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Research Article

Lyapunov-type inequality for an anti-periodic fractional boundary value problem of the riesz-caputo derivative

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ABSTRACT

This paper concerns with a Lyapunov-type inequality for the Riesz-Caputo fractional boundary value problem with anti-periodic boundary conditions. As an application for the obtained inequality, a lower bound for the eigenvalues of anti-periodic fractional boundary problems of the Riesz-Caputo derivative has been obtained.

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INTRODUCTION

Recently, many physical phenomena in applications and sciences can be modelled by fractional calculus. Some examples can be found in physics [1], bioengineering [2], engineering [3−6]. Additionally, anti-periodic fractional differential equations reflect the physical phenomena in the real-word applications and have been recently drawn to many researchers' attention, see [7−9, 10] and the references therein. Unlike the other fractional operators, the main feature of the Riesz fractional operator is that it uses both left and right fractional derivatives that hold non-local memory effects. This property of the Riesz derivative is important in the mathematical modelling in physical processes on a finite domain because the present states depend both on the past and future memory effects. However, the commonly used fractional derivatives are Riemann-Lioville and Caputo derivatives in the literature and these operators depend only on past or future information, so reflect only one-sided memory effect [5]. A variety of papers are devoted to numerical solutions of the fractional calculus, specifically in the anomalous diffusion that involves the Riesz derivate [11,12]. Recently, there are papers on existence of solutions and positive solutions in the sense of Riesz-Caputo derivative [13−16].

On the other hand, Lyapunov-type inequalities for fractional boundary value problems have been investigated in many papers [17−20] and references therein. However, the obtained results have been proved in the sense of the

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Riemann-Liouville and Caputo fractional derivatives. To the best of knowledge, there is no result on Lyapunov-type inequality for fractional boundary value problem (FBVP) of the Riesz-Caputo differential equation. To fill this gap, we derive a Lyapunov-type inequality for the following FBVP subject to anti-periodic boundary conditions

$$\begin{cases} {}^{RC}D_a^{v}u(\eta) + r(\eta)u(\eta) = 0, \ v \in (1,2], \ a \le \eta \le b, \\ u(a) + u(b) = 0 = u'(a) + u'(b), \end{cases} \tag{1}$$

where ${}^{RC}D_a^{v}$ is the Riesz-Caputo derivative defined below and $r \in C\,[a,b]$.

This paper is organized as follows. In Section 2, we collect some definitions and related results. We present our main results and an example as an application of the main result in Section 3. Finally, we provide some conclusions and future directions in Section 4.

PRELIMINARIES

We recall some definitions and results related problems considered in this paper.

Definition 2.1. *[4] Let* $v > 0$*. The left and right Riemann-Liouville fractional integral of a function* $f \in C\,[a,b]$ *of order v defined as, respectively*

$$I_a^{v} f(\eta) = \frac{1}{\Gamma(v)} \int_a^{\eta} (\eta - s)^{v-1} f(s)ds, \qquad \eta \in [a,b].$$

$${}_bI^{v} f(\eta) = \frac{1}{\Gamma(v)} \int_{\eta}^{b} (s - \eta)^{v-1} f(s)ds, \qquad \eta \in [a,b].$$

Definition 2.2. *(Riesz Fractional İntegral) Let* $v > 0$*. The Riesz fractional integral of a function* $f \in C\,[a,b]$ *of order v defined as*

$${}_bI_a^{v} f(\eta) = \frac{1}{2\Gamma(v)} \int_a^{b} |\eta - s|^{v-1} f(s)ds, \qquad \eta \in [a,b].$$

Note that the Riesz fractional integral operator can be written as

$$I_a^{v} f(\eta) = \frac{1}{2}\left(I_a^{v} f(\eta) + {}_bI^{v} f(\eta)\right) \tag{2}$$

Definition 2.3. *[4] Let* $v \in (n, n+1]$, $n \in \mathbb{N}$*. The left and right Caputo fractional derivative of a function* $f \in C^{n+1}\,[a,b]$ *of order v defined as, respectively*

$${}_a^{C}D_{\eta}^{v} f(\eta) = \frac{1}{\Gamma(n+1-v)} \int_a^{\eta} (\eta - s)^{n-v} f^{(n+1)} ds = (I_a^{n+1-v} D^{n+1} u(\eta)).$$

$${}_{\eta}^{C}D_b^{v} f(\eta) = \frac{(-1)^{n+1}}{\Gamma(n+1-v)} \int_{\eta}^{b} (s - \eta)^{n-v} f^{(n+1)} ds = (-1)^{n+1}({}_bI^{n+1-v} D^{n+1} u(\eta)).$$

where Du is the ordinary derivative of a function u.

