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Research Article

Some new properties of octonionic matrices

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ABSTRACT

In this study, firstly, real, complex and quaternion combinations of octonionic matrices are defined. In terms of these defined combinations, basic operations of octonionic matrices are given. Later, algebraic structures of octonionic matrix set are obtained. In addition, the modulus structures of the octonionic matrix set on the real, complex and quaternion matrix sets are examined. Bases and dimension of octonionic matrix sets with modulus structure have been found. Finally, special octonionic matrices and transpose, conjugate, trace of octonionic matrices are given.

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INTRODUCTION

Octonions are non-associative algebras. They form the largest normed division algebra. The octonions were discovered independently by Graves and Cayley [1]. The set of octonions can be written in the form:

$$O=\left\{A=a_0e_0+\sum_{j=1}^{7}a_je_j : a_j\in\mathbb{R}, 0\leq j\leq 7\right\}$$

where a_i's are real numbers (coefficients of octonions), e_i's $(0 \leq j \leq 7)$are the octonion units (basis elements of octonions), and $e_0 = +1$ is the multiplicative scalar element. These octonion units satisfy the following properties:

$$\begin{aligned}
&e_0e_j = e_je_0 = e_j,\ \ 1\leq j\leq 7\\
&e_je_k = -\delta_{jk}e_0+f_{jkl}e_l, 1\leq j,k,l\leq 7,\\
&j\neq k\neq l, j\neq 0, k\neq 0, l\neq 0,
\end{aligned}$$

where δ_{jk} is the usual Kronecker delta symbol and f_{jkl} are completely antisymmetric tensor and they are equal to 1 for following combinations [2]:

$$f_{jkl}=+1;\ \forall(jkl)=(123),(471),(257),(165),(624),(543),(736).$$

Dray and Manogue discussed the eigenvalue problem for 2×2 and 3×3 octonionic Hermitian matrices. In both cases, they gave the general solution for real eigenvalues, and they showed there are also solutions with non-real eigenvalues [3]. Dray and Manogue described the use of Mathematica in analyzing octonionic eigenvector problem,

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and in particular its use in proving a generalized orthogonality property for which no other proof is known [4]. The eigenvalue problem of symmetric 3×3 octonionic matrix has been analyzed by Okubo [5]. Gillow-Wiles and Dray showed that any 3-component octonionic vector which is purely imaginary, but not quaternionic, is an eigenvector of a 6-parameter family of Hermitian octonionic matrices, with imaginary eigenvalue equal to the associator of its elements [9]. Serôdio et.al. studied how some operations defined on the octonions change the set of eigenvalues of the matrix obtained if these operations are performed after or before the matrix representation [12]. Octonions and octonionic matrices have applications in fields such as string theory, special relativity and quantum logic.

Tian gave a complete investigation to real matrix representations of octonions, and considered their various applications to octonions as well as matrices of octonions [6]. Daboul and Delbourgo defined a special matrix multiplication among a special subset of 2N×2N matrices, and studied the resulting (non-associative) algebras and their subalgebras. They derived the conditions under which these algebras become alternative non-associative and when they become associative [13].

The determinant of octonionic matrices and its properties were given by Li and Yuan [7]. Nieminen gave two-by-two random matrix theory with matrix representations of octonions [8]. Karataş and Halıcı investigated octonions and their special vector matrix representation. Theye gave some geometrical definitions and properties related with them [10]. Split-type octonion matrix was given by Bektaş, [11].

Octonions

Let us first give some fundamental notions of the octonions. The real octonion A is defined by $A = a_0e_0 + \sum_{i=1}^{7} a_ie_i$, where a_i's are the real number components of the octonions, e_i's $(i = 1,2,\dots,7)$ are the unit octonions basis elements,and $e_0 = +1$ is the scalar element [14]. The multiplication rules of these unit octonion basis elements are given by :

Table 1. The multiplication table of the unit cctonion basis elements

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

The set of octonions is denoted by $\mathbb{O}$. The sum operation on this set is defined as follows:

$$A + B = \sum_{i=0}^{7} (a_i + b_i)e_i.$$

Another operation on the set of octonion is the conjugate operation. The conjugate of the octonion A is denoted by $\overline{A}$ A and is defined as follows:

$$\overline{A} = a_0e_0 - a_1e_1 - a_2e_2 - a_3e_3 - a_4e_4 - a_5e_5 - a_6e_6 - a_7e_7$$
$$= a_0e_0 - \sum_{i=1}^{7} a_ie_i,$$

where $\overline{e_0} = e_0$ and $\overline{e_j} = -e_j$ $(j = 1,\dots,7)$, [15]. Besides that, the octonion A has real part and vectorial part. They are called the real (S_A), and vectorial (V_A) parts of the octonion A, [15, 16]:

$$S_A = \frac{1}{2}\left(A + \overline{A}\right) = a_0e_0, \quad V_A = \frac{1}{2}\left(A - \overline{A}\right) = \sum_{i=1}^{7} a_ie_i.$$

Thus, an octonion A can be written by $A = S_A + V_A$. The multiplication of $A, B \in O$,is defined by

$$A \times B = S_AS_B - g(V_A, V_B) + S_AV_B + S_BV_A + V_A \wedge V_B, \quad (1)$$

where g

$$g: O \times O \to \mathbb{R}, g(A,B) = \frac{1}{2}\left(A \times \overline{B} + B \times \overline{A}\right) = \sum_{i=0}^{7} a_ib_i,$$

is symmetric, non-degenerate real-valued bilinear form and is called the octonionic inner product.

If $A + \overline{A} = 0$, then the octonion A is called the spatial (pure) octonion. The norm of the octonion A is denoted by

$$\|A\| = \sqrt{A \times \overline{A}} = \sqrt{\sum_{i=0}^{7} a_i^2}.$$

If $\|A_0\| = 1$, then A_0 is called the unit octonion [15]-[17]. The inverse of an octonion is defined by

$$A^{-1} = \frac{\overline{A}}{\|A\|^2}, \ A \neq 0.$$

If A and B octonions, then $(B \times A^{-1}) \times A = B$ or $A^{-1} \times (A \times B) = B$, [20].

Some New Properties of Octonionic Matrices

In this Section, we will investigate some new properties of octonion matrices. We can list some of these new properties as follows:

1) Real, complex and quaternion coefficient matrices representations of octonionic matrices.
2) Basic operations on octonionic matrices.

3) Conjugate, transpose, conjugate transpose, inverse and trace of octonionic matrices.
4) Algebraic structures of the set of octonionic matrices.
5) Special defined octonionic matrix and their properties.

Definition 1 Octonion matrix is defined by $\overset{\frown}{A} = [\,\overset{\frown}{a}_{rs}]$, where $\overset{\frown}{a}_{rs} = \sum_{i=0}^{7} a_{rs}{}^{i} e_i \in \mathbb{O}, a_{rs}{}^{i} \in \mathbb{R}$, $(1 \le r \le m,\ 1 \le s \le n)$. The set of octonionic matrices is denoted by $M_{m\times n}(\mathbb{O})$. If $m = n$, the set of square octonionic matrices is denoted by $M_n(\mathbb{O})$, [6].

Definition 2 Let $\overset{\frown}{A} = [\overset{\frown}{a}_{rs}]$ and $\overset{\frown}{B} = [\overset{\frown}{b}_{rs}] \in M_{m\times n}(\mathbb{O})$ $(1 \le r \le m,\ 1 \le s \le n)$ be two octonionic matrices, then the product of two octonionic matrices given as follows:

$$\overset{\frown}{A} \oplus \overset{\frown}{B} = [\overset{\frown}{a}_{rs}] + [\overset{\frown}{b}_{rs}] = [\overset{\frown}{a}_{rs} + \overset{\frown}{b}_{rs}] \in M_{m\times n}(\mathbb{O}).$$

Thus, we get

$$\oplus : M_{m\times n}(\mathbb{O}) \times M_{m\times n}(\mathbb{O}) \to M_{m\times n}(\mathbb{O})$$

$$(\overset{\frown}{A}, \overset{\frown}{B}) \mapsto \overset{\frown}{A} \oplus \overset{\frown}{B} = [\overset{\frown}{a}_{rs} + \overset{\frown}{b}_{rs}] = \begin{bmatrix} \overset{\frown}{a}_{11} + \overset{\frown}{b}_{11} & \overset{\frown}{a}_{12} + \overset{\frown}{b}_{12} & \cdots & \overset{\frown}{a}_{1n} + \overset{\frown}{b}_{1n} \\ \overset{\frown}{a}_{21} + \overset{\frown}{b}_{21} & \overset{\frown}{a}_{22} + \overset{\frown}{b}_{22} & \cdots & \overset{\frown}{a}_{2n} + \overset{\frown}{b}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overset{\frown}{a}_{m1} + \overset{\frown}{b}_{m1} & \overset{\frown}{a}_{m2} + \overset{\frown}{b}_{m2} & \cdots & \overset{\frown}{a}_{mn} + \overset{\frown}{b}_{mn} \end{bmatrix}_{m\times n}.$$

