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Research Article

Solutions of multiplicative linear differential equations via the multiplicative power series method

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ABSTRACT

In this study, fundamental concepts of multiplicative analysis is given. Also, definitions of multiplicative Taylor series (MTS), multiplicative Taylor polynomials (MTP), and multiplicative power series (MPS) are given. Solutions of higher-order multiplicative linear differential equations (MLDE) are investigated with the help of the MPS method. Applications of MPS method are done for first, second, and third-order multiplicative ordinary linear differential equations.

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INTRODUCTION

Geometric analysis, which is a non-Newtonian analysis, was first said by Dick Stanley as multiplicative analysis [1]. Addition and subtraction in classical analysis correspond to multiplication and division operations in geometric analysis. This is why geometric analysis is called multiplicative analysis. In the following years, some studies on multiplicative analysis were carried out by Duff Campell [2]. In 2008, Bashirov, Kurpinar and Özyapıcı, defined the basic concepts of multiplicative analysis and some applications were given in [3]. Misirli and Gurefe [4] developed the multiplicative Adams Bashforth-Moulton methods to obtain numerical solutions of multiplicative differential equations. Bashirov and Riza studied multiplicative differentiation for complex-valued functions [5]. Yalcin, Celik and Gokdogan defined the multiplicative Laplace transform and found the solution of some multiplicative linear differential equations using the multiplicative Laplace transform [6]. Bhat et al. defined the multiplicative Fourier transform and investigated the solution of multiplicative differential equations with the help of the multiplicative Fourier transform [7]. Bhat et al. defined the multiplicative Sumudu transform and found the solution of some multiplicative differential equations using the multiplicative Sumudu transform [8]. Yalçın and Dedeturk presented a multiplicative differential transform method to find the numerical solution of first and second-order multiplicative ordinary differential equations [9]. Some studies presented in recent years have shown that multiplicative analysis is a different alternative

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to classical analysis in some problems encountered in science and engineering. For more details see [4, 6, 9-20].

Çakmak and Başar [21] defined the non-Newtonian real numbers, non-Newtonian integers, non-Newtonian absolute value, non-Newtonian distance. In this article, these definitions are adapted to multiplicative calculus which is also a non-Newtonian calculus (with the generator function $\alpha = exp(x)$).

In this work, definitions of multiplicative Taylor series (MTS), multiplicative Taylor polynomials (MTP) and multiplicative power series (MPS) multiplicative power series method (MPSM) are given. The third-order multiplicative linear differential equation (MLDE) is solved via MPSM beside the solutions of first and second-order multiplicative linear differential equations (MLDE). These studies are supported with numerical examples.

MATERIALS AND METHODS

Multiplicative Analysis and Multiplicative Analytic Functions

In this section, we will give some basic definitions and properties of the multiplicative analysis which can be found in [2, 5, 11, 14]. Readers can find some important operations, concepts and theorems that are used here from Bashirov, et al. [3, 11] and also from the article of Yalçın and Dedeturk [20].

Here after we will represent natural numbers starting from zero with $\mathbb{N} = \{0,1,2,...\}$, positive real numbers with $\mathbb{R}^+$ and negative real numbers with $\mathbb{R}^-$. And also for Euclidean distance, we will use the notation $d(x,y) = |y-x|$.

Definition 1. The exponential numbers are defined as

i) $\mathbb{R}_{\exp} = \{e^x \mid x \in \mathbb{R}\}$ is the set of exponential real numbers,

ii) $\mathbb{Z}_{\exp} = \{e^x \mid x \in \mathbb{Z}\}$ is the set of exponential integers,

iii) $\mathbb{R}^+_{\exp} = \{e^x \mid x \in \mathbb{R}^+\}$ is the set of exponential positive real numbers,

iv) $\mathbb{R}^-_{\exp} = \{e^x \mid x \in \mathbb{R}^-\}$is the set of exponential negative real numbers

Definition 2. Exponential arithmetic (geometric arithmetic named by Çakar and Başar) is the arithmetic whose domain is $\mathbb{R}_{\exp}$ and whose operations are defined as follows. For $x, y \in \mathbb{R}_{\exp}$

i) exp-addition: $x \oplus y = \exp[\ln x + \ln y]$

ii) exp-subtraction: $x \ominus y = \exp[\ln x - \ln y]$

iii) exp-multiplication: $x \odot y = \exp[\ln x \cdot \ln y]$

iv) exp-division: $x \oslash y = \exp[\ln x \div \ln y]$

Definition 3. The multiplicative absolute value of an exponential real number x in $A \subset \mathbb{R}_{\exp}$ is defined as

$$|x|_* = e^{|\ln(x)|}.$$

From the definition above we have

$$|x|_* = \begin{cases} x, & x > 1, \\ 1, & x = 1, \\ \dfrac{1}{x}, & 0 < x < 1. \end{cases}$$

Lemma 1. *For $x \in \mathbb{R}_{\exp}$ and $n \in \mathbb{R}$ the following equality holds* $|x^n|_* = |x|_*^{|n|}$

Proof. Let $x \in \mathbb{R}_{\exp}$ and $n \in \mathbb{R}$. Then we have

$$|x^n|_* = e^{|\ln x^n|} = e^{|n \cdot \ln x|} = e^{|\ln x| \cdot |n|} = \left(e^{|\ln x|}\right)^{|n|} = |x|_*^{|n|}$$

Definition 4. The multiplicative distance between two exponential real numbers x and y is defined by

$$d_*(x,y) = |y \ominus x|_*$$

which is equal to

$$d_*(x,y) = \left|\frac{y}{x}\right|_*.$$

Definition 5. Let $A \subset \mathbb{R}$, $f: A \to \mathbb{R}_{\exp}$ be a function and $a \in A$. The multiplicative limit of f at the element $a \in A$is, if it exists, the unique number $L \in \mathbb{R}_{\exp}$ if and only if for every number $\varepsilon > 1$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that $d_*(f(x), L) < \varepsilon$ whenever $0 < d(x,a) < \delta$. And it is represented by

$$\overset{*}{\lim_{x \to a}} f(x) = L$$

Definition 6. Let $A \subset \mathbb{R}$, $f: A \to \mathbb{R}_{\exp}$ be a function and $a \in A$. The function f is said to be multiplicative continuous at the point $a \in A$ if and only if

$$\overset{*}{\lim_{x \to a}} f(x) = f(a).$$

Definition 7. Let $A \subset \mathbb{R}$, $f: A \to \mathbb{R}_{\exp}$ be a function. The *multiplicative derivative* of the function f is given by:

$$\frac{d^* f}{dx}(x) = f^*(x) = \lim_{h \to 0} \left(\frac{f(x+h)}{f(x)}\right)^{1/h}.$$

Assuming that f is a positive function and using properties of the classical derivative, the multiplicative derivative can be written as

$$f^*(x) = \exp(\ln \circ f)'(x)$$

for $(\ln \circ f)(x) = \ln[f(x)]$.