Definition 2.3. *Let* $v \in (n, n+1]$, $n \in \mathbb{N}$*. The Riesz-Caputo fractional derivative* ${}_a^{RC}D_b^{v}$ *of a function* $f \in C^{n+1}\,[a,b]$ *of order v defined as*

$${}_a^{RC}D_b^{v} f(\eta) = \frac{1}{\Gamma(n+1-v)} \int_a^{b} |\eta - s|^{n-v} f^{(n+1)} ds = \frac{1}{2}\left({}_a^{C}D_{\eta}^{v} f(\eta) + (-1)^{n+1}\, {}_{\eta}^{C}D_b^{v} f(\eta)\right).$$

In the case when $v \in (1,2]$, we then have

$${}_a^{RC}D_b^{v} f(\eta) = \frac{1}{2}\left({}_a^{C}D_{\eta}^{v} f(\eta) + {}_{\eta}^{C}D_b^{v} f(\eta)\right)$$

Lemma 2.5. *[15] If* $h \in C\,[a,b]$ *then the following fractional differential equation*

$$\begin{cases} {}_a^{RC}D_b^{v}u(\eta) + h(\eta) = 0, \ v \in (1,2], \quad a \le \eta \le b, \\ u(a) + u(b) = 0 = u'(a) + u'(b), \end{cases}$$

has the solution u(η)

$$u(\eta) = -\frac{b-a}{4\Gamma(v-1)} \int_a^{b} (b-s)^{\eta-2} h(s)ds + \frac{1}{2\Gamma(v)} \int_a^{\eta} (\eta - s)^{v-1} h(s)ds + \frac{1}{2\Gamma(v)} \int_{\eta}^{b} (s-\eta)^{v-1} h(s)ds.$$

Remark 1. *In [15], the authors defined the Riesz fractional integral as*

$${}_bI_a^{v} f(\eta) = \frac{1}{\Gamma(v)} \int_a^{b} |\eta - s|^{v-1} f(s)ds, \ \ \eta \in [a,b], \ \ v > 0.$$

and the Riesz-Caputo fractional derivative of order $v \in (1,2]$ *as*

$${}_a^{RC}D_b^{v} f(\eta) = \frac{1}{2}\left({}_a^{C}D_{\eta}^{v} f(\eta) + {}_{\eta}^{C}D_b^{v} f(\eta)\right). \tag{5}$$

Therefore, they have proved that the following FBVP

$$\begin{cases} {}_a^{RC}D_b^{v}u(\eta) + h(\eta) = 0, \ v \in (1,2], \quad 0 \le \eta \le T, \\ u(0) + u(T) = 0 = u'(0) + u'(T), \end{cases}$$

has the solution u(η)

$$u(\eta) = -\frac{T}{2\Gamma(v-1)} \int_a^{b} (b-s)^{\eta-2} h(s)ds + \frac{1}{\Gamma(v)} \int_a^{\eta} (\eta - s)^{v-1} h(s)ds + \frac{1}{\Gamma(v)} \int_{\eta}^{b} (s-\eta)^{v-1} h(s)ds.$$

However, we will be consistent on the definitions and will continue to use Definition 2.2 for the Riesz-Caputo integral in this work.

Theorem 2.6. *[21] Let* $v \in (1,2]$ *If there exists a non-zero continuous solution of the following fractional boundary value problem*

$${}^{C}D_a^{v}u(x) + r(x)u(x) = 0, \qquad a \le x \le b,$$

$$u(a) + y(b) = 0 = u'(a) + u'(b),$$

where r is a continuous function, then

$$\int_a^b (b-t)^{\nu-2}|r(t)|\,dt > \frac{4}{(3-\nu)(b-a)}. \quad (6)$$

The inequality (6) is a generalization of the following celebrated Lyapunov inequlity.

Theorem 2.7. *[22] Let $r \in C\,[a,b]$ If there exists a nonzero solution u of the following boundary value problem*

$$u''(x) + r(x)u(x) = 0, \qquad a \le x \le b,$$
$$u(a) = y(b) = 0,$$

then

$$\int_a^b |r(t)|\,dt > \frac{4}{b-a}. \quad (7)$$

RESULTS AND DISCUSSION

In this section, we provide a Lyapunov-type inequality for FBVP (1) similar to the one given by (6). To do this, we first convert FBVP (1) to the integral equation in the next lemma.