Definition 3 Let $k \in \mathbb{R}$ and $\overset{\frown}{A} = [\overset{\frown}{a}_{rs}] \in M_{m\times n}(\mathbb{O})$ be an octonionic matrix, then the multiplication of a real number and an octonionic matrix defined as follows

$$\odot : \mathbb{R} \times M_{m\times n}(\mathbb{O}) \to M_{m\times n}(\mathbb{O})$$
$$(k, \overset{\frown}{A}) \mapsto k \odot \overset{\frown}{A} = [k\overset{\frown}{a}_{rs}]_{m\times n}.$$

Definition 4 Let $\overset{\frown}{A} = [\overset{\frown}{a}_{rs}] = [\sum_{i=0}^{7} a_{rs}{}^{i} e_i] = [a_{rs}{}^{0}] + [a_{rs}{}^{1}]e_1 + \ldots + [a_{rs}{}^{7}]e_7$ be an octonionic matrix. The octonionic matrix is written in real combination as

$$\overset{\frown}{A} = \sum_{i=0}^{7} A_i e_i$$

where $A_0 = [a_{rs}{}^{0}], A_1 = [a_{rs}{}^{1}], \ldots, A_7 = [a_{rs}{}^{7}] \in M_{m\times n}(\mathbb{R})$ $(1 \le r \le m,\ 1 \le s \le n)$.

Definition 5 Let $\overset{\frown}{A} = [\overset{\frown}{a}_{rs}] = \sum_{i=0}^{7} A_i e_i \in M_{m\times n}(\mathbb{O})$ be an octonionic matrix. The octonionic matrix as a combination of four complex matrices is written as

$$\overset{\frown}{A} = A_0 + A_1e_1 + (A_2 + A_3e_1)e_2 + (A_4 + A_5e_1)e_4 + (A_6 - A_7e_1)e_6 = \hat{A}_1 + \hat{A}_2e_2 + \hat{A}_3e_4 + \hat{A}_4e_6$$

where $\hat{A}_1 = A_0 + A_1e_1, \hat{A}_2 = A_2 + A_3e_1, \hat{A}_3 = A_4 + A_5e_1, \hat{A}_4 = A_6 - A_7e_1 \in M_{m\times n}(\mathbb{C})$.

Definition 6 Let $\overset{\frown}{A} = \sum_{i=0}^{7} A_i e_i \in M_{m\times n}(\mathbb{O})$ be an octonionic matrix. The octonionic matrix as a combination of two quaternionic matrices is written as

$$\overset{\frown}{A} = \sum_{i=0}^{3} A_i e_i + (\sum_{i=4}^{7} A_i e_{i-4})e_4 = \tilde{A}_1 + \tilde{A}_2 e_4$$

where $\tilde{A}_1 = \sum_{i=0}^{3} A_i e_i, \tilde{A}_2 = \sum_{i=4}^{7} A_i e_{i-4} \in M_{m\times n}(\mathbb{H})$.

Definition 7 Let $\overset{\frown}{A} = [\overset{\frown}{a}_{rs}], \overset{\frown}{B} = [\overset{\frown}{b}_{rs}] \in M_{m\times n}(\mathbb{O})$ be given. If $\overset{\frown}{a}_{rs} = \overset{\frown}{b}_{rs}$, then it is called $\overset{\frown}{A}$ is equal to $\overset{\frown}{B}$, and written $\overset{\frown}{A} = \overset{\frown}{B}$.

Remark 1 Let $\overset{\frown}{A} = \sum_{i=0}^{7} A_i e_i$ and $\overset{\frown}{B} = \sum_{i=0}^{7} B_i e_i \in M_{m\times n}(\mathbb{O})$ be given. $\overset{\frown}{A} = \overset{\frown}{B} \Leftrightarrow \forall i = 0,1,\ldots,7, A_i = B_i$.

Remark 2 Let $\overset{\frown}{A} = \hat{A}_1 + \hat{A}_2e_2 + \hat{A}_3e_4 + \hat{A}_4e_6$ and $\overset{\frown}{B} = \hat{B}_1 + \hat{B}_2e_2 + \hat{B}_3e_4 + \hat{B}_4e_6 \in M_{m\times n}(\mathbb{O})$ be given. $\overset{\frown}{A} = \overset{\frown}{B} \Leftrightarrow \forall i = 1,\ldots,4, \hat{A}_i = \hat{B}_i$.

Remark 3 Let $\overset{\frown}{A} = \tilde{A}_1 + \tilde{A}_2e_4$ and $\overset{\frown}{B} = \tilde{B}_1 + \tilde{B}_2e_4 \in M_{m\times n}(\mathbb{O})$ be given. $\overset{\frown}{A} = \overset{\frown}{B} \Leftrightarrow \forall i = 1,2, \tilde{A}_i = \tilde{B}_i$.

Definition 8 Let $\overset{\frown}{A}, \overset{\frown}{B} \in M_{m\times n}(\mathbb{O})$ $(1 \le r \le m,\ 1 \le s \le n)$ be two octonionic matrices. The addition operation of octonionic matrices as follows:

$$\overset{\frown}{A} + \overset{\frown}{B} = [\overset{\frown}{a}_{rs}] + [\overset{\frown}{b}_{rs}] = [\overset{\frown}{a}_{rs} + \overset{\frown}{b}_{rs}],$$

$$\overset{\frown}{A} + \overset{\frown}{B} = (\sum_{i=0}^{7} A_i + B_i)e_i,$$

$$\overset{\frown}{A} + \overset{\frown}{B} = (\hat{A}_1 + \hat{B}_1) + (\hat{A}_2 + \hat{B}_2)e_2 + (\hat{A}_3 + \hat{B}_3)e_4 + (\hat{A}_4 + \hat{B}_4)e_6,$$

and

$$\overset{\frown}{A} + \overset{\frown}{B} = (\tilde{A}_1 + \tilde{B}_1) + (\tilde{A}_2 + \tilde{B}_2)e_4 \in M_{m\times n}(\mathbb{O}).$$

The Properties of the Addition Operation of the Octonionic Matrices

Let $\overset{\frown}{A}, \overset{\frown}{B}, \overset{\frown}{C} \in M_{m\times n}(\mathbb{O})$ and $\overset{\frown}{0} \in M_{m\times n}(\mathbb{O})$, then the following properties are satisfied

1) $(\overset{\frown}{A} +, \overset{\frown}{B}) + \overset{\frown}{C} = \overset{\frown}{A} + (, \overset{\frown}{B} + \overset{\frown}{C})$,
2) $\overset{\frown}{A} + \overset{\frown}{0} = \overset{\frown}{0} + \overset{\frown}{A} = \overset{\frown}{A}$,
3) There is only one $\overset{\frown}{A}' = -\overset{\frown}{A} \in M_{m\times n}(\mathbb{O})$ such that $\overset{\frown}{A} + \overset{\frown}{A}' = \overset{\frown}{A}' + \overset{\frown}{A} = \overset{\frown}{0}$,
4) $\overset{\frown}{A} + \overset{\frown}{B} = \overset{\frown}{B} + \overset{\frown}{A}$.

Corollarly 1 $(M_{m\times n}(\mathbb{O}), +)$ is an Abelian group.

Definition 9 Let $\overset{\frown}{A} = [\overset{\frown}{a}_{rs}] \in M_{m\times n}(\mathbb{O}), \overset{\frown}{B} = [\overset{\frown}{b}_{st}] \in M_{n\times p}(\mathbb{O})$ $(1 \le r \le m, 1 \le s \le n, 1 \le t \le p)$ be two octonionic matrices. The multiplication of the octonionic matrices defined by

$$\overset{\frown}{A}\,\overset{\frown}{B} = [\sum_{s=1}^{n} \overset{\frown}{a}_{rs} \times \overset{\frown}{b}_{st}] \in M_{m\times p}(\mathbb{O})$$

The multiplication operation can be written as follows:

$$\cdot: M_{m\times n}(\mathbb{O}) \times M_{n\times p}(\mathbb{O}) \to M_{m\times p}(\mathbb{O})$$

$$(\overset{\frown}{A}, \overset{\frown}{B}) \mapsto \overset{\frown}{A}\,\overset{\frown}{B} = \left[\sum_{s=1}^{n} \overset{\frown}{a}_{rs} \times \overset{\frown}{b}_{st}\right]_{m\times p}.$$