Definition 8. If the multiplicative derivative f^* as a function also has a multiplicative derivative, then the multiplicative derivative of f^* is called the second-order multiplicative derivative of f and it is represented by f^{**}. Similarly, we can define n^{th} order multiplicative derivative of f with the notation $f^{*(n)}$. With n times repetition of the multiplicative differentiation operation, a positive f function has an n^{th} order multiplicative derivative at the point x which is defined as

$$f^{*(n)}(x) = \exp(\ln \circ f)^{(n)}(x).$$

Theorem 2. *If a positive function f is differentiable with the multiplicative derivative at the point x, then it is differentiable in the classical sense and the relation between these two derivatives can be shown as*

$$f'(x) = f(x)\ln f^*(x).$$

Theorem 3. *(Multiplicative Mean Value Theorem) If the function f is continuous on* $[a,b]$ *and is multiplicatively differentiable on* (a,b)*, then there exist* $a < c < b$ *such that*

$$f^*(c) = \left[\frac{f(b)}{f(a)}\right]^{\frac{1}{b-a}}.$$

Definition 9. Let $\{c_k\}_{k\in\mathbb{N}}$ be a positive series such that $c_k > 0$, $\forall k \in \mathbb{N}$. The infinite product

$$\prod_{k=0}^{\infty}(c_k)^{(x-x_0)^k}$$

is called a multiplicative power series centered at the point $x = x_0$

Lemma 4. *Let the multiplicative power series*

$$\prod_{k=0}^{\infty}(c_k)^{(x-x_0)^k}$$

is given, where $\{c_k\}_{k\in\mathbb{N}}$ *is a positive series such that* $c_k > 0$, $\forall k \in \mathbb{N}$. *Then the multiplicative power series is convergent if the power series*

$$\sum_{k=0}^{\infty}[(\ln c_k)\cdot(x-x_0)^k]$$

is convergent.

Proof. Suppose

$$\sum_{k=0}^{\infty}[(\ln c_k)\cdot(x-x_0)^k]$$

is convergent then

$$\exp\sum_{k=0}^{\infty}[(\ln c_k)\cdot(x-x_0)^k]$$

is convergent, too. The last expression is equal to

$$\begin{aligned}\exp\sum_{k=0}^{\infty}[(\ln c_k)\cdot(x-x_0)^k] &= \prod_{k=0}^{\infty}\exp\,[(\ln c_k)\cdot(x-x_0)^k]\\ &= \prod_{k=0}^{\infty}[\exp(\ln c_k)]^{(x-x_0)^k}\\ &= \prod_{k=0}^{\infty}(c_k)^{(x-x_0)^k}\end{aligned}$$

which is the given multiplicative power series. Thus, we have shown the convergence.

Definition 10. [3] Let $f(x)$ be a positive function that has multiplicative derivatives of any order on the open interval (a,b) and let $x_0 \in (a,b)$. Then the multiplicative series

$$\prod_{k=0}^{\infty}\left[f^{*(k)}(x_0)\right]^{\frac{(x-x_0)^k}{k!}} \tag{1}$$

is called the multiplicative Taylor series of $f(x)$ at $x = x_0$

Definition 11. Let $f(x)$ be a positive function that has multiplicative derivatives up to order m on the open interval (a,b) and let $x_0 \in (a,b)$. Then the product

$$\mathcal{P}_m(x) = \prod_{k=0}^{m}\left[f^{*(k)}(x_0)\right]^{\frac{(x-x_0)^k}{k!}}$$

is called the m-th degree multiplicative Taylor polynomial of $f(x)$ at $x = x_0$

Theorem 5. [3] *Let* $x_0 \in (a,b)$ *and* $f:(a,b) \to \mathbb{R}^+$ *be* $m + 1$ *times multiplicative differentiable function on the open interval* (a,b) *and* $f^{*(m)}(x)$ *is multiplicative continuous on the closed interval* $[a,b]$. *Then* $\forall x \in (a,b)$ *there exists some*

$$\xi = \xi(x) \in K = \{\tilde{x} \mid min(x_0, x) < \tilde{x} < max(x_0, x)\}$$

such that

$$f(x) = \prod_{k=0}^{m}\left[f^{*(k)}(x_0)\right]^{\frac{(x-x_0)^k}{k!}}\cdot\left[f^{*(m+1)}(\xi)\right]^{\frac{(x-x_0)^{m+1}}{(m+1)!}} \tag{2}$$

The last term on the right side of equality

$$\left[f^{*(m+1)}(\xi)\right]^{\frac{(x-x_0)^{m+1}}{(m+1)!}}$$

is called the multiplicative truncation error of the multiplicative Taylor polynomial which is m-th degree approximation to the multiplicative Taylor series given in (1).

Theorem 6. *Let* $x_0 \in (a,b)$ *and* $f:(a,b) \to \mathbb{R}^+$ *be* $m + 1$ *times multiplicative differentiable function on the open* *interval (a,b) and $f^{*(m)}(x)$ is multiplicative continuous on the closed interval $[a,b]$. An upper bound for error which is obtained by truncating the Taylor series expansion given in* (1) *by its* $(m+1)$ *th term is given by*

$$U_m(x) = M^{\frac{|x-x_0|^{m+1}}{(m+1)!}}$$

where $M = \max_{x\in K}\left|f^{*(m+1)}(x)\right|_*$.
Thus $f(x)$ is bounded between

$$\frac{\mathcal{P}_m(x)}{U_m(x)} \le f(x) \le U_m(x)\cdot\mathcal{P}_m(x).$$

Proof. Let f has the Taylor series expansion given in (1), then we have

$$f(x) = \prod_{k=0}^{m}\left[f^{*(k)}(x_0)\right]^{\frac{(x-x_0)^k}{k!}}\cdot\left[f^{*(m+1)}(\xi)\right]^{\frac{(x-x_0)^{m+1}}{(m+1)!}}$$

for some

$$\xi = \xi(x) \in K = \{\tilde{x} \mid \min(x_0,x) < \tilde{x} < \max(x_0,x)\}.$$

From here we can write

$$\begin{aligned} f(x) &= \mathcal{P}_m(x)\cdot\left[f^{*(m+1)}(\xi)\right]^{\frac{(x-x_0)^{m+1}}{(m+1)!}} \\ \frac{f(x)}{\mathcal{P}_m(x)} &= \left[f^{*(m+1)}(\xi)\right]^{\frac{(x-x_0)^{m+1}}{(m+1)!}} \end{aligned}$$