Lemma 3.1. *Let $\nu \in (1,2]$ and $r \in C\,[a,b]$, $a < b$, Then u is a solution of FBVP (1) if and only if, u is the solution of the following integral equation*

$$u(\eta) = \int_a^b (b-s)^{\nu-2} G(\eta,s) r(s) u(s)\,ds, \quad (8)$$

where the Green's function G(t, s) is given by

$$G(t,s) = \begin{cases} \frac{1}{2\Gamma(\nu)}(t-s)^{\nu-1}(b-s)^{2-\nu} - \frac{b-a}{4\Gamma(\nu-1)}, & a \le s \le t \le b, \\ \frac{1}{2\Gamma(\nu)}(s-t)^{\nu-1}(b-s)^{2-\nu} - \frac{b-a}{4\Gamma(\nu-1)}, & a \le t \le s \le b. \end{cases} \quad (9)$$

*Proof***.** By the use of Lemma 2.5, we can rewrite the solution u of FBVP (1) as follows

$$\begin{aligned} u(\eta) &= -\frac{b-a}{4\Gamma(\nu-1)}\int_a^b (b-s)^{\eta-2} r(s)u(s)\,ds + \frac{1}{2\Gamma(\nu)}\int_a^\eta (\eta-s)^{\nu-1} r(s)u(s)\,ds \\ &\quad + \frac{1}{2\Gamma(\nu)}\int_\eta^b (s-\eta)^{\nu-1} r(s)u(s)\,ds \\ &= \int_a^\eta (b-s)^{\eta-2}\left[\frac{1}{2\Gamma(\nu)}(\eta-s)^{\nu-1}(b-s)^{2-\eta} - \frac{b-a}{4\Gamma(\nu-1)}\right] r(s)u(s)ds \\ &\quad + \int_\eta^b (b-s)^{\eta-2}\left[\frac{1}{2\Gamma(\nu)}(s-\eta)^{\nu-1}(b-s)^{2-\eta} - \frac{b-a}{4\Gamma(\nu-1)}\right] r(s)u(s)ds \\ &= \int_a^b (b-s)^{\eta-2} G(\eta,s) r(s)u(s)ds \end{aligned}$$

This completes the proof.

Next, we find a bound for the Green's function *G(t, s)* in (9).

Lemma 3.2. *The Green functions G(t, s) given by (9) obeys the following bound:*

$$|G(t,s)| \le \frac{(3-\nu)(b-a)}{4\Gamma(\nu)}, \qquad a \le t,s \le b.$$

Proof. We first prove the case when $a \le s \le t \le b$. Let the function *g(t, s)* be defined by

$$g(t,s) = \frac{1}{2\Gamma(\nu)}(t-s)^{\nu-1}(b-s)^{2-\nu} - \frac{b-a}{4\Gamma(\nu-1)}, \quad a \le s \le t \le b.$$

For a fixed $s \in [a,t]$ the second derivative of *g(t, s)* with respect to the first variable *t* reveals that

$$\frac{\partial^2 g(t,s)}{\partial t^2} = \frac{(\nu-1)(\nu-2)}{2}(t-s)^{\nu-3}(b-s)^{2-\nu} \le 0, \quad t > s,$$

which implies that $\frac{\partial g(t,s)}{\partial t}$ is a decreasing function of *t*. Since $\frac{\partial g(t,s)}{\partial t}|_{t=b} = \frac{(\nu-1)}{2} > 0$, we infer that $\frac{\partial g(t,s)}{\partial t}$ is positive for fixed *s* and $t \in [s,b]$. This concludes that *g(t, s)* is an increasing function of *t*. Thus, we have that

$$\max_{s\le t\le b}|g(t,s)| = \max\{|g(s,s)|, |g(b,s)|\}.$$

Clearly, we have

$$g(s,s)| = \frac{(\nu-1)(b-a)}{4\Gamma(\nu)}. \quad (10)$$

Obviously, $g(b,s) = \frac{b-s}{2\Gamma(\nu)} - \frac{b-a}{4\Gamma(\nu-1)}$ is a decreasing function of *s*. Thus,

$$\max_{a\le s\le b}|g(b,s)| = \max\{|g(b,a)|, |g(b,b)|\}$$

Now, we have

$$|g(b,a)| = \frac{(b-a)}{2\Gamma(\nu)} - \frac{(b-a)}{4\Gamma(\nu-1)} = \frac{(3-\nu)(b-a)}{4\Gamma(\nu)}, \quad (11)$$