If $\sum_{s=1}^{n} \overset{\frown}{a}_{rs} \times \overset{\frown}{b}_{st} = \overset{\frown}{c}_{rt}$ $(1 \le r \le m, 1 \le t \le p)$, then we get

$$\begin{aligned}
\overset{\frown}{c}_{11} &= \textstyle\sum_{s=1}^{n} \overset{\frown}{a}_{1s} \times \overset{\frown}{b}_{s1} = \overset{\frown}{a}_{11} \times \overset{\frown}{b}_{11} + \overset{\frown}{a}_{12} \times \overset{\frown}{b}_{21} + \cdots + \overset{\frown}{a}_{1n} \times \overset{\frown}{b}_{n1}\\
\overset{\frown}{c}_{12} &= \textstyle\sum_{s=1}^{n} \overset{\frown}{a}_{1s} \times \overset{\frown}{b}_{s2} = \overset{\frown}{a}_{11} \times \overset{\frown}{b}_{12} + \overset{\frown}{a}_{12} \times \overset{\frown}{b}_{22} + \cdots + \overset{\frown}{a}_{1n} \times \overset{\frown}{b}_{n2}\\
&\vdots\\
\overset{\frown}{c}_{1p} &= \textstyle\sum_{s=1}^{n} \overset{\frown}{a}_{1s} \times \overset{\frown}{b}_{sp} = \overset{\frown}{a}_{11} \times \overset{\frown}{b}_{1p} + \overset{\frown}{a}_{12} \times \overset{\frown}{b}_{2p} + \cdots + \overset{\frown}{a}_{1n} \times \overset{\frown}{b}_{np}.
\end{aligned}$$

On the other hand, we can write the multiplication as follows:

$$\overset{\frown}{A}\,\overset{\frown}{B} = \begin{bmatrix} \overset{\frown}{a}_{11} & \overset{\frown}{a}_{12} & \cdots & \overset{\frown}{a}_{1n}\\ \overset{\frown}{a}_{21} & \overset{\frown}{a}_{22} & \cdots & \overset{\frown}{a}_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \overset{\frown}{a}_{m1} & \overset{\frown}{a}_{m2} & \cdots & \overset{\frown}{a}_{mn} \end{bmatrix} \begin{bmatrix} \overset{\frown}{b}_{11} & \overset{\frown}{b}_{12} & \cdots & \overset{\frown}{b}_{1p}\\ \overset{\frown}{b}_{21} & \overset{\frown}{b}_{22} & \cdots & \overset{\frown}{b}_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ \overset{\frown}{b}_{n1} & \overset{\frown}{b}_{n2} & \cdots & \overset{\frown}{b}_{np} \end{bmatrix} = \left[\overset{\frown}{c}_{rt}\right] = \overset{\frown}{C}.$$

Remark 4 Let $\overset{\frown}{A} = \sum_{i=0}^{7} A_i e_i \in M_{m\times n}(\mathbb{O})$, $\overset{\frown}{B} = \sum_{i=0}^{7} B_i e_i \in M_{n\times p}(\mathbb{O})$ be two octonionic matrices. Here, $A_i \in M_{m\times n}(\mathbb{R})$ and $B_i \in M_{n\times p}(\mathbb{R})$ $(\forall i = 0,1,\ldots,7)$ real matrices. Then, the multiplication operation in terms of the real combination of two octonionic matrices can be defined as follows:

$$\begin{aligned}
\overset{\frown}{A}\overset{\frown}{B} &= \left(\textstyle\sum_{i=0}^{7} A_i e_i\right)\left(\textstyle\sum_{i=0}^{7} B_i e_i\right)\\
&= A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3 - A_4B_4 - A_5B_5 - A_6B_6 - A_7B_7\\
&+(A_0B_1 + A_1B_0 + A_2B_3 - A_3B_2 + A_4B_5 - A_5B_4 + A_7B_6 - A_6B_7)e_1\\
&+(A_0B_2 + A_2B_0 + A_3B_1 - A_1B_3 + A_4B_6 - A_6B_4 + A_5B_7 - A_7B_5)e_2\\
&+(A_0B_3 + A_3B_0 + A_1B_2 - A_2B_1 + A_4B_7 - A_7B_4 + A_6B_5 - A_5B_6)e_3\\
&+(A_0B_4 + A_4B_0 + A_5B_1 - A_1B_5 + A_6B_2 - A_2B_6 + A_7B_3 - A_3B_7)e_4\\
&+(A_0B_5 + A_5B_0 + A_1B_4 - A_4B_1 + A_3B_6 - A_6B_3 + A_7B_2 - A_2B_7)e_5\\
&+(A_0B_6 + A_6B_0 + A_1B_7 - A_7B_1 + A_2B_4 - A_4B_2 + A_5B_3 - A_3B_5)e_6\\
&+(A_0B_7 + A_7B_0 + A_2B_5 - A_5B_2 + A_3B_4 - A_4B_3 + A_6B_1 - A_1B_6)e_7.
\end{aligned}$$

Lemma 1 Let $\hat{A} = A_1 + A_2 e_1 \in M_n(\mathbb{C})$ and $A_1, A_2 \in M_n(\mathbb{R})$ be given. Then, the following statements hold

1) $e_4\hat{A} = \overline{\hat{A}}e_4$,
2) $e_6\hat{A} = \overline{\hat{A}}e_6$,
3) $e_1(e_1e_4) = (e_1e_1)e_4 = -e_4$,
4) $e_1(e_1e_6) = (e_1e_1)e_6 = -e_6$.

Proof.

1) $e_4\hat{A} = e_4(A_1 + A_2e_1) = e_4A_1 + e_4(A_2e_1) = A_1e_4 + A_2(e_4e_1) = A_1e_4 - A_2e_5$ and $\overline{\hat{A}}e_4 = (A_1 - A_2e_1)e_4 = A_1e_4 - A_2(e_1e_4) = A_1e_4 - A_2e_5$. Hence, we get $e_4\hat{A} = \overline{\hat{A}}e_4$. Similarly, other statements can be proved.

Lemma 2 Let $\tilde{A} = A_1 + A_2e_1 + A_3e_2 + A_4e_3 \in M_n(\mathbb{H})$ and $A_1, A_2, A_3, A_4 \in M_n(\mathbb{R})$ be given. Then, the following statements hold

1) $\tilde{A}e_4 = e_4\overline{\tilde{A}}$,
2) $(\tilde{A}e_4)e_4 = \tilde{A}(e_4e_4) = -\tilde{A}$,
3) $\tilde{A}(\tilde{B}e_4) \neq (\tilde{A}\tilde{B})e_4$, in general
4) $\tilde{A}(\tilde{B}e_4) \neq (\tilde{B}\tilde{A})e_4$, in general.

Proof.
1)

$$\begin{aligned}
\tilde{A}e_4 &= (A_1 + A_2e_1 + A_3e_2 + A_4e_3)e_4\\
&= A_1e_4 + A_2(e_1e_4) + A_3(e_2e_4) + A_4(e_3e_4)\\
&= A_1e_4 + A_2e_5 + A_3e_6 + A_4e_7
\end{aligned}$$

and

$$\begin{aligned}
e_4\overline{\tilde{A}} &= e_4(A_1 - A_2e_1 - A_3e_2 - A_4e_3)\\
&= A_1e_4 - A_2(e_4e_1) - A_3(e_4e_2) - A_4(e_4e_3)\\
&= A_1e_4 - A_2(-e_5) - A_3(-e6) - A_4 - (e_7)\\
&= A_1e_4 + A_2e_5 + A_3e_6 + A_4e_7.
\end{aligned}$$

4)

$$\begin{aligned}
\tilde{A}(\tilde{B}e_4) &= (A_1 + A_2e_1 + A_3e_2 + A_4e_3)(B_1e_4 + B_2e_5 + B_3e_6 + B_4e_7)\\
&= A_1B_1e_4 + A_1B_2e_5 + A_1B_3e_6 + A_1B_4e_7\\
&+A_2B_1(e_1e_4) + A_2B_2(e_1e_5) + A_2B_3(e_1e_6) + A_2B_4(e_1e_7)\\
&+A_3B_1(e_2e_4) + A_3B_2(e_2e_5) + A_3B_3(e_2e_6) + A_3B_4(e_2e_7)\\
&+A_4B_1(e_3e_4) + A_4B_2(e_3e_5) + A_4B_3(e_3e_6) + A_4B_4(e_3e_7)\\
&= (A_1B_1 - A_2B_2 - A_3B_3 - A_4B_4)e_4 + (A_1B_2 + A_2B_1 - A_3B_4 + A_4B_3)e_5\\
&+(A_1B_3 + A_2B_4 + A_3B_1 - A_4B_2)e_6 + (A_1B_4 - A_2B_3 + A_3B_2 + A_4B_1)e_7
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{B}\tilde{A})e_4 &= [(B_1 + B_2e_1 + B_3e_2 + B_4e_3)(A_1 + A_2e_1 + A_3e_2 + A_4e_3)]e_4\\
&= (B_1A_1 + B_1A_2e_1 + B_1A_3e_2 + B_1A_4e_3)e_4 + (B_2A_1e_1 - B_2A_2 + B_2A_3e_3 - B_2A_4e_2)e_4\\
&+= (B_3A_1e_2 - B_3A_2e_3 - B_3A_3 + B_3A_4e_1)e_4 + (B_4A_1e_3 + B_4A_2e_2 - B_4A_3e_1 - B_4A_4)e_4\\
&= (B_1A_1 - B_2A_2 - B_3A_3 - B_4A_4)e_4 + (B_1A_2 + B_2A_1 - B_3A_4 - B_4A_3)e_5\\
&+(B_1A_3 - B_2A_4 + B_3A_1 + B_4B_2)e_6 + (B_1A_4 + B_2A_3 - B_3A_2 + B_4A_1)e_7
\end{aligned}$$

Similarly, other statements can be proved.