Taking the multiplicative absolute value of both sides we have

$$\begin{aligned} \left|\frac{f(x)}{\mathcal{P}_m(x)}\right|_* &= \left|\left[f^{*(m+1)}(\xi)\right]^{\frac{(x-x_0)^{m+1}}{(m+1)!}}\right|_* \\ &= \left|f^{*(m+1)}(\xi)\right|_*^{\left|\frac{(x-x_0)^{m+1}}{(m+1)!}\right|} \\ &\le M^{\frac{|x-x_0|^{m+1}}{(m+1)!}} \end{aligned}$$

where

$$M = \max_{x\in K}\left|f^{*(m+1)}(x)\right|_*$$

Thus the following inequality is valid:

$$\left|\frac{f(x)}{\mathcal{P}_m(x)}\right|_* \le U_m(x)$$

So, we get

$$\frac{\mathcal{P}_m(x)}{U_m(x)} \le f(x) \le U_m(x)\cdot\mathcal{P}_m(x).$$

Definition 12. [20] Let $x_0 \in (a,b)$, $N(x_0) \subset (a,b)$ be a neighborhood of x_0 and $f(x)$ be a positive function defined on (a,b). In this case, $f(x)$ is said to be multiplicative-analytic at x_0 if $f(x)$ can be expressed as a multiplicative series of natural powers of $(x - x_0)$ for all $x \in N(x_0)$. In other words, $f(x)$ can be expressed as follows:

$$f(x) = \prod_{n=0}^{\infty}(c_n)^{(x-x_0)^n},\ (c_n \in \mathbb{R}^+)\ . \tag{3}$$

Also note that, there exists $\delta > 0$ such that this series is convergent for all t satisfying $|x - x_0| < \delta$ and divergent for $|x - x_0| > \delta$. δ is the radius of convergence of the series.

Lemma 7. *The multiplicative derivative of the multiplicative power series defined in* (3) *is*

$$\frac{d^*}{dx}\left[\prod_{n=0}^{\infty}(c_n)^{(x-x_0)^n}\right] = \prod_{n=0}^{\infty}(c_{n+1})^{(n+1)(x-x_0)^n}.$$

Definition 13. Let the function $P: \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be defined by

$$P(r,s) = \begin{cases} 1, & s = 0 \text{ and } r \ge 0, \\ \dfrac{r!}{(r-s)!}, & 0 < s \le r, \\ 0, & 0 \le r < s. \end{cases}$$

Lemma 8. *Suppose function $f(x)$ has multiplicative power series expansion as $f(x) = \prod_{n=0}^{\infty}(c_n)^{x^n}$. Then $f(x)$ has multiplicative derivatives as multiplicative power series shown below.*

$$\begin{aligned} f^*(x) &= \prod_{n=0}^{\infty}(c_{n+1})^{(n+1)x^n}, \\ f^{**}(x) &= \prod_{n=0}^{\infty}(c_{n+2})^{(n+2)(n+1)x^n}, \\ f^{***}(x) &= \prod_{n=0}^{\infty}(c_{n+3})^{(n+3)(n+2)(n+1)x^n}, \\ &\vdots \\ f^{*(j)}(x) &= \prod_{n=0}^{\infty}\left(c_{n+j}\right)^{P(n+j,j)x^n}. \end{aligned}$$

Definition 14. [20] Let $x_0 \in (a,b)$ and the functions $a_k(x)$ be analytic at x_0 for $k \in \mathbb{N}$. In this case, the point $x_0 \in (a,b)$ is said to be a multiplicative-ordinary point of the equation

$$y^{*(n)} \cdot \left(y^{*(n-1)}\right)^{a_{n-1}(x)} \cdot \dots \cdot (y^{*})^{a_1(x)} \cdot y^{a_0(x)} = 1. \quad (4)$$

If a point $x_0 \in (a,b)$is not a multiplicative-ordinary point, then it is said to be multiplicative singular point.

With the theorem below, the existence of a multiplicative power series solution of a multiplication second-order multiplicative homogeneous linear differential equation at a multiplicative-ordinary point is guaranteed.

Theorem 9. [20] *Let $a_1(x)$, $a_0(x)$ be analytic functions at x_0, in other words, they can be expanded as classical Taylor series at a neighborhood $B(x_0, \delta)$ of x_0:*

$$\begin{aligned} a_1(x) &= \sum_{k=0}^{\infty} A_{1,k} (x - x_0)^k, \\ a_0(x) &= \sum_{k=0}^{\infty} A_{0,k} (x - x_0)^k \end{aligned}$$

where $\delta > 0$ is the minimum of the radii of convergence of these series, and let x_0 be a multiplicative-ordinary point of the equation

$$y^{**}(y^{*})^{a_1(x)} y^{a_0(x)} = 1$$

Then, there exists a solution to the equation above as multiplicative power series

$$y = \prod_{n=0}^{\infty} (c_n)^{(x-x_0)^n}$$

in the same neighborhood $B(x_0, \delta)$ of x_0.

Suppose the multiplicative non-homogeneous linear differential equation

$$y^{*(n)} \cdot \left(y^{*(n-1)}\right)^{a_{n-1}(x)} \cdot \dots \cdot (y^{*})^{a_1(x)} \cdot y^{a_0(x)} = f(x)$$

is given. If y_p is a particular solution of this non-homogeneous equation and y_h is the solution of the corresponding homogeneous equation given in (4), then we can write the general solution y as the multiplication of these two solutions, namely

$$y = y_h \cdot y_p$$

Multiplicative Power Series Method for Higher Order Multiplicative Linear Differential Equations

Theorem 10. *Let $x = 0$ be a multiplicative ordinary point of the m-th order multiplicative differential equation*

$$y^{*(m)} \cdot \left(y^{*(m-1)}\right)^{a_{m-1}(x)} \cdot \dots \cdot (y^{*})^{a_1(x)} \cdot y^{a_0(x)} = f(x)$$

and let

$$a_j(x) = \sum_{k=0}^{\infty} \left(A_{j,k}\right) x^k, \ 0 \le j \le m-1$$

are power series of exponents $a_j(x)$, respectively. And also suppose that

$$f(x) = \prod_{n=0}^{\infty} (F_n)^{x^n}$$

is the multiplicative power series of $f(x)$. Then the multiplicative power series solution

$$y(x) = \prod_{n=0}^{\infty} (Y_n)^{x^n}$$

has bases Y_n, $n \ge 0$ which can be calculated by the recurrence relations

$$\left[\prod_{k=0}^{n}\left(\prod_{j=k}^{k+m-1} \left(Y_j\right)^{P(j,j-k)A_{j-k,n-k}}\right)\right] \cdot (Y_{n+m})^{P(n+m,m)} = F_n, \quad (5)$$

for $n \ge 0$ where Y_k, $0 \le k \le m-1$ are given constants.

