (4), we have $\zeta e(t) + \zeta k(t) \ge 0$. It implies that $\zeta e(t) \perp \zeta k(t)$. Hence, ζ is $\perp$-preserving. For all $t \in [0,1]$ and $e(t) \perp k(t)$, we get

$$\Delta(\zeta e, \zeta k, r) = e^{-\frac{(\zeta e(t)+\zeta k(t))^3}{r}}. \tag{5}$$

Also,

$$\zeta e(t) + \zeta k(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,e(s)) ds + \frac{2t}{(2-k^2)\Gamma(\alpha)} \int_0^k \left(\int_0^s (s-m)^{\alpha-1} f(m,e(m)) dm \right) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,k(s)) ds + \frac{2t}{(2-k^2)\Gamma(\alpha)} \int_0^k \left(\int_0^s (s-m)^{\alpha-1} f(m,k(m)) dm \right) ds.$$

From the fact that $e(t) + k(t) \ge 0$, we can take $e(t) + k(t) = |e(t) + k(t)|$, since ζ is $\perp$-preserving, which means that $\zeta e(t) + \zeta k(t) = |\zeta e(t) + \zeta k(t)|$. We have

$$|\zeta e(t) + \zeta k(t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,e(s)) ds + \frac{2t}{(2-k^2)\Gamma(\alpha)} \int_0^k \left(\int_0^s (s-m)^{\alpha-1} f(m,e(m)) dm \right) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,k(s)) ds + \frac{2t}{(2-k^2)\Gamma(\alpha)} \int_0^k \left(\int_0^s (s-m)^{\alpha-1} f(m,k(m)) dm \right) ds \right|$$

$$\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,e(s)) + f(s,k(s))| ds + \frac{2t}{(2-k^2)\Gamma(\alpha)} \int_0^k \left(\int_0^s (s-m)^{\alpha-1} |f(m,e(m)) + f(m,k(m))| dm \right) ds$$

$$\le \frac{L|e+k|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{2L|e+k|}{\Gamma(\alpha)} \int_0^k \left(\int_0^s (s-m)^{\alpha-1} dm \right) ds$$

$$\le \frac{L|e+k|}{\Gamma(\alpha+1)} + \frac{2k^{\alpha+1} L|e+k|\Gamma(\alpha)}{(2-k^2)\Gamma(\alpha+2)}$$

$$\le L|e+k| \left(\frac{1}{\Gamma(\alpha+1)} + \frac{2k^{\alpha+1}\Gamma(\alpha)}{(2-k^2)\Gamma(\alpha+2)} \right) = LЛ|e+k|.$$

From the fact $LЛ < 1$ and (5), we get

$$\Delta(\zeta e, \zeta k, r) = e^{-\frac{(\zeta e(t)+\zeta k(t))^3}{r}} \ge e^{-\frac{(LЛ(e+k))^3}{r}} \ge e^{-\frac{(e+k)^3}{r}} = \Delta(e,k,r).$$

It implies that ζ is an $\perp$-contraction.

Suppose that $\{e_n\}$ is an O-sequence in K such that $\{e_n\}$ converges to $e \in$ K. Because ζ is $\perp$-preserving, $\{\zeta e_n\}$ is an O-sequence for each $n \in \mathbb{N}$. Also, because ζ is an $\perp$-contraction, we have

$$\Delta(\zeta e_n(t), \zeta e(t), qr) \geq \Delta(e_n(t), e(t), r).$$

As $\lim_{n\to\infty} \Delta(e_n(t), e(t), r)$ exists and is finite for all $r > 0$. it is clear that $\lim_{n\to\infty} \Delta(\zeta e_n(t),\ \zeta e(t), qr)$ exists and is finite.

Hence, ζ is $\perp$-continuous. Thus, all the conditions of Theorem 2.1 are satisfied, and so $e(t)$ is the unique fixed point of ζ.

CONCLUSION

In this manuscript, we introduced the notion of orthogonal fuzzy b-metric like spaces as a combination of orthogonal sets and fuzzy b-metric like spaces. This new setting has many applications and opens the door to extend and generalize some known related fixed point results in (fuzzy) b-metric like spaces. At the end, we solve a fractional differential equation and we gave some concrete examples illustrating the new concepts. This work can be extend in the structure of orthogonal control fuzzy metric-like spaces, Intutionistic fuzzy b-metric-like spaces, neutrosophic metric-like spaces etc.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

[1] Schweizer B, Sklar A. Statistical metric spaces. Pacific J Math 1960;10:313-334. [CrossRef]

[2] Alghamdi MA, Hussain N, Salimi P. Fixed point and coupled fixed point theorems in b-metric-like spaces. J Inequal Appl 2013;2013:199. [CrossRef]

[3] Shukla S, Abbas M. Fixed point results in fuzzy metric-like spaces. Iran J Fuzzy Syst 2014;11:81–92.