Example 1 Let $\overset{\frown}{A} = \begin{bmatrix} e_7 & e_2\\ e_6 & e_5 \end{bmatrix}$, $\overset{\frown}{B} = \begin{bmatrix} e_4\\ e_3 \end{bmatrix}$ be two octonionic matrices. Let's find the product of these matrices and calculate it in terms of real, complex and quaternion combinations. According to the definition of multiplication $\overset{\frown}{A}\,\overset{\frown}{B} = \left[\sum_{s=1}^{2} \overset{\frown}{a}_{rs} \times \overset{\frown}{b}_{st}\right]_{m\times p}$, we get

$$\begin{aligned}
\overset{\frown}{A}\,\overset{\frown}{B} = \textstyle\sum_{s=1}^{2} \mathrm{a_{rs}} \times \mathrm{b_{st}} &= \mathrm{a_{r1}} \times \mathrm{b_{1t}} + \mathrm{a_{r2}} \times \mathrm{b_{2t}}\\
&= \begin{bmatrix} e_1 - e_3\\ -e_3 + e_6\\ \end{bmatrix}.
\end{aligned}$$

The multiplication operation in terms of the real combination of two octonionic matrices A and B can be defined as follows:

$$\overset{\frown}{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} e_2 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} e_5 + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} e_6 + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} e_7$$

and

$$\overset{\frown}{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e_3 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_4$$

Hence, we get

$$\overset{\frown}{A}\,\overset{\frown}{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_1 - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e_2 - \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_3 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e_6$$

The multiplication operation in terms of the complex combination of two octonionic matrices A and B can be defined as follows:

$$\begin{aligned} \overset{\frown}{A}\,\overset{\frown}{B} &= (\hat{A}_1 + \hat{A}_2 e_2 + \hat{A}_3 e_4 + \hat{A}_4 e_6)(\hat{B}_1 + \hat{B}_2 e_2 + \hat{B}_3 e_4 + \hat{B}_4 e_6) \\ &= \hat{A}_1\hat{B}_1 - \hat{A}_2\overline{\hat{B}_2} + (\hat{A}_1\hat{B}_2 + \hat{A}_2\overline{\hat{B}_1})e_2 + (\hat{A}_1\hat{B}_3)e_4 + (\hat{A}_1\hat{B}_4)e_6 \\ &+(\hat{A}_2 e_2)(\hat{B}_3 e_4) + (\hat{A}_2 e_2)(\hat{B}_4 e_6) + (\hat{A}_3 e_4)(\hat{B}_1) + (\hat{A}_3 e_4)(\hat{B}_2 e_2) \\ &+(\hat{A}_3 e_4)(\hat{B}_3 e_4) + (\hat{A}_3 e_4)(\hat{B}_4 e_6) + (\hat{A}_4 e_6)(\hat{B}_1) + (\hat{A}_4 e_6)(\hat{B}_2 e_2) \\ &+(\hat{A}_4 e_6)(\hat{B}_3 e_4) + (\hat{A}_4 e_6)(\hat{B}_4 e_6) \\ &= \begin{bmatrix} e_1 - e_3 \\ -e_3 + e_6 \end{bmatrix}. \end{aligned}$$

Similarly, the product of $\overset{\frown}{A}$ and $\overset{\frown}{B}$ can be calculated in terms of quaternion combinations.

Theorem 1(The Properties of the Product of the Octonionic Matrices)

1) Let $\overset{\frown}{A} \in M_{m\times n}(\mathbb{H})$, $\overset{\frown}{B} \in M_{n\times o}(\mathbb{H})$ and $\overset{\frown}{C} \in M_{o\times p}(\mathbb{H})$ be given. $(\overset{\frown}{A}\overset{\frown}{B})\,\overset{\frown}{C} \neq \overset{\frown}{A}(\overset{\frown}{B}\overset{\frown}{C})$, in general,

3) Let $\overset{\frown}{A} \in M_{m\times n}(\mathbb{H})$ and $\overset{\frown}{B}, \overset{\frown}{C} \in M_{p\times m}(\mathbb{H})$ be given. $(\overset{\frown}{B} + \overset{\frown}{C})\,\overset{\frown}{A} \neq \overset{\frown}{B}\,\overset{\frown}{A} + \overset{\frown}{C}\,\overset{\frown}{A}$, in general,

4) Let $\overset{\frown}{A}, \overset{\frown}{B} \in M_n(\mathbb{H})$ be given. $\overset{\frown}{A}\overset{\frown}{B} \neq \overset{\frown}{B}\,\overset{\frown}{A}$, in general.

Corollarly 2 $(M_{m\times n}(\mathbb{O}), +, \cdot)$ is not a ring.

Conjugate of Octonionic Matrices

Definition 10 Let $\overset{\frown}{A} = [\overset{\frown}{a}_{rs}] \in M_{m\times n}(\mathbb{O})$ $(1 \leq r \leq m,\ 1 \leq s \leq n)$ be an octonionic matrix. Then, the conjugate of octonionic matrix $\overset{\frown}{A}$ is defined as $\overline{\overset{\frown}{A}} = [\overline{\overset{\frown}{a}_{rs}}] = [\overline{\sum_{i=0}^{7} a_{rs}{}^{i} e_i}] = [a_{rs}{}^{0} - \sum_{i=1}^{7} a_{rs}{}^{i} e_i]$.

Remark 5 The conjugate of octonionic matrix $\overset{\frown}{A} = \sum_{i=0}^{7} A_i e_i \in M_{m\times n}(\mathbb{O})$ is $\overline{\overset{\frown}{A}} = A_0 - \sum_{i=1}^{7} A_i e_i$.

Remark 6 The conjugate of octonionic matrix $\overset{\frown}{A} = \hat{A}_1 + \hat{A}_2 e_2 + \hat{A}_3 e_4 + \hat{A}_4 e_6 \in M_{m\times n}(\mathbb{O})$ is $\overline{\overset{\frown}{A}} = \overline{\hat{A}_1} - \hat{A}_2 e_2 - \hat{A}_3 e_4 - \hat{A}_4 e_6$.

Proof. Let $\hat{A}_1 = A_0 + A_2 e_1, \hat{A}_2 = A_2 + A_3 e_1, \hat{A}_3 = A_4 + A_5 e_1, \hat{A}_4 = A_6 - A_7 e_1 \in M_{m\times n}(\mathbb{C})$, then we get

$$\begin{aligned} \overline{\overset{\frown}{A}} &= \overline{(\hat{A}_1 + \hat{A}_2 e_2 + \hat{A}_3 e_4 + \hat{A}_4 e_6)} \\ &= \overline{((A_0 + A_2 e_1) + (A_2 + A_3 e_1)e_2 + (A_4 + A_5 e_1)e_4 + (A_6 - A_7 e_1)e_6)} \\ &= \overline{(\textstyle\sum_{i=0}^{7} A_i e_i)} \\ &= A_0 - \textstyle\sum_{i=1}^{7} A_i e_i \\ &= (A_0 - A_1 e_1) - (A_2 + A_3 e_1)e_2 - (A_4 + A_5 e_1)e_4 - (A_6 - A_7 e_1)e_6 \\ &= \overline{\hat{A}_1} - \hat{A}_2 e_2 - \hat{A}_3 e_4 - \hat{A}_4 e_6. \end{aligned}$$

Remark 7 The conjugate of octonionic matrix $\overset{\frown}{A} = \tilde{A}_1 + \tilde{A}_2 e_4 \in M_{m\times n}(\mathbb{O})$ is $\overline{\overset{\frown}{A}} = \overline{\tilde{A}_1} - \tilde{A}_2 e_4$, where $\tilde{A}_1 = \sum_{i=0}^{3} A_i e_i$ and $\tilde{A}_2 = \sum_{i=4}^{7} A_i e_{i-4}$.