Proof. We can write the given equation in a more compact form as

$$y^{*(m)} \cdot \prod_{j=0}^{m-1} \left(y^{*(j)}\right)^{a_j(x)} = f(x).$$

Using power series expansion of the functions we have

$$\prod_{n=0}^{\infty} (Y_{n+m})^{P(n+m,m)x^n} \cdot \left\{\prod_{j=0}^{m-1}\left[\prod_{n=0}^{\infty}\left(Y_{n+j}\right)^{P(n+j,j)x^n}\right]^{\sum_{k=0}^{\infty} A_{j,k}x^k}\right\} = \prod_{n=0}^{\infty} (F_n)^{x^n}$$

$$\prod_{n=0}^{\infty} (Y_{n+m})^{P(n+m,m)x^n} \cdot \left\{\prod_{j=0}^{m-1}\left[\prod_{n=0}^{\infty}\left(Y_{n+j}\right)^{P(n+j,j)x^n \sum_{k=0}^{\infty} A_{j,k}x^k}\right]\right\} = \prod_{n=0}^{\infty} (F_n)^{x^n}$$

Changing the order of the products in the set parentheses, we get

$$\prod_{n=0}^{\infty} (Y_{n+m})^{P(n+m,m)x^n} \cdot \left\{\prod_{n=0}^{\infty}\left[\prod_{j=0}^{m-1}\left(Y_{n+j}\right)^{P(n+j,j) \sum_{k=0}^{\infty} A_{j,k}x^{k+n}}\right]\right\} = \prod_{n=0}^{\infty} (F_n)^{x^n}$$

Using the equality

$$\left(Y_{n+j}\right)^{P(n+j,j)\sum_{k=0}^{\infty} A_{j,k}x^{k+n}} = \prod_{k=0}^{\infty} \left(Y_{n+j}\right)^{P(n+j,j)A_{j,k}x^{k+n}} \quad (6)$$

in the last equation (6) we have

$$\prod_{n=0}^{\infty}(Y_{n+m})^{P(n+m,m)x^n}\cdot\left\{\prod_{n=0}^{\infty}\left[\prod_{j=0}^{m-1}\left(\prod_{k=0}^{\infty}(Y_{n+j})^{P(n+j,j)A_{j,k}x^{k+n}}\right)\right]\right\}=\prod_{n=0}^{\infty}(F_n)^{x^n}$$
$$\prod_{n=0}^{\infty}(Y_{n+m})^{P(n+m,m)x^n}\cdot\left\{\prod_{n=0}^{\infty}\left[\prod_{j=0}^{m-1}\prod_{k=0}^{\infty}(Y_{n+j})^{P(n+j,j)A_{j,k}x^{k+n}}\right]\right\}=\prod_{n=0}^{\infty}(F_n)^{x^n}$$

Subsequently, we change the order of the products with indices J and k to write

$$\prod_{n=0}^{\infty}(Y_{n+m})^{P(n+m,m)x^n}\cdot\left\{\prod_{n=0}^{\infty}\left[\prod_{k=0}^{\infty}\prod_{j=0}^{m-1}(Y_{n+j})^{P(n+j,j)A_{j,k}x^{k+n}}\right]\right\}=\prod_{n=0}^{\infty}(F_n)^{x^n}$$
$$\prod_{n=0}^{\infty}(Y_{n+m})^{P(n+m,m)x^n}\cdot\left\{\prod_{n=0}^{\infty}\prod_{k=0}^{\infty}\left(\prod_{j=0}^{m-1}(Y_{n+j})^{P(n+j,j)A_{j,k}x^{k+n}}\right)\right\}=\prod_{n=0}^{\infty}(F_n)^{x^n}$$

Afterward, if we shift the index of the product with index k we have

$$\prod_{n=0}^{\infty}(Y_{n+m})^{P(n+m,m)x^n}\cdot\left\{\prod_{n=0}^{\infty}\prod_{k=n}^{\infty}\left(\prod_{j=0}^{m-1}(Y_{n+j})^{P(n+j,j)A_{j,k-n}x^{k}}\right)\right\}=\prod_{n=0}^{\infty}(F_n)^{x^n}.$$

Now, changing the order of the products with indices n and k we get

$$\prod_{n=0}^{\infty}(Y_{n+m})^{P(n+m,m)x^n}\cdot\left\{\prod_{k=0}^{\infty}\prod_{n=0}^{k}\left(\prod_{j=0}^{m-1}(Y_{n+j})^{P(n+j,j)A_{j,k-n}x^{k}}\right)\right\}=\prod_{n=0}^{\infty}(F_n)^{x^n}\qquad(7)$$

For getting harmony with the index of the first product on the left side, we use the index n instead of k in the first of the products which are in the set parentheses. And also, we use the index k instead of n in the second of the products which are in the set parentheses Thus, the equation (7) can be rewritten as

$$\prod_{n=0}^{\infty}(Y_{n+m})^{P(n+m,m)x^n}\cdot\left\{\prod_{n=0}^{\infty}\prod_{k=0}^{n}\left(\prod_{j=0}^{m-1}(Y_{k+j})^{P(k+j,j)A_{j,n-k}x^{n}}\right)\right\}=\prod_{n=0}^{\infty}(F_n)^{x^n},$$
$$\prod_{n=0}^{\infty}[(Y_{n+m})^{P(n+m,m)}]^{x^n}\cdot\prod_{n=0}^{\infty}\left[\prod_{k=0}^{n}\prod_{j=0}^{m-1}(Y_{k+j})^{P(k+j,j)A_{j,n-k}}\right]^{x^n}=\prod_{n=0}^{\infty}(F_n)^{x^n}$$

We can multiply the two products on the left with the same index n to get

$$\prod_{n=0}^{\infty}\left\{(Y_{n+m})^{P(n+m,m)}\cdot\left[\prod_{k=0}^{n}\left(\prod_{j=0}^{m-1}(Y_{k+j})^{P(k+j,j)A_{j,n-k}}\right)\right]\right\}^{x^n}=\prod_{n=0}^{\infty}(F_n)^{x^n}$$

Finally, we equate the terms of the products of the opposite sides and get the recurrence relation

$$(Y_{n+m})^{P(n+m,m)}\cdot\left[\prod_{k=0}^{n}\left(\prod_{j=0}^{m-1}(Y_{k+j})^{P(k+j,j)A_{j,n-k}}\right)\right]=F_n,n\geq 0$$
$$(Y_{n+m})^{P(n+m,m)}\cdot\left[\prod_{k=0}^{n}\left(\prod_{j=k}^{k+m-1}(Y_j)^{P(j,j-k)A_{j-k,n-k}}\right)\right]=F_n,n\geq 0$$

This ends the proof.

The recurrence relations in (5) for $n\geq 0$ can also be expressed as

$$\prod_{k=0}^{n}Y_k^{\left(\sum_{j=0}^{m-1}P(k,j)A_{j,n-k+j}\right)}\prod_{k=n+1}^{n+m-1}Y_k^{\left(\sum_{j=k-n}^{m-1}P(k,j)A_{j,n-k+j}\right)}\cdot Y_{n+m}^{P(n+m,m)}=F_n$$

Solution of First-Order Multiplicative Linear Differential Equations Via MPSM

Corollary 11. *Let $x = 0$ be the multiplicative ordinary point of the first order multiplicative linear differential equation $(y^*)\cdot y^{a(x)}=f(x)$ and let*

$$a(x)=\sum_{k=0}^{\infty}A_k\,x^k,$$
$$f(x)=\prod_{n=0}^{\infty}(F_n)^{x^n}$$

be the power series of $a(x)$ and $f(x)$, respectively. Then the multiplicative power series solution

$$y(x)=\prod_{n=0}^{\infty}(Y_n)^{x^n}$$

have bases Y_n, $n\geq 0$ which can be calculated by the recurrence relations

$$\left[\prod_{k=0}^{n}(Y_k)^{A_{n-k}}\right]\cdot(Y_{n+1})^{n+1}=F_n,\ n\geq 0\qquad(8)$$

where Y_0 is a given constant.