[4] Shukla S, Gopal D, Roldán-López-deHierro AF. Some fixed point theorems in 1-M-complete fuzzy metric-like spaces. Int J Gen Syst 2016;45:272–281. [CrossRef]

[5] Hammad HA, Aydi H, Sen MDL. Generalized dynamic process for an extended multi-valued F–contraction in metric-like spaces with applications. Alexandria Eng J 2020;59:3799–3808. [CrossRef]

[6] Hussain N, Roshan JR, Parvaneh V, Kadelburg Z. Fixed points of contractive mappings in b-metric-like spaces. Sci World J 2014;2014:598165. [CrossRef]

[7] Chen C, Wen L, Dong J, Gu Y. Fixed point theorems for generalized F-contractions in b-metric-like spaces. J Nonlinear Sci Appl 2016;9:2161-2174. [CrossRef]

[8] Chen C, Dong J, Zhu C. Some fixed point theorems in b-metric-like spaces. Fixed Point Theory Appl 2015;2015:122. [CrossRef]

[9] Raki D, Mukheimer A, Došenovic T, Mitrovic ZD, Radenovic S. On some new fixed point results in fuzzy b-metric spaces. J Inequal Appl 2020;2020:99. [CrossRef]

[10] Nădăban S. Fuzzy b-metric spaces. Int J Comput Commun Control 2016;11:273–281. [CrossRef]

[11] Kramosil I, Michlek J. Fuzzy metric and statistical metric spaces. Kybernetika 1975;11:336–344.

[12] Zadeh LA. Fuzzy sets. Inf Control 1965;3:338-353. [CrossRef]

[13] Došenovic T, Javaheri A, Sedghi S, Shobe N. Coupled fixed point theorem in b-fuzzy metric spaces. Novi Sad J Math 2017;47:77-88. [CrossRef]

[14] Sedghi S, Shobe N. Common fixed point theorem in b-fuzzy metric space. Nonlinear Funct Anal Appl 2012;17:349–359.

[15] George A, Veeramani P. On some results of analysis for fuzzy metric spaces. Fuzzy Sets Syst 1997;90:365-368. [CrossRef]

[16] Imdad M, Ali J, Hasan M. Common fixed point theorems in fuzzy metric spaces employing common property (E.A.). Math Comput Modelling 2012;55:770–778. [CrossRef]

[17] Harandi A. Metric-likes paces. partial metric spaces and fixed point. Fixed Point Theory Appl 2012;2012:12. [CrossRef]

[18] Baleanu D, Rezapour S, Mohammadi H. Some existence results on nonlinear fractional differential equations. Philos Trans A Math Phys Eng Sci 2013;371:20120148. [CrossRef]

[19] Sudsutad W, Tariboon J. Boundary value problems for fractional differential equations with three-point fractional integral boundary conditions. Adv Differ Equ 2012;2012:93. [CrossRef]

[20] Aydi H, Felhi A, Sahmim S. On common fixed points for (α, ψ)-contractions and generalized cyclic contractions in b-metric like spaces and consequences. J Nonlinear Sci Appl 2016;9:2492–2510. [CrossRef]

[21] Eshaghi M, Ramezani M, Sen MDL, Cho YJ. On orthogonal sets and Banach's fixed point theorem. Fixed Point Theory 2017;18:569–578.

[22] Yang Q, Bai C. Fixed point theorem for orthogonal contraction of Hardy-Rogers-type mapping on O-complete metric spaces. AIMS Math 2020;5:5734-5742. [CrossRef]

[23] Baghani H. A new contractive condition related to Rhoades's open question. Indian J Pure Appl Math 2020;51:565–578. [CrossRef]

[24] Javed K, Aydi H, Uddin F, Arshad M. On orthogonal partial metric spaces with an application. J Math 2021;2021:6624595. [CrossRef]

[25] Sezen MS. Some special functions in orthogonal fuzzy bipolar metric spaces and their fixed point applications. Num Methods Partial Differ Equations 2020;36:1606–1614.

[26] Grabiec M. Fixed points in fuzzy metric spaces. Fuzzy Sets Syst 1988;3:385-389. [CrossRef]

[27] Javed k, Uddin F, Aydi H, Arshad M, Ishtiaq U, Alsamire H. On fuzzy b-metric-like spaces. J Funct Spaces 2021;2021:6623467. [CrossRef]

[28] Liu F, Din A, Haung L, Yusuf A. Stochastic optimal control analysis for the hepatitis B epidemic model. Results Phys 2021;26:104010. [CrossRef]

[29] Alqahtani RT, Yusuf A, Agarwal RP. Mathematical analysis of oxygen uptake rate in continuous process under caputo derivative. Mathematics 2021;9:1716. [CrossRef]

[30] Zha TH, Castillo O, Jahanshahi H, Yusuf A, Alassafi MO, Alsaadi FE, et al. A fuzzy-based strategy to suppress the novel coronavirus (2019-ncov) massive outbreak. Appl Comput Math 2021;20:160–176.

[31] Ahmed I, Goufo EFD, Yusuf A, Kumam P, Chaipanya P, Nonlaopon K. An epidemic prediction from analysis of a combined HIV-COVID-19 co-infection model via ABC fractional operator. Alexandria Eng J 2021;60:2979-2995. [CrossRef]

[32] Qureshi S, Yusuf A, Aziz S. On the use of mohand ıntegral transform for solvıng fractıonal-order classıcal caputo dıfferentıal equatıons. J Appl Math Comput Mech 2020;19:99–109. [CrossRef]

[33] Qureshi S, Yusuf A. A new third order convergent numerical solver for continuous dynamical systems. J King Saud Univ Sci 2019;32:543–548. [CrossRef]

[34] Uddin F, Javed K, Aydi H, Ishtiaq U, Arshad M. Control fuzzy metric spaces via orthogonality with an application. J Math 2021;2021:6625230. [CrossRef]