Theroem 2 (The Properties of Conjugate)

Let $\overset{\frown}{A}, \overset{\frown}{B} \in M_{m\times n}(\mathbb{O})$, $\overset{\frown}{C} \in M_{n\times r}(\mathbb{O})$, $K \in \mathbb{O}$ be given. Then, the following properties satisfied.

1) $\overline{\left(\overline{\overset{\frown}{A}}\right)} = \overset{\frown}{A}$,

2) $\overline{K\overset{\frown}{A}} \neq \overline{\overset{\frown}{A}K}$,

3) $\overline{(\overset{\frown}{A} + \overset{\frown}{B})} = \overline{\overset{\frown}{A}} + \overline{\overset{\frown}{B}}$,

4) $\overline{\overset{\frown}{A}\overset{\frown}{C}} \neq \overline{\overset{\frown}{A}}\ \overline{\overset{\frown}{C}}$ (in general)

Proof. (1), (2), and (3) can be easily shown. Now we will prove one condition of (4):

4)

$$\begin{aligned} \overset{\frown}{A}\overset{\frown}{C} &= (\textstyle\sum_{i=0}^{7} A_i e_i)(\sum_{i=0}^{7} C_i e_i) \\ &= A_0 C_0 - \textstyle\sum_{i=1}^{7} A_i C_i e_i + (A_0 C_1 + A_1 C_0 + \ldots + A_7 C_6 - A_6 C_7)e_1 \\ &+(A_0 C_2 + A_2 C_0 + \ldots + A_5 C_7 - A_7 C_5)e_2 \\ &+\ldots+(A_0 C_7 + A_7 C_0 + \ldots + A_6 C_1 - A_1 C_6)e_7 \end{aligned}$$

and

$$\begin{aligned} \overline{\overset{\frown}{A}\overset{\frown}{C}} &= A_0 C_0 - \sum_{i=1}^{7} A_i C_i e_i - (A_0 C_1 + A_1 C_0 + \ldots + A_7 C_6 - A_6 C_7)e_1 \\ &= -(A_0 C_2 + A_2 C_0 + \ldots + A_5 C_7 - A_7 C_5)e_2 - \ldots - (A_0 C_7 + A_7 C_0 + \ldots + A_6 C_1 - A_1 C_6)e_7. \end{aligned}$$

On the other hand,

$$\begin{aligned} \overline{\overset{\frown}{A}}\ \overline{\overset{\frown}{C}} &= (A_0 - \textstyle\sum_{i=1}^{7} A_i e_i)(B_0 - \sum_{i=1}^{7} C_i e_i) \\ &= A_0 C_0 - \textstyle\sum_{i=1}^{7} A_i C_i e_i - (A_0 C_1 + A_1 C_0 + \ldots + A_7 C_6 - A_6 C_7)e_1 \\ &-(A_0 C_2 + A_2 C_0 + \ldots + A_5 C_7 - A_7 C_5)e_2 \\ &-\ldots-(A_0 C_7 + A_7 C_0 + \ldots + A_6 C_1 - A_1 C_6)e_7. \end{aligned}$$

Then, ve get $\overline{\overset{\frown}{A}\overset{\frown}{C}} \neq \overline{\overset{\frown}{A}}\ \overline{\overset{\frown}{C}}$.

Corollarly 3 For $\overline{\overset{\frown}{A}} = \sum_{i=0}^{7} A_i e_i \in M_n(\mathbb{O})$, we have $(\overline{\overset{\frown}{A}})^2 \neq \overline{(\overset{\frown}{A}^2)}$.

Transpose of Octonionic Matrices

Definition 11 Let $\overset{\frown}{A} = [\overset{\frown}{a}_{rs}] \in M_{m\times n}(\mathbb{O})$ $(1 \leq r \leq m,\ 1 \leq s \leq n)$ be an octonionic matrix. The transpose of octonionic matrix $\overset{\frown}{A}$ is defined as $\overset{\frown}{A}{}^t = [\overset{\frown}{a}_{sr}] \in M_{n\times m}(\mathbb{O})$.

Remark 8 Let $\widehat{A} = \sum_{i=0}^{7} A_i e_i \in M_{m\times n}(\mathbb{O})$ be given. Then, the transpose of octonionic matrix $\widehat{A}$ is $\widehat{A}$ is $\widehat{A}^t = \sum_{i=0}^{7} A_i^t e_i$.

Proof. For $\widehat{A} = [\widehat{a}_{rs}] \in M_{m\times n}(\mathbb{H})$ and $\widehat{a}_{rs} = \sum_{i=0}^{7} a_{rs}^i e_i$, then we get

$$\begin{aligned}\widehat{A}^t &= \textstyle\sum_{i=0}^{7} a_{sr}^i e_i \\ &= \textstyle\sum_{i=0}^{7} A_i^t e_i\end{aligned}$$

Remark 9 Let $\widehat{A} = \hat{A}_1 + \hat{A}_2 e_2 + \hat{A}_3 e_4 + \hat{A}_4 e_6 \in M_{m\times n}(\mathbb{O})$ be given. Then, the transpose of octonionic matrix $\widehat{A}$ is $\widehat{A}^t = \hat{A}_1^t + \hat{A}_2^t e_2 + \hat{A}_3^t e_4 + \hat{A}_4^t e_6$.

Proof. Let $\hat{A}_1 = A_0 + A_1 e_1,\ \hat{A}_2 = A_2 + A_3 e_1\ \hat{A}_3 = A_4 + A_5 e_1\ \hat{A}_4 = A_6 - A_7 e_1 \in M_{m\times n}(\mathbb{C})$ be given. Then we have

$$\begin{aligned}\widehat{A}^t &= (\hat{A}_1 + \hat{A}_2 e_2 + \hat{A}_3 e_4 + \hat{A}_4 e_6)^t \\ &= ((A_0 + A_1 e_1) + (A_2 + A_3 e_1) e_2 + (A_4 + A_5 e_1) e_4 + (A_6 - A_7 e_1) e_6)^t \\ &= (A_0^t + A_1^t e_1) + (A_2^t + A_3^t e_1) e_2 + (A_4^t + A_5^t e_1) e_4 + (A_6^t - A_7^t e_1) e_6 \\ &= \hat{A}_1^t + \hat{A}_2^t e_2 + \hat{A}_3^t e_4 + \hat{A}_4^t e_6.\end{aligned}$$

Remark 10 Let $\widehat{A} = \tilde{A}_1 + \tilde{A}_2 e_4 \in M_{m\times n}(\mathbb{O})$ be given. Then, the transpose of octonionic matrix $\widehat{A}$ is $\widehat{A}^t = \hat{A}_1^t + \hat{A}_2^t e_4$.

Theroem 3 (The Properties of Transpose Operation)

Let $\widehat{A}, \widehat{B} \in M_{m\times n}(\mathbb{O}), \widehat{C} \in M_{n\times r}(\mathbb{O})$ and $K \in \mathbb{O}$ be given. Then, the following properties are hold:

1) $(\widehat{A} + \widehat{B})^t = \widehat{A}^t + \widehat{B}^t$,

2) $(\widehat{A}^t)^t = \widehat{A}$,

3) $(K\widehat{A})^t = K\widehat{A}^t$,

4) $(\widehat{A}\widehat{C}) \neq \widehat{C}^t \widehat{A}^t$ (in general).

Proof. (1), (2), and (4) can be easily shown. Now we will prove one condition of (3):

3) Let $\widehat{A} = [\widehat{a}_{rs}] \in M_{m\times n}(\mathbb{O})$ and $K \in \mathbb{O}$ be given. Then we get

$$(K\widehat{A})^t = (K[\widehat{a}_{rs}])^t = [K\widehat{a}_{rs}]^t = [K\widehat{a}_{sr}] = K[\widehat{a}_{sr}] = K\widehat{A}^t.$$

Conjugate Transpose of Octonionic Matrices

Definition 12 Let $\widehat{A} = [\widehat{a}_{rs}] \in M_{m\times n}(\mathbb{O})$ $(1 \le r \le m,\ 1 \le s \le n)$ be given. The conjugate transpose of octonionic matrix $\widehat{A}$ is $\left(\overline{\widehat{A}}\right)^t \in M_{n\times m}(\mathbb{O})$.

Remark 11 Let $\widehat{A} = \sum_{i=0}^{7} A_i e_i \in M_{m\times n}(\mathbb{O})$ be given. Then, the conjugate transpose of octonionic matrix $\widehat{A}$ is $\left(\overline{\widehat{A}}\right)^t = A_0^t - \sum_{i=1}^{7} A_i^t e_i$.

Remark 12 Let $\widehat{A} = \hat{A}_1 + \hat{A}_2 e_2 + \hat{A}_3 e_4 + \hat{A}_4 e_6 \in M_{m\times n}(\mathbb{O})$ be given. Then, the conjugate transpose of octonionic matrix $\widehat{A}$ is $\left(\overline{\widehat{A}}\right)^t = \left(\overline{\hat{A}}_1\right)^t - \hat{A}_2^t e_2 + \hat{A}_3^t e_4 + \hat{A}_4^t e_6$.