The bases Y_k of the above corollary can be calculated from the following recurrence relations.

$$\begin{aligned}Y_1\cdot Y_0^{A_0}&=F_0\\ Y_2^2\cdot Y_1^{A_0}\cdot Y_0^{A_1}&=F_1\\ Y_3^3\cdot Y_2^{A_0}\cdot Y_1^{A_1}\cdot Y_0^{A_2}&=F_2\\ &\vdots\\ \left[\prod_{k=0}^{n}(Y_k)^{A_{n-k}}\right]\cdot(Y_{n+1})^{n+1}&=F_n\\ &\vdots\end{aligned}$$

From these equations the bases Y_1, Y_2, Y_3 can be given in terms of Y_0 and F_k, $k\geq 0$ as follows

$$
\begin{aligned}
Y_1 &= F_0 \cdot Y_0^{-A_0} \\
Y_2 &= \left[F_1^{\frac{1}{2}} \cdot F_0^{\frac{-1}{2}A_0}\right] \cdot Y_0^{\frac{1}{2}(A_0^2 - A_1)} \\
Y_3 &= \left[F_2^{\frac{1}{3}} \cdot F_1^{-\frac{1}{2\cdot 3}A_0} \cdot F_0^{\frac{1}{2\cdot 3}A_0^2 - \frac{1}{3}A_1}\right] \cdot Y_0^{\frac{-1}{2\cdot 3}A_0^3 + \frac{1}{2}A_0 \cdot A_1 - \frac{1}{3}A_2}.
\end{aligned}
$$

We see that the bases Y_1, Y_2, ... of the solution are formed with the product of two parts. The ones which are the product of F_k's take a role in forming a particular solution to the non-homogeneous equation, namely y_p. The other parts with the Y_0 for the solution of the corresponding homogeneous equation, namely y_h. So the third-order approximate solution, $\tilde{y}$ can be written as the product of an approximation of the particular solution to the non-homogeneous equation, namely $\tilde{y}_p$ and an approximation of the solution of the corresponding homogeneous equation, namely $\tilde{y}_h$:

$$
\begin{aligned}
y &\approx \tilde{y} = \tilde{y}_h \cdot \tilde{y}_p = \prod_{k=0}^{3} (Y_k)^{x_k} = Y_0 \cdot Y_1^x \cdot Y_2^{x^2} \cdot Y_3^{x^3} \\
\tilde{y} &= \left\{Y_0 \cdot \left(Y_0^{-A_0}\right)^x \cdot \left(Y_0^{\frac{1}{2}(A_0^2 - A_1)}\right)^{x^2} \cdot \left(Y_0^{-\frac{1}{2\cdot 3}A_0^3 + \frac{1}{2}A_0 \cdot A_1 - \frac{1}{3}A_2}\right)^{x^3}\right\} \cdot \\
&\left\{(F_0)^x \cdot \left[F_1^{\frac{1}{2}} \cdot F_0^{\frac{-1}{2}A_0}\right]^{x^2} \cdot \left[F_2^{\frac{1}{3}} \cdot F_1^{-\frac{1}{2\cdot 3}A_0} \cdot F_0^{\frac{1}{2\cdot 3}A_0^2 - \frac{1}{3}A_1}\right]^{x^3}\right\} \\
&= \left\{(Y_0)^{1 - A_0 x + (A_0^2 - A_1)x^2/2 + (-A_0^3 + 3A_0 \cdot A_1 - 2A_2)x^3/6}\right\} \cdot \\
&\left\{(F_0)^{x - A_0 x^2/2 + (A_0^2 - 2A_1)x^3/6} \cdot (F_1)^{x^2/2 - A_0 x^3/6} \cdot (F_2)^{x^3/3}\right\}
\end{aligned}
$$

Example 1. *Suppose the first-order multiplicative linear differential equation*

$$
(y^*) \cdot y^{4x+6} = e^{12x^2 + 26x + 15} \tag{9}
$$

is given. We want to find the fourth-order approximate solution of this equation by the multiplicative power series method.

Solution 1. Here $a(x) = 4x + 6$ which is the power of y, has the Taylor series expansion below

$$
\begin{aligned}
a(x) &= \sum_{n=0}^{\infty} A_n x^n = A_0 + A_1 x + \sum_{n=2}^{\infty} A_n x^n \\
4x + 6 &= 6 + 4x + \sum_{n=2}^{\infty} 0\, x^n.
\end{aligned}
$$

So, we see that the coefficients of the Taylor series are

$$
A_0 = 6, A_1 = 4, A_n = 0, n \geq 2.
$$

And the right-hand side function $f(x) = e^{12x^2 + 26x + 15}$ has the multiplicative Taylor series expansion

$$
\begin{aligned}
f(x) &= \prod_{n=0}^{\infty} (F_n)^{x^n} = F_0 \cdot (F_1)^x \cdot (F_2)^{x^2} \cdot \prod_{n=3}^{\infty} (F_n)^{x^n} \\
e^{12x^2 + 26x + 15} &= e^{15} \cdot (e^{26})^x \cdot (e^{12})^{x^2} \cdot \prod_{n=3}^{\infty} 1^{x^n}
\end{aligned}
$$

Thus, we show that the bases of the multiplicative Taylor series are

$$
F_0 = e^{15}, F_1 = e^{26}, F_2 = e^{12}, F_n = 1, n \geq 3.
$$

$$
y(x) \approx \tilde{y}(x) = \prod_{n=0}^{4} (Y_n)^{x^n}
$$

is to be found. From the recurrence relations in (8), we can calculate Y_k for $k = 1,2,3,4$ as below.

$$
\begin{aligned}
Y_1 \cdot Y_0^{A_0} &= F_0, \\
Y_1 \cdot Y_0^{6} &= e^{15}, \\
Y_1 &= e^{15} \cdot Y_0^{-6},
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
Y_2^2 \cdot Y_1^{A_0} \cdot Y_0^{A_1} &= F_1, \\
Y_2^2 \cdot (e^{15} \cdot Y_0^{-6})^6 \cdot Y_0^4 &= F_1, \\
Y_2^2 \cdot e^{90} \cdot Y_0^{-36} \cdot Y_0^4 &= e^{26}, \\
Y_2 &= e^{-32} \cdot Y_0^{16}
\end{aligned}
$$

and

$$
\begin{aligned}
Y_3^3 \cdot Y_2^{A_0} \cdot Y_1^{A_1} \cdot Y_0^{A_2} &= F_2 \\
Y_3^3 \cdot (e^{-32} \cdot Y_0^{16})^6 \cdot (e^{15} \cdot Y_0^{-6})^4 \cdot Y_0^0 &= e^{12} \\
Y_3^3 \cdot e^{-132} \cdot Y_0^{72} &= e^{12} \\
Y_3 &= e^{48} \cdot Y_0^{-24}
\end{aligned}
$$

and

$$
\begin{aligned}
Y_4^4 \cdot Y_3^{A_0} \cdot Y_2^{A_1} \cdot Y_1^{A_2} \cdot Y_0^{A_3} &= F_3 \\
Y_4^4 \cdot (e^{48} \cdot Y_0^{-24})^6 \cdot (e^{-32} \cdot Y_0^{16})^4 \cdot (Y_1)^0 \cdot (Y_0)^0 &= 1 \\
Y_4^4 \cdot e^{160} \cdot Y_0^{-80} &= 1 \\
Y_4 &= e^{-40} \cdot Y_0^{20}
\end{aligned}
$$