and

$$|g(b,b)| = \frac{(\nu-1)(b-a)}{4\Gamma(\nu)} = |g(s,s)|.$$

Since $\nu \in (1,2]$, it holds that $0 < \nu - 1 \le 3 - \nu$, so from (10), we find

$$|g(b,b)| = |g(s,s)| \le |g(b,a)|.$$

Therefore we obtain that

$$\max_{a\le s,t\le b}|g(t,s)| = \frac{(3-\nu)(b-a)}{4\Gamma(\nu)}. \quad (12)$$

Exactly the same argument shows that the function *q(x, t)* defined by

$$q(t,s) = \frac{1}{2\Gamma(\nu)}(s-t)^{\nu-1}(b-s)^{2-\nu} - \frac{(b-a)}{4\Gamma(\nu-1)}, \quad a \le t \le s \le b,$$

has the following bound:

$$\max_{a\leq t,s\leq b}|q(t,s)| = \frac{(3-v)(b-a)}{4\Gamma(v)}. \tag{13}$$

Thus, we conclude from (12) and (13) that

$$G(t,s) \leq \frac{(3-v)(b-a)}{4\Gamma(v)},$$

which finishes the proof.

Now, we are ready to state and prove the main theorem of this paper.

Theorem 3.3. *Let* $v \in (1,2]$ *and* $r \in C\,[a,b]$ *If there exists a non-zero solution of the following FBVP*

$${}^{RC}D_a^v u(\eta) + r(\eta)u(\eta) = 0, \qquad v \in (1,2], \qquad a \leq \eta \leq b,$$

$$u(a) + y(b) = 0 = u'(a) + u'(b),$$

then we have the following Lyapunov-type inequality

$$\int_a^b (b-\mu)^{v-2}|r(\mu)|\,d\mu > \frac{4\Gamma(v)}{(3-v)(b-a)}. \tag{14}$$

Proof. Let $C\,[a, b]$ be the Banach space with maximum norm $\|u\| = max_{\mu\in[a,b]}|u(\mu)|$.

By Lemma 3.1, we infer that if u solves the FBVP, then it also solves the following integral equation

$$u(\eta) = \int_a^b (b-\mu)^{v-2} G(\eta,\mu) r(\mu) u(\mu) d\mu, \quad a \leq \eta \leq b.$$

We then have

$$|u(\eta)| \leq \int_a^b (b-\mu)^{v-2}|G(\eta,\mu)|\,|r(\mu)||u(\mu)|d\mu, \quad a \leq \eta \leq b.$$

Nontrivialness of the solution u implies that $r(\mu) > 0$ on some subinterval of [a, b]. Also, from Lemma 3.2, we get $|G(\eta,\mu)| \leq \frac{(3-v)(b-a)}{4\Gamma(v)}$ on this subinterval. Now, taking the maximum norm of both sides and the above-mentioned arguments give that

$$\|u\| < \frac{(3-v)(b-a)}{4\Gamma(v)} \int_a^b (b-\mu)^{v-2}\,|r(\mu)|\|u\|d\mu,$$

or equivalently

$$\int_a^b (b-\mu)^{v-2}\,|r(\mu)|d\mu > \frac{4\Gamma(v)}{(3-v)(b-a)},$$

which is the inequality in (14).

Remark 2. *If we let* $v \to 2$, *we get the classical Lyapunov inequality (7) subject to the anti-periodic boundary condition given in [7](Corollary 2.5 for n=2) and [23].*

As an application of the inequality (14), we find a bound on the eigenvalues of FBVP.

Example 1. *Let* $v \in (1,2]$ *If a non-zero solution of the following FBVP*

$${}^{RC}D_a^v u(\eta) + \lambda u(\eta) = 0, \qquad a \leq \eta \leq b,$$

$$u(a) + u(b) = 0 = u'(a) + u'(b),$$

then the eigenvalue $\lambda \in \mathbb{R}$ *must obey*

$$\lambda > \frac{4(v-1)\Gamma(v)}{(3-v)(b-a)^v}.$$

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CONCLUSION

In this paper, we consider a linear Riesz-Caputo fractional boundary value problem with anti-periodic boundary conditions. We firstly establish the Green's function corresponding to the boundary value problem and then derive a Lyapunov-type inequality for the boundary value problems. Furthermore, we provide a lower bound for the eigenvalues of the FBVP associated with the non trvial solution.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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