Proof. Let $\overline{\overline{A}} = \overline{\hat{A}}_1 - \hat{A}_2 e_2 - \hat{A}_3 e_4 - \hat{A}_4 e_6$ be conjugate of octonionic matrix $\widehat{A}$, then we get

$$\begin{aligned}\left(\overline{\widehat{A}}\right)^t &= \left(\overline{\hat{A}}_1 - \hat{A}_2 e_2 - \hat{A}_3 e_4 - \hat{A}_4 e_6\right)^t \\ &= (\overline{\hat{A}}_1)^t - \hat{A}_2^t e_2 + \hat{A}_3^t e_4 + \hat{A}_4^t e_6.\end{aligned}$$

Remark 13 Let Let $\widehat{A} = \tilde{A}_1 + \tilde{A}_2 e_4 \in M_{m\times n}(\mathbb{O})$ be given. Then, the conjugate transpose of octonionic matrix $\widehat{A}$ is $\left(\overline{\widehat{A}}\right)^t = \left(\overline{\tilde{A}}_1\right)^t - \tilde{A}_2^t e_4$.

Thereom 4 (The Properties of Conjugate Transpose Operation)

Let $\widehat{A}, \widehat{B} \in M_{m\times n}(\mathbb{O}), \forall \widehat{C} \in M_{n\times r}(\mathbb{O})$ and $K \in \mathbb{O}$ be given. Then, the following properties are hold:

1) $\left(\overline{K\widehat{A}}\right)^t = \left(\overline{\widehat{A}}\right)^t \overline{K}$,

2) $\left(\overline{\widehat{A}}\right)^t = \overline{\widehat{A}^t}$,

3) $\left(\overline{\widehat{A} + \widehat{B}}\right)^t = \left(\overline{\widehat{A}}\right)^t + \left(\overline{\widehat{B}}\right)^t$,

4) $\left(\overline{\widehat{A}\widehat{C}}\right)^t \neq \left(\overline{\widehat{C}}\right)^t \left(\overline{\widehat{A}}\right)^t$.

Proof. (2), (3), and (4) can be easily shown. Now we will prove one condition of (1):

1) From the 2-th property of the conjugate operation, $\overline{(K\widehat{A})} = \overline{\widehat{A}}\,\overline{K}$ So, we get

$$\left(\overline{K\widehat{A}}\right)^t = (\overline{\widehat{A}}\,\overline{K})^t = \left(\overline{\widehat{A}}\right)^t \overline{K}.$$

Trace of Octonionic Matrices

Definition 13 Let $\widehat{A} = [\widehat{a}_{rs}] = \sum_{i=0}^{7} A_i e_i \in M_n(\mathbb{O})$ be given. The sum of the elements of the octonionic square matrix A on the principal diagonal is called the trace of the matrix $\widehat{A}$ and denoted by $iz(\widehat{A})$.

Remark 14 The trace of octonionic matrix $\widehat{A} = \sum_{i=0}^{7} A_i e_i \in M_n(\mathbb{H})$ is

$$\begin{aligned}iz(\widehat{A}) &= \textstyle\sum_{r=1}^{n} \widehat{a}_{rr} = \sum_{r=1}^{n} (\sum_{r=0}^{7} a_{rr}^i) \\ &= \textstyle\sum_{r=1}^{n} (a_{rr}^0 + \ldots + a_{rr}^7) = \sum_{r=1}^{n} a_{rr}^0 + \ldots + \sum_{r=1}^{n} a_{rr}^7\end{aligned}$$

or

$$iz(\widehat{A}) = iz(A_0) + iz(A_1)e_1 + \ldots + iz(A_7)e_i$$
$$= \sum_{i=0}^{7} iz(A_i)e_i.$$

Theorem 5 (The Properties of Trace of Octonionic Matrix)

Let $\widehat{A} = \sum_{i=0}^{7} A_i e_i$, $\widehat{B} = \sum_{i=0}^{7} B_i e_i \in M_n(\mathbb{O})$ ve $K \in \mathbb{O}$ be given. Then, thefollowing properties are hold:

1) $iz(\widehat{A} + \widehat{B}) = iz(\widehat{A}) + iz(\widehat{B})$,
2) $iz(\widehat{A}\widehat{B}) \neq iz(\widehat{A})iz(\widehat{B})$,
3) $iz(\widehat{A}\widehat{B}) \neq iz(\widehat{B}\widehat{A})$ (in general),
4) $iz(\widehat{A}K) = iz(\widehat{A})K$ or $iz(K\widehat{B}) = Kiz(\widehat{A})$,
5) $iz(\widehat{A^t}) = iz(\widehat{A})$.

Proof. (1), (3), and (4) can be easily shown. Now we will prove one condition of (2):

2) We know that

$$\begin{aligned}\widehat{A}\widehat{B} &= A_0B_0 - \textstyle\sum_{i=1}^{7} A_iB_i + (A_0B_1 + A_1B_0 + \ldots + A_7B_6 - A_6B_7)e_1 \\ &+ \ldots + (A_0B_7 + A_7B_0 + \ldots + A_6B_1 - A_1B_6)e_7).\end{aligned}$$

From here, we get

$$\begin{aligned}iz(\widehat{A}\widehat{B}) &= iz(A_0B_0) - iz(\textstyle\sum_{i=1}^{7} A_iB_i) \\ &+ iz(A_0B_1)e_1 + iz(A_1B_0)e_1 + \ldots + iz(A_7B_6)e_1 - iz(A_6B_7)e_1 \\ &+ \ldots + iz(A_0B_7)e_7 + iz(A_7B_0)e_7 + \ldots + iz(A_6B_1)e_7 - iz(A_1B_6)e_7.\end{aligned} \tag{2}$$

Besides, we know that $iz(\widehat{A}) = \sum_{i=0}^{7} iz(A_i)e_i$ and $iz(\widehat{B}) = \sum_{i=0}^{7} iz(B_i)e_i$.

On the other hand, we find that

$$\begin{aligned}iz(\widehat{A})iz(\widehat{B}) &= iz(A_0)iz(B_0) - \textstyle\sum_{i=1}^{7} iz(A_i)iz(B_i) \\ &+ iz(A_0)iz(B_1)e_1 + iz(A_1)iz(B_0)e_1 + \ldots + iz(A_7)iz(B_6)e_1 - iz(A_6)iz(B_7)e_1 \\ &+ \ldots + iz(A_0)iz(B_7)e_7 + iz(A_7)iz(B_0)e_7 + \ldots + iz(A_6)iz(B_1)e_7 - iz(A_1)iz(B_6)e_7\end{aligned} \tag{3}$$

The properties of the trace function of real matrix is $iz(AB) \neq iz(A)iz(B)$. So, the equations (2) and (3) is not equal. Finally, we get

$$iz(\widehat{A}\widehat{B}) \neq iz(\widehat{A})iz(\widehat{B}).$$

Inverse of Octnionic Matrices

Definition 14 Let $\widehat{A}\widehat{B} = I_n$ for any octonionic matrix $\widehat{A} \in M_n(\mathbb{O})$, then the square octonionic matrix $\widehat{B}$ is called right inverse of $\widehat{A}$. If $\widehat{B}\widehat{A} = I_n$, then the square octonionic matrix $\widehat{B}$ is called left inverse of $\widehat{A}$.

Theorem 6 Let $\widehat{A} \in M_{m\times n}(\mathbb{O})$, $\widehat{B} \in M_{n\times r}(\mathbb{O})$.

1) If $\widehat{A}$ is right(left) inverse matrix, then $\left(\left(\overline{\widehat{A}}\right)^t\right)^{-1} = \left(\overline{\widehat{A^{-1}}}\right)^t$,

2) If $\widehat{A}$ is right(left) inverse matrix, then $\left(\overline{\widehat{A}}\right)^{-1} \neq \overline{\widehat{A^{-1}}}$ (in general),

3) If $\widehat{A}$ is right(left) inverse matrix, then $\left(\widehat{A^t}\right)^{-1} \neq \left(\widehat{A^{-1}}\right)^t$ (in general).

Proof.