And we get an approximate solution written below

$$
\begin{aligned}
y(x) &\approx \tilde{y}(x) = \prod_{n=0}^{4} (Y_n)^{x^n} = Y_0 \cdot Y_1^x \cdot Y_2^{x^2} \cdot Y_3^{x^3} \cdot Y_4^{x^4} \\
\tilde{y}(x) &= Y_0 \cdot (e^{15} \cdot Y_0^{-6})^x \cdot (e^{-32} \cdot Y_0^{16})^{x^2} \cdot (e^{48} \cdot Y_0^{-24})^{x^3} \cdot (e^{-40} \cdot Y^{20})^{x^4} \\
\tilde{y}(x) &= Y_0^{(1 - 6x + 16x^2 - 24x^3 + 20x^4)} \cdot e^{(15x - 32x^2 + 48x^3 - 40x^4)}.
\end{aligned}
$$

In this approximate solution

$$
\begin{aligned}
\tilde{y}_p &= e^{15x - 32x^2 + 48x^3 - 40x^4}, \\
\tilde{y}_h &= Y_0^{1 - 6x + 16x^2 - 24x^3 + 20x^4}
\end{aligned}
$$

are approximations of particular and homogeneous solutions, respectively. Moreover, if we write $\tilde{y}_p$, the approximation of a particular solution, instead of y in the equation (9) we get

$$\tilde{y}_p = e^{15x-32x^2+48x^3-40x^4}$$
$$\tilde{y}_p^* = e^{15-64x+144x^2-160x^3}$$
$$\tilde{y}_p^{4x+6} = e^{90x-132x^2+160x^3-32x^4}$$
$$\tilde{y}_p^* \cdot \tilde{y}_p^{4x+6} = e^{15+26x+12x^2\%+0x^3-32x^4}$$
$$\tilde{y}_p^* \cdot \tilde{y}_p^{4x+6} \cong e^{12x^2+26x+15}$$

which is an approximation of the right-hand side function of equation (9). Also if we write $\tilde{y}_h$, the approximation of the homogeneous solution, instead of y in the equation (9) we get

$$\tilde{y}_h = Y_0^{1-6x+16x^2-24x^3+20x^4}$$
$$\tilde{y}_h^* = Y_0^{-6+32x-72x^2+80x^3}$$
$$\tilde{y}_h^{4x+6} = Y_0^{+6-32x+72x^2-80x^3-24x^4}$$
$$\tilde{y}_h^* \cdot \tilde{y}_h^{4x+6} = Y_0^{0+0x+0x^2+0x^3-24x^4} \cong Y_0^0$$
$$\tilde{y}_h^* \cdot \tilde{y}_h^{4x+6} \cong 1$$

Solution of 2nd Order Multiplicative Linear Differential Equations Via MPSM

Corollary 12. *Let $x = 0$ be a multiplicative ordinary point of the second-order multiplicative differential equation*

$$y^{**} \cdot (y^*)^{a_1(x)} \cdot y^{a_0(x)} = f(x) \qquad (10)$$

and let

$$a_0(x) = \sum_{k=0}^{\infty} (A_{0,k})\, x^k, a_1(x) = \sum_{k=0}^{\infty} (A_{1,k})\, x^k$$

are power series of $a_0(x)$ and $a_1(x)$, respectively. And also suppose that

$$f(x) = \prod_{n=0}^{\infty} (F_n)^{x^n}$$

is the multiplicative power series of $f(x)$. Then the multiplicative power series solution

$$y(x) = \prod_{n=0}^{\infty} (Y_n)^{x^n}$$

have bases Y_n, $n \geq 0$ which can be calculated by the recurrence relations

$$\left(\prod_{k=0}^{n} (Y_k)^{k(A_{1,n-k+1})+A_{0,k}}\right) \cdot Y_{n+1}^{(n+1)A_{1,0}} \cdot Y_{n+2}^{(n+2)(n+1)} = F_n, n \geq 0 \qquad (11)$$

where Y_0, Y_1 are given constants.

Example 2. *Suppose second order multiplicative non-homogeneous linear differential equation below and the initial values*

$$y^{**} \cdot (y^*)^2 \cdot y^x = e^{14x^2+3x+28}, y(0) = e^3, y^*(0) = e^{14}$$

are given. We want to solve this Cauchy problem with the multiplicative power series method.

Solution 2. $x = 0$ is a multiplicative ordinary point of this equation, since we can expand $a_0(x) = x$, $a_1(x) = 2$ as power series centered at $x = 0$ and also we can expand

$$f(x) = e^{14x^2+3x+28}$$

as multiplicative power series like below

$$\begin{aligned} a_1(x) &= 2 \Rightarrow A_{1,0} = 2, A_{1,n} = 0, n \geq 1 \\ a_0(x) &= x \Rightarrow A_{0,0} = 0, A_{0,1} = 1, A_{0,n} = 0, n \geq 2 \\ f(x) &= e^{14x^2+3x+28} = (e^{28})^1 \cdot (e^3)^x \cdot (e^{14})^{x^2} \\ F_0 &= e^{28}, F_1 = e^3, F_2 = e^{14}, F_n = 1, n \geq 3 \end{aligned}$$

Let

$$y(x) = \prod_{n=0}^{\infty} (Y_n)^{x^n}$$

be the power series solution of $y(x)$. We find Y_0 and Y_1 as below

$$\begin{aligned} y(0) &= e^3 \\ Y_0 \cdot \prod_{n=1}^{\infty} (Y_n)^0 &= e^3 \\ Y_0 &= e^3, \end{aligned} \quad \text{and} \quad \begin{aligned} y^*(0) &= e^{14} \\ \prod_{n=0}^{\infty} (Y_{n+1})^{(n+1)x^n}\Bigg|_{x=0} &= e^{14} \\ Y_1^1 \cdot \prod_{n=1}^{\infty} (Y_{n+1})^0 &= e^{14} \\ Y_1 &= e^{14}. \end{aligned}$$

We will use the relation in equation (11) to find Y_2, Y_3 ... and so on.