1) Let $\widehat{A}$ is right(left) inverse matrix, then $\widehat{A}\widehat{A}^{-1} = I$ or $\widehat{A}^{-1}\widehat{A} = I$. If we take conjugate transpose of both side of the equation $\widehat{A}^{-1}\widehat{A} = I$, we get

$$\left(\overline{\widehat{A^{-1}}}\right)^t \left(\overline{\widehat{A}}\right)^t = \bar{I}^t = I.$$

Similarly, if we take conjugate transpose of both side of the equation $\widehat{A}\widehat{A}^{-1} = I$, we get

$$\left(\overline{\widehat{A}}\right)^t \left(\overline{\widehat{A^{-1}}}\right)^t = \bar{I}^t = I.$$

Thus, we have $\left(\left(\overline{\widehat{A}}\right)^t\right)^{-1} = \left(\overline{\widehat{A^{-1}}}\right)^t$

Special Defined Octonionic Matrices

Definition 15 Let $\widehat{A} = [\widehat{a}_{rs}] \in M_n(\mathbb{O})$ be square octonion matrix. If $\widehat{A}\left(\overline{\widehat{A}}\right)^t = \left(\overline{\widehat{A}}\right)^t\widehat{A}$, then $\widehat{A}$ is called normal matrix.

Definition 16 Let $\widehat{A} = [\widehat{a}_{rs}] \in M_n(\mathbb{O})$ be square octonion matrix. If If $\left(\overline{\widehat{A}}\right)^t = \widehat{A}$, the $\widehat{A}$ is called Hermitien matrix.

Remark 15 Let $\widehat{A} = \sum_{i=0}^{7} A_i e_i \in M_n(\mathbb{O})$, $\forall A_i \in M_n(\mathbb{R})$, $0 \leq i \leq 7$ be a square octonionic matrix. If $\widehat{A}$ is Hermitien matrix, then, $A_i = A_i^t$, $0 \leq i \leq 7$.

Proof. Let's first we find octonioni matrix $\left(\overline{\widehat{A}}\right)^t$:

$$\left(\overline{\widehat{A}}\right)^t = (\overline{\textstyle\sum_{i=0}^{7} A_i e_i})^t = \left(\overline{\textstyle\sum_{i=0}^{7} A_i e_i}\right)^t = A_0^t - \textstyle\sum_{i=1}^{7} A_i e_i^t$$

From the definition of the Hermitien matrix, we get $\left(\overline{\widehat{A}}\right)^t = \widehat{A}$ and $A_i = A_i^t$, $0 \leq i \leq 7$.

Remark 16 Let $\widehat{A} = \hat{A}_1 + \hat{A}_2 e_2 + \hat{A}_3 e_4 + \hat{A}_4 e_6$, $\in M_n(\mathbb{O})$, $\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{A}_4 \in M_n(\mathbb{C})$ be given.

If $\tilde{A}$ is Hermitien matrix, then $\left(\hat{A}_1\right)^t = \hat{A}_1,\ \hat{A}_2^t = -\hat{A}_2,\ \hat{A}_3^t = -\hat{A}_3,\ \hat{A}_4^t = -\hat{A}_4$.

Algebraic Structures of the Set of Octonionic Matrices

Vector space structure on $\mathbb{R}$

Definition 17 Let $k \in \mathbb{R}$ and $\overset{\frown}{A} = [\overset{\frown}{a}_{rs}] \in M_{m\times n}(\mathbb{O})$ be a real octonionic matrix. Then the multiplication of real number and a real octonionic matrix is defined as

$$k \odot \overset{\frown}{A} = k \odot [\overset{\frown}{a}_{rs}] = [k\overset{\frown}{a}_{rs}]_{m\times n}.$$

Let $k, l \in \mathbb{R}$ and $\overset{\frown}{A}, \overset{\frown}{B} \in M_{m\times n}(\mathbb{O})$ and $1_{\mathbb{R}}$ be unit element for $\mathbb{R}$.Then the multiplication of real number and a real octonionic matrix provides the following properties:

1) $k \odot \left(\overset{\frown}{A} + \overset{\frown}{B}\right)=\left(k \odot \overset{\frown}{A}\right)+\left(k \odot \overset{\frown}{B}\right)$,
2)$(k + l) \odot \overset{\frown}{A} = \left(k \odot \overset{\frown}{A}\right)+\left(l \odot \overset{\frown}{A}\right)$,
3)$(kl) \odot \overset{\frown}{A}=k \odot \left(l \odot \overset{\frown}{A}\right)$,
4) $1_{\mathbb{R}} \odot \overset{\frown}{A}=\overset{\frown}{A}$.

Corollary 4 $\{M_{m\times n}(\mathbb{O}),\oplus, \mathbb{R}, +,\cdot,\odot\}$ is a vector space.

The basis of the vector space $\{M_{m\times n}(\mathbb{O}),\oplus, \mathbb{R}, +,\cdot,\odot\}$ is the following set S_1.

$$S_1 = \left\{ \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{m\times n}, \begin{bmatrix} e_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{m\times n}, \dots, \begin{bmatrix} e_7 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{m\times n}, \dots, \right.$$
$$\begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{m\times n}, \begin{bmatrix} 0 & 0 & \dots & e_1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{m\times n}, \dots, \begin{bmatrix} 0 & 0 & \dots & e_7 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{m\times n}, \dots,$$
$$\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}_{m\times n}, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_1 & 0 & \dots & 0 \end{bmatrix}_{m\times n}, \dots, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_7 & 0 & \dots & 0 \end{bmatrix}_{m\times n}, \dots,$$
$$\left. \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{m\times n}, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_1 \end{bmatrix}_{m\times n}, \dots, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_7 \end{bmatrix}_{m\times n} \right\}$$

Corollary 5 Let the set $M_{m\times n}(\mathbb{O})$ be given.
1) The system S_1 is linear independent,
2) $M_{m\times n}(\mathbb{O}) = Sp\{S_1\}$.

Because of the above corollarly, the matrix system S_1 is standart basis of the vector space $M_{m\times n}(\mathbb{O})$. So, $boy(M_{m\times n}(\mathbb{O})) = 8mn$.

Left (right) module structure on $\mathbb{C}$

Definition 18 Let $\overset{\frown}{A} = [\overset{\frown}{a}_{rs}] \in M_{m\times n}(\mathbb{O})$ be a octonionic matrix and $z \in \mathbb{C}$ be a complex. The left or right multiplication of a complex number and a octonion matrix is defined as $z\overset{\frown}{A} = z[\overset{\frown}{a}_{rs}] = [z\overset{\frown}{a}_{rs}] \in M_{m\times n}(\mathbb{O})$ or $\overset{\frown}{A}z = [\overset{\frown}{a}_{rs}]z = [\overset{\frown}{a}_{rs}z] \in M_{m\times n}(\mathbb{O})$. Left and right scalar multiplication operations are scaler operations. These operations are not equal. So, in general $z\overset{\frown}{A} \neq \overset{\frown}{A}z$. Let $\overset{\frown}{A} \in M_{m\times n}(\mathbb{O}),\ \overset{\frown}{B} \in M_{n\times r}(\mathbb{O})$ be octonionic matrices, and $z, t \in \mathbb{C}$ be quaternions. Then, we get

1) $\left(z\overset{\frown}{A}\right)\overset{\frown}{B} \neq z(\overset{\frown}{A}\overset{\frown}{B})$,
2) $\left(\overset{\frown}{A}z\right)\overset{\frown}{B} \neq \overset{\frown}{A}(z\overset{\frown}{B})$,
3) $(zt)\overset{\frown}{A} \neq z(t\overset{\frown}{A})$.

Corollary 6 $\{M_{m\times n}(\mathbb{O}),\oplus, \mathbb{C}, +,\cdot,\odot\}$ is not left (right) modules.

Left (right) module structure on $\mathbb{H}$

Definition 19 Let $\overset{\frown}{A} = [\overset{\frown}{a}_{rs}] \in M_{m\times n}(\mathbb{O})$ be a octonionic matrix and $p \in \mathbb{H}$ be a quaternion. The left or right multiplication of a quaternion and a octonion matrix is defined as $p\overset{\frown}{A} = p[\overset{\frown}{a}_{rs}] = [p\overset{\frown}{a}_{rs}] \in M_{m\times n}(\mathbb{O})$ or $\overset{\frown}{A}p = [\overset{\frown}{a}_{rs}]p = [\overset{\frown}{a}_{rs}p] \in M_{m\times n}(\mathbb{O})$. Left and right scalar multiplication operations are scaler operations. These operations are not equal. So, in general $p\overset{\frown}{A} \neq \overset{\frown}{A}p$. Let $\overset{\frown}{A} \in M_{m\times n}(\mathbb{O}),\ \overset{\frown}{B} \in M_{n\times r}(\mathbb{O})$ be octonionic matrices, and $p, r \in \mathbb{H}$ be quaternions. Then, we get

1) $\left(p\overset{\frown}{A}\right)\overset{\frown}{B} \neq p(\overset{\frown}{A}\overset{\frown}{B})$,
2) $\left(\overset{\frown}{A}p\right)\overset{\frown}{B} \neq \overset{\frown}{A}(p\overset{\frown}{B})$,
3) $(pr)\overset{\frown}{A} \neq p(r\overset{\frown}{A})$.

Corollary 7 $\{M_{m\times n}(\mathbb{O}),\oplus, \mathbb{H}, +,\cdot,\odot\}$ is not left (right) modules.

Left (right) module structure on $M_n(\mathbb{R})$

Definition 20 Let $A = [a_{rs}] \in M_n(\mathbb{R})$ and $\overset{\frown}{A} = [\overset{\frown}{a}_{rs}] \in M_n(\mathbb{O})$. Then, the left product of a square real matrix and an square octonionic matrix is defined as

$$A\Theta\overset{\frown}{A} = [a_{rs}]_{n\times n}\Theta[\overset{\frown}{a}_{rs}]_{n\times n} = \left[\textstyle\sum_{s=1}^{n} a_{rs}\overset{\frown}{a}_{rs}\right]_{n\times n}.$$

Let $A, B \in M_n(\mathbb{R})$ and $\overset{\frown}{A}, \overset{\frown}{B} \in M_{m\times n}(\mathbb{O})$ and $1_{M_n(\mathbb{R})}$ be unit element of $M_n(\mathbb{R})$. Then the multiplication of real matrix and a real octonionic matrix provides the following properties:

1) $A \odot \left(\overset{\frown}{A} + \overset{\frown}{B}\right)=\left(A \odot \overset{\frown}{A}\right)+\left(B \odot \overset{\frown}{B}\right)$,
2)$(A + B) \odot \overset{\frown}{A} = \left(k \odot \overset{\frown}{A}\right)+\left(l \odot \overset{\frown}{A}\right)$,
3)$(AB) \odot \overset{\frown}{A}=A \odot \left(B \odot \overset{\frown}{A}\right)$,
4) $1_{M_n(\mathbb{R})} \odot \overset{\frown}{A}=\overset{\frown}{A}$.

Corollary 8 $\{M_n(\mathbb{O}),\oplus, M_n(\mathbb{R}),+,\cdot,\Theta\}$ is a vector space.

The basis of the vector space $\{M_n(\mathbb{O}),\oplus, M_n(\mathbb{R}),+,\cdot,\Theta\}$ is the following set S_2.

$$S_2=\left\{=\left\{\begin{bmatrix}1&0&\dots&0\\0&0&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&0\end{bmatrix}_{n\times n},\begin{bmatrix}e_1&0&\dots&0\\0&e_1&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&e_1\end{bmatrix}_{n\times n},\dots,\begin{bmatrix}e_7&0&\dots&0\\0&e_7&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&e_7\end{bmatrix}_{n\times n}\right\}\right\}$$

Corollary 9 Let the set $M_n(\mathbb{O})$ be given.

1) The system S_2 is linear independent,

2) $M_n(\mathbb{O}) = Sp\{S_2\}$.

Because of the above corollarly, the matrix system S_2 is standart basis of the vector space $M_n(\mathbb{O})$. So, $boy(M_n(\mathbb{O})) = 8$.

Left (right) module structure on $M_n(\mathbb{C})$

Definition 21 Let $\hat{C} = [\hat{c}_{rs}] \in M_n(\mathbb{C})$ and $\overset{\frown}{A} = [\overset{\frown}{a}_{rs}] \in M_n(\mathbb{O})$. Then, the left product of a square complex matrix and an square octonionic matrix is defined as

$$\hat{C}\Theta\overset{\frown}{A} = [\hat{c}_{rs}]_{n\times n}\Theta[\overset{\frown}{a}_{rs}]_{n\times n} = \left[\textstyle\sum_{s=1}^{n} \hat{c}_{rs}\overset{\frown}{a}_{rs}\right]_{n\times n}.$$

Since the octonionic multiplication operation does not provide the associative property, then

$\{M_n(\mathbb{O}),\oplus, M_n(\mathbb{C}),+,\cdot,\Theta\}$ is not a left(right) module.

Similarly, the right product of a square complex matrix and a square octonionic matrix can be defined.

Left (right) module structure on $M_n(\mathbb{H})$

Definition 22 Let $\tilde{A} = [\tilde{a}_{rs}] \in M_n(\mathbb{H})$ and $\overset{\frown}{A} = [\overset{\frown}{a}_{rs}] \in M_n(\mathbb{O})$. Then, the left multiplication of square quaternion matrix and a square octonionic matrix is defined as

$$\tilde{A}\Theta\overset{\frown}{A} = [\tilde{a}_{rs}]_{n\times n}\Theta[\overset{\frown}{a}_{rs}]_{n\times n} = \left[\textstyle\sum_{s=1}^{n} \tilde{a}_{rs}\overset{\frown}{a}_{rs}\right]_{n\times n}.$$

Since the octonionic multiplication operation does not provide the associative property, then

$\{M_n(\mathbb{O}),\oplus, M_n(\mathbb{H}),+,\cdot,\Theta\}$ is not a left (right) module.

Similarly, the right product of a square quaternionic matrix and an square octonionic matrix can be defined.

CONCLUSION

In the first section of study, octonions and their basic algebraic operations has examined. In the second section of study, octonionic coefficient matrices is defined. According to the definition of the octonionic matrices real, complex, quaternionic coefficient octonionic matrix has been obtained. After, addition, multiplication, conjugate, transpose, conjugate transpose and trace operations have been defined for each definition of octonionic matrices. Finally, algebraic structures of set of octonionic matrices have been searched. The objective of this study is to research octonionic matrix and their properties. The octonic matrix concept can also be studied using sedenions.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

[1] Cayley A. XXVIII. On Jacobi's Elliptic functions, in reply to the Rev. Brice Bronwin; and on Quaternions; To the editors of the Philosophical Magazine and Journal. Lond Edinb Dublin Philos Mag J Sci 1845;26:208–211. [CrossRef]

[2] Sabinin LV, Sbitneva L, Shestakov IP. Non-associative Algebra and its Applications. Boca Raton, Florida: CRC Press; 2006. p. 235. [CrossRef]

[3] Dray T, Manogue CA. The octonionic eigenvalue problem. Adv Appl Clifford Algebras 1998;8:341–364. [CrossRef]

[4] Dray T, Manogue CA. Finding octonionic eigenvectors using mathematica. Comput Phys Commun 1998;115:536–547. [CrossRef]

[5] Okubo S. Eigenvalue problem for symmetric 3× 3 octonionic matrix. Adv Appl Clifford Algebras 1999;9:131–176. [CrossRef]

[6] Tian Y. Matrix representations of octonions and their applications. Adv Appl Clifford Algebras 2000;10:61- 90. [CrossRef]

[7] Xingmin L, Hong Y. The determinant of octonionic matrices and its properties. Acta Math. Sinica (Chin Ser) 2008;51:947-954. [Chinese]

[8] Nieminen JM. Two-by-two random matrix theory with matrix representations of octonions. Journal of mathematical physics 2010;51:053510. [CrossRef]

[9] Gillow-Wiles H, Dray, T. Finding 3× 3 hermitian matrices over the octonions with imaginary eigenvalues. Adv Appl Clifford Algebras 2010;20:247–254. [CrossRef]

[10] Karataş A, Halici S. Vector matrix representation of octonions and their geometry. Commun Fac Sci Univ Ank Ser A1 Math Stat 2018;67:161–167. [CrossRef]

[11] Bektaş Ö. Split-type octonion matrix. Math Methods Appl Sci 2019;42:5215–5232. [CrossRef]

[12] Serôdio R, Beites P, Vitória J. Eigenvalues of matrices related to the octonions. 4open 2019;2:16. [CrossRef]

[13] Daboul J, Delbourgo R. Matrix representation of octonions and generalizations. J Math Phys 1999;40:4134–4150. [CrossRef]

[14] Cayley A. Memoir on the Theory of Matrice. Philosophical Transactions of the Royal Society of London 1858;148:17–37. [CrossRef]

[15] Fenn R. Geometry, Springer Undergraduate Mathematics Series. London: Springer; 2007.

[16] Lounesto P. Octonions and triality. Adv Appl Cliff Alg 2001;11:191–213. [CrossRef]

[17] Lounesto P. Clifford Algebras and Spinors. Cambridge, UK: Cambridge University Press; 1997. [CrossRef]

[18] Massey W. Cross products of vectors in higher dimensional euclidean space. Amer Math Monthly 1983;90:697–701. [CrossRef]

[19] Calabi E. Construction and properties of some 6-dimensional almost complex manifolds. Trans Amer Math Soc 1958;87:407–458. [CrossRef]

[20] Ward JP. Quaternions and Cayley Numbers Algebra and Applications. London: Kluwer Academic Publishers; 1997. [CrossRef]

[21] Bektaş Ö. Oktoniyonik Eğriler ve Karakteristik Özellikleri (doktora tezi). İstanbul: Yıldız Teknik Üniversitesi, Fen Bilimler, Enstitüsü; 2015. [Turkish]