i) For $n = 0$, we get

$$\begin{aligned} Y_2^2 \cdot Y_1^{A_{1,0}} \cdot Y_0^{A_{0,0}} &= F_0 \\ Y_2^2 \cdot Y_1^2 \cdot Y_0^0 &= e^{28} \\ Y_2^2 \cdot (e^{14})^2 \cdot Y_0^0 &= e^{28} \\ Y_2 &= 1 \end{aligned}$$

ii) Taking $n = 1$, we have

$$\begin{aligned} Y_3^6 \cdot Y_2^{2A_{1,0}} \cdot Y_1^{A_{1,1}+A_{0,0}} \cdot Y_0^{A_{0,1}} &= F_1 \\ Y_3^6 \cdot 1^{2\cdot 2} \cdot (e^{14})^{0+0} \cdot (e^3)^1 &= e^3 \\ Y_3 &= 1 \end{aligned}$$

iii) For $n = 2$, Y_4 is calculated as

$$\begin{aligned} Y_4^{12} \cdot Y_3^{3A_{1,0}} \cdot Y_2^{2A_{1,1}+A_{0,0}} \cdot Y_1^{A_{1,2}+A_{0,1}} \cdot Y_0^{A_{0,2}} &= F_2 \\ Y_4^{12} \cdot 1^{3\cdot 2} \cdot 1^{2\cdot 0+0} \cdot (e^{14})^{0+1} \cdot (e^3)^0 &= e^{14} \\ Y_4 &= 1 \end{aligned}$$

iv) Letting $n = 3$, we find

$$\begin{aligned} Y_5^{20} \cdot Y_4^{4a_0} \cdot Y_3^{3A_{1,1}+A_{0,0}} \cdot Y_2^{2A_{1,2}+A_{0,1}} \cdot Y_1^{A_{1,3}+A_{0,2}} \cdot Y_0^{A_{0,3}} &= F_3 \\ Y_5^{20} \cdot 1 \cdot 1 \cdot 1 \cdot (e^{14})^0 \cdot (e^3)^0 &= 1 \\ Y_5 &= 1 \end{aligned}$$

For $n \geq 4$, we have

$$Y_{n+2}^{(n+2)(n+1)} \cdot Y_{n+1}^{(n+1)A_{1,0}} \cdot \prod_{k=0}^{n} (Y_{n-k})^{(n-k)(A_{1,k+1})+A_{0,k}} = F_n$$

$$\begin{aligned} F_n &= Y_{n+2}^{(n+2)(n+1)} \cdot Y_{n+1}^{(n+1)A_{1,0}} \cdot \prod_{k=0}^{n} (Y_{n-k})^{(n-k)(A_{1,k+1})+A_{0,k}} \\ F_n &= Y_{n+2}^{(n+2)(n+1)} \cdot Y_{n+1}^{(n+1)A_{1,0}} \cdot (Y_n)^{nA_{1,1}+A_{0,0}} \cdot \\ &\quad (Y_{n-1})^{(n-1)A_{1,2}+A_{0,1}} \cdot \prod_{k=2}^{n} (Y_{n-k})^{(n-k)A_{1,k+1}+A_{0,k}} \\ 1 &= Y_{n+2}^{(n+2)(n+1)} \cdot Y_{n+1}^{2(n+1)} \cdot (Y_n)^0 \cdot (Y_{n-1})^1 \cdot \prod_{k=2}^{n} (Y_{n-k})^0 \\ 1 &= Y_{n+2}^{(n+2)(n+1)} \cdot Y_{n+1}^{2(n+1)} \cdot (Y_{n-1}) \\ Y_{n+2} &= \left[Y_{n-1} \cdot (Y_{n+1})^{2(n+1)}\right]^{-1/[(n+2)(n+1)]} \end{aligned}$$

v) For $n = 4$

$$\begin{aligned} Y_6 &= [Y_3 \cdot (Y_5)^{10}]^{-1/30} \\ Y_6 &= 1 \end{aligned}$$

vi) For $n = 5$

$$\begin{aligned} Y_7 &= [Y_4 \cdot (Y_6)^{12}]^{-1/42} \\ Y_7 &= 1. \end{aligned}$$

Continuing this process, we get

$$Y_n = 1 \text{ for } n \geq 8.$$

Thus, we have

$$\begin{aligned} Y_0 &= e^3, \\ Y_1 &= e^{14}, \\ Y_n &= 1, n \geq 2 \end{aligned}$$

Using these bases, we write the solution as below

$$\begin{aligned} y(x) &= \prod_{n=0}^{\infty} (Y_n)^{x^n} = Y_0 \cdot Y_1^x \cdot \prod_{n=2}^{\infty} (Y_n)^{x^n} \\ y(x) &= e^3 \cdot (e^{14})^x \cdot \prod_{n=2}^{\infty} 1^{x^n} \\ y(x) &= e^{14x+3}. \end{aligned}$$

Solution of 3rd Order Multiplicative Linear Differential Equations Via MPSM

Corollary 13. *Let $x = 0$ be a multiplicative ordinary point of the third-order multiplicative differential equation*

$$y^{***} \cdot (y^{**})^{a_2(x)} \cdot (y^*)^{a_1(x)} \cdot y^{a_0(x)} = f(x)$$

and let

$$a_j(x) = \sum_{k=0}^{\infty} (A_{j,k})\, x^k, 0 \leq j \leq 2$$

are power series of exponents $a_j(x)$, respectively. And also suppose that

$$f(x) = \prod_{n=0}^{\infty} (F_n)^{x^n}$$

is the multiplicative power series of $f(x)$. Then the multiplicative power series solution

$$y(x) = \prod_{n=0}^{\infty} (Y_n)^{x^n}$$

has bases Y_n, $n \geq 0$ which can be calculated by the recurrence relations

$$\left\{\prod_{k=0}^{n} Y_k^{\left(A_{0,n-k}+k\cdot A_{1,n-k+1}+k(k-1)\cdot A_{2,n-k+2}\right)}\right\} \cdot Y_{n+1}^{\left((n+1)A_{1,0}+(n+1)n\cdot A_{2,1}\right)} \cdot Y_{n+2}^{(n+2)(n+1)\cdot A_{2,0}} \cdot Y_{n+3}^{(n+3)(n+2)(n+1)} = F_n \tag{12}$$

for $n \geq 0$where Y_k, $0 \leq k \leq 2$ are given constants.

Example 3. *Suppose the third-order multiplicative non-homogeneous linear differential equation below and the initial values*

$$\begin{cases} y^{***} \cdot (y^{**})^5 \cdot (y^*)^3 \cdot y^2 = e^{8x+10}, \\ y(0) = e^{-1}, y^*(0) = e^4, y^{**}(0) = 1 \end{cases}$$

are given. We want to solve this Cauchy problem with the multiplicative power series method.

Solution 3. $x = 0$ is a multiplicative ordinary point of this equation, since we can expand $a_0(x) = 2$, $a_1(x) = 3$,

$a_2(x) = 5$ as power series centered at $x = 0$ and also we can expand $f(x) = e^{8x+10}$ as multiplicative power series like below

$$\begin{aligned} a_0(x) &= 2 \Rightarrow A_{0,0} = 2, A_{0,n} = 0, n \geq 1 \\ a_1(x) &= 3 \Rightarrow A_{1,0} = 3, A_{1,n} = 0, n \geq 1 \\ a_2(x) &= 5 \Rightarrow A_{2,0} = 5, A_{2,n} = 0, n \geq 1 \\ f(x) &= e^{8x+10} = (e^{10})^1 \cdot (e^8)^x \\ F_0 &= e^{10}, F_1 = e^8, F_n = 1, n \geq 2 \end{aligned}$$

Let $y(x) = \prod_{n=0}^{\infty}(Y_n)^{x^n}$
be the power series solution of $y(x)$. We find Y_0, Y_1 and Y_2 as below

$$\begin{aligned} y(0) &= e^{-1}, \text{ and} \\ Y_0 \cdot \prod_{n=1}^{\infty}(Y_n)^0 &= e^{-1}, \\ Y_0 &= e^{-1}, \end{aligned} \qquad \begin{aligned} y^*(0) &= e^4, \\ \prod_{n=0}^{\infty}(Y_{n+1})^{(n+1)x^n}\Big|_{x=0} &= e^4, \\ Y_1 \cdot \prod_{n=1}^{\infty}(Y_{n+1})^0 &= e^4, \\ Y_1 &= e^4, \end{aligned} \quad \text{and} \quad \begin{aligned} y^{**}(0) &= 1, \\ \prod_{n=0}^{\infty}(Y_{n+2})^{(n+2)(n+1)x^n}\Big|_{x=0} &= 1, \\ Y_2^2 \cdot \prod_{n=0}^{\infty}(Y_{n+2})^{x^n} &= 1, \\ Y_2 &= 1. \end{aligned}$$

We will use the relation

$$(Y_{n+3})^{(n+3)(n+2)(n+1)} \cdot \prod_{k=0}^{n}\left[\prod_{j=k}^{k+2}(Y_j)^{P(j,j-k)A_{j-k,n-k}}\right] = F_n,\ n \geq 0$$

to find Y_{n+3} for $n \geq 0$.

i) For $n = 0$, we get

$$\begin{aligned} Y_0^{A_{0,0}} \cdot Y_1^{A_{1,0}} \cdot Y_2^{2A_{2,0}} \cdot Y_3^6 &= F_0 \\ (e^{-1})^2 \cdot (e^4)^3 \cdot 1^{10} \cdot Y_3^6 &= e^{10} \\ Y_3 &= 1 \end{aligned}$$

ii) Taking $n = 1$, we have

$$\begin{aligned} Y_0^{A_{0,1}} \cdot Y_1^{A_{0,0}+A_{1,1}} \cdot Y_2^{2A_{1,0}+2A_{2,1}} \cdot Y_3^{6A_{2,0}} \cdot Y_4^{24} &= F_1 \\ (e^{-1})^0 \cdot (e^4)^{2+0} \cdot 1^{2\cdot3+2\cdot0} \cdot 1^6 \cdot Y_4^{24} &= e^8 \\ Y_4 &= 1 \end{aligned}$$

iii)

$$\begin{aligned} y(x) &= \prod_{n=0}^{\infty}(Y_n)^{x^n} = Y_0 \cdot Y_1^x \cdot \prod_{n=2}^{\infty}(Y_n)^{x^n} \\ y(x) &= e^{-1} \cdot (e^4)^x \cdot 1 \cdots = e^{4x-1} \end{aligned}$$

CONCLUSION

In this paper, the definitions of the multiplicative Taylor series, m-th degree multiplicative Taylor polynomial, truncation error of the multiplicative Taylor series of a positive function are defined. Also, an upper bound for this truncation error is calculated. Consequently, we used the multiplicative Taylor series method (MTSM) to find the numerical solution of higher-order multiplicative linear differential equations (MLDE). Solutions of first, second and third-order multiplicative linear differential equations (MLDE) are given by MPSM. Then, the applicability of this method is supported by numerical examples.

AUTHORSHIP CONTRIBUTIONS

As being the first author (lead author) Numan Yalçın has contributed the idea, design, literature review, analysis and writing part of the article. Mutlu Dedetürk has contributed to the idea, control and examples part of the article.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

[1] Stanley D. A multiplicative calculus. PRIMUS 1999;9:310–326. [CrossRef]

[2] Campbell D. Multiplicative calculus and student projects. PRIMUS 1999;9:327–332. [CrossRef]

[3] Bashirov AE, Kurpinar EM, Ozyapici A. Multiplicative calculus and its applications. J Math Anal Appl 2008;337:36–48. [CrossRef]

[4] Misirli E, Gurefe Y. Multiplicative Adams Bashforth Moulton methods. Numer Algorithms 2011;57:425–439. [CrossRef]

[5] Bashirov A, Riza M. On complex multiplicative differentition. TWMS J Appl Eng Math 2018;1:75–85.

[6] Yalçın N, Çelik E. The solution of multiplicative non-homogeneous linear differential equations. J Appl Math Comput 2018;2:27–36. [CrossRef]

[7] Bhat AH, Majid J, Shah TR, Wani IA, Jain R. Multiplicative fourier transform and ıts applications to multiplicative differential Equations. J Comput Math Sci 2019;10:375–383. [CrossRef]

[8] Bhat AH, Majid J, Shah TR, Wani IA, Jain R. Multiplicative Sumudu transform and its applications. JETIR 2019;6:579–589.

[9] Yalçın N, Dedeturk M. Solutions of multiplicative ordinary differential equations via the multiplicative differential transform method. AIMS Math 2021;6:3393–3409. [CrossRef]

[10] Aniszewska D. Multiplicative Runge-Kutta method. Nonlinear Dyn 2007;50:265–272. [CrossRef]

[11] Bashirov AE, Misirli E, Tandogdu Y, Ozyapici A. On modeling with multiplicative differential equations. Appl Math 2011;26:425–438. [CrossRef]

[12] Florack L, Assen H. Multiplicative calculus in biomedical image analysis. J Math Imaging Vis 2012;42:64–75. [CrossRef]

[13] Grossman M, Katz R. Non-Newtonian Calculus. Pigeon Cove, Massachusetts, USA: Lee Press; 1972.

[14] Ozyapici A, Misirli E. Exponential approximations on multiplicative calculus. Proceed Jangjeon Math Soc 2009;12:227–236.

[15] Ozyapici A, Bilgehan B. Finite product representation via multiplicative calculus and its applications to exponential signal processing. Numer Algorithms 2016;71:475–489. [CrossRef]

[16] Riza M, Ozyapici A, Misirli E. Multiplicative finite difference methods, Q Appl Math 2009;67:745–754. [CrossRef]

[17] Yalcin, N, Celik E, Gokdogan A. Multiplicative Laplace transform and its applications. Optik (Stuttg) 2016;127:20. [CrossRef]

[18] Yalcin N, Celik E. Solution of multiplicative homogeneous linear differential equations with constant exponentials. New Trends Math Sci 2018;6:58–67. [CrossRef]

[19] Yalçın N, Çelik E. Multiplicative Cauchy-Euler and legendre differential equation. Gumushane Univ J Sci Technol 2019;9:382–373. [Turkish]

[20] Yalçın N. The solutions of multiplicative Hermite differential equation and multiplicative Hermite polynomials, Rendiconti del Circolo Matematico di Palermo Series 2. New York: Springer; 2021. [CrossRef]

[21] Çakmak AF, Başar F. Some new results on sequence spaces with respect to non-Newtonian calculus. J Inequal Appl 2012;2012:228. [CrossRef]