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Research Article

Some new inequalities for LR-(p, $\hbar$)-convex interval-valued functions by means of pseudo order relation

Muhammad Bilal KHAN[1], Muhammad Aslam NOOR[1], Khalida Inayat NOOR[1], Vildan ÇETKIN[2,*]

[1]*Department of Mathematics, COMSATS University Islamabad, 22060, Pakistan*
[2]*Department of Mathematics, Kocaeli University, Kocaeli, 41380, Türkiye*

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ABSTRACT

Being a critical part of classical analysis, some of the convex functions and inequalities have drawn much attention recently because both concepts establish a strong relationship. As a familiar extension of classical one, the interval-valued analysis is frequently used to the research of control theory, mathematical economy and so on. Motivated by the importance of convexity and inequality, our aim is to consider new class of convex interval-valued functions is known as LR-(p, $\hbar$)-convex interval-valued functions through pseudo order relation($\leq_p$). This order relation is defined on interval space. By using this concept, firstly we obtain Hermite-Hadamard (HH-) and Hermite- Hadamard-Fejér (HH-Fejér) type inequalities through pseudo order relation. Secondly, we present some new versions of discrete Jensen and Schur type inequalities via LR-(p, $\hbar$)-convex interval-valued functions. Moreover, we have shown that our results include a wide class of new and known inequalities for LR-$\hbar$-convex-IVFs and their variant forms as special cases. Under some mild restrictions, we have proved that the inclusion relation "$\subseteq$" coincident to pseudo order relation "$\leq_p$" when the interval-valued function is LR-(p, $\hbar$)-convex or LR-(p, $\hbar$)-concave. Results obtained in this paper can be viewed as improvement and refinement of previously known results.

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INTRODUCTION

The convex analysis has played an important and fundamental part in development of various fields of applied and pure science. In last few decades much attention has been given in studying and distinguishing diverse directions of classical idea of convexity. In classical approach, a real valued function $\mathcal{F}: K \to \mathbb{R}$ is called convex function on K if

$$\mathcal{F}(\varrho x + (1-\varrho)y) \leq \varrho\mathcal{F}(x) + (1-\varrho)\mathcal{F}(y), \quad (1)$$

for all $x, y \in K$, $\varrho \in [0, 1]$. If f is convex then,$-f$ is concave. Recently, many extensions and generalizations of

*Corresponding author.
*E-mail address: vildan.cetkin@kocaeli.edu.tr
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convex set and convex functions have been established such as harmonic convexity [3], $\hbar$-convexity [30], quasi convexity [23], Schur convexity [8, 9], strong convexity [31, 32], p-convexity [36], $(p, \hbar)$-convexity [16], generalized convexity [33] and the main generalization of convex function is discrete Jensen inequality [1] because it plays a critical in probability theory, in optimization theory and among others field of sciences. For more useful details, see [3, 4, 5, 8, 19, 22] and the references are therein.

The concept of convexity establishes strong relationship with integral problem. Therefore, this field of research has attracted many authors to contribute their role. Therefore, many inequalities have been introduced as applications of convex functions. The representative results include Gagliardo-Nirenberg-type inequality [29], Hardy-type inequality [10], Ostrowski-type inequality [18], Olsen-type inequality [20], and the most well-known inequality of, namely, the Hermite-Hadamard inequality (HH-inequality, in short) [21]. The HH-inequality is an interesting outcome in convex analysis which is formulated for convex function. In [17], Fejér considered the major generalizations of HH-inequality which is known as HH-Fejér inequality. This inequality basically depends upon the convex and symmetric function. With the assistance of Fejér-inequality, many inequalities can be obtained through special symmetric functions for convex functions.

It is also familiar fact that interval analysis [25] and fuzzy analysis [34] are considered to be two different fields of mathematics that provide tools to deal with data uncertainty. In general, interval analysis is typically used to deal with the models whose data are composed of inaccuracies that may occur from certain kinds of measurements. On the other hand, without complete knowledge on the problem, the fuzzy analysis can be used to deal with the models that were obtained. Moreover, it plays an important role in the study of a broad-based class problems in pure mathematics and applied sciences including operational analysis, computer science, managements sciences, artificial intelligence, control engineering and decision makings. The convex analysis has played an important and fundamental part in development of various fields of applied and pure science. Similarly, the notions of convexity and non-convexity play a vital role in optimization under interval and fuzzy domain. Therefore, several classical discrete and integral inequalities have been generalized not only to the environment of the IVF and fuzzy-IVFs by Costa [13], Costa and Roman-Flores [14], Roman-Flores et al. [27], and Chalco-Cano et al. [11, 12], but also to more general set valued maps by Nikodem et al. [26], and Matkowski and Nikodem [24]. In particular, Zhang et al. [35] derived the new version of Jensen's inequalities for set-valued and fuzzy set-valued functions by means of a pseudo order relation and proved that these Jensen's inequalities generalized form of Costa Jensen's inequalities [13]. Motivated by the above literature, Zhao et al. [37] introduced $\hbar$-convex interval-valued functions ($\hbar$-convex IVFs, in short) and demonstrated that the HH-type inequalities and Jensen HH-type inequalities for $\hbar$-convex IVFs. Besides, Yanrong An et al. [2] defined the class of $(\hbar_1, \hbar_2)$-convex IVFs and established interval HH-inequality for $(\hbar_1, \hbar_2)$-convex IVFs. For further review of the literature on the applications and properties of generalized convex functions and HH-inequalities, see [6, 7, 11, 15, 28, 38-47] and the references therein. Inspired by Costa and Roman-Flores [14], and Zhang et al. [35], we present discrete interval inequalities and interval HH- inequalities for LR-$(p, \hbar)$-convex IVFs by means of pseudo order relation.

Inspired by the ongoing research work, the main aim of this paper is to introduce the class of LR-$(p, \hbar)$-convex IVFs and to establish inequalities of Jensen, schur, HH- and HH- Fejér type for LR-$(p, \hbar)$-convex IVFs by means of pseudo order relation via Riemann integrals. The main results of this paper also obtain some applications.

PRELIMINARIES

In this section, we first give some definitions, preliminary notations and results which will be helpful for further study. Then, we define new definitions and properties of the LR-$(p, \hbar)$-convex IVFs.

For the basic notions and definitions on the interval analysis, we use literature [three]. Let $\mathbb{R}_I$ be the space of all closed and bounded intervals of R and $\varpi \in \mathbb{R}_I$ be defined by

$$\varpi = [\varpi_*, \varpi^*] = \{x \in \mathrm{R} \mid \varpi_* \leq x \leq \varpi^*\}, (\varpi_*, \varpi^* \in \mathbb{R}).$$

If $\varpi_* = \varpi^*$ then, ϖ is said to be degenerate. In this article, all intervals will be non-degenerate intervals. If $\varpi_* \geq 0$, then $[\varpi_*, \varpi^*]$ is called positive interval. The set of all positive interval is denoted by $\mathbb{R}_I^+$ and defined as $\mathbb{R}_I^+ = \{[\varpi_*, \varpi^*]: [\varpi_*, \varpi^*] \in \mathbb{R}_I \text{ and } \varpi_* \geq 0\}$.

Let $\rho \in \mathbb{R}$ and $\rho\varpi$ be defined by

$$\rho.\varpi = \begin{cases} [\rho\varpi_*, \rho\varpi^*] \text{ if } \rho \geq 0, \\ [\rho\varpi^*, \rho\varpi_*] \text{ if } \rho < 0. \end{cases}$$

Then the Minkowski difference $\eta - \varpi$, addition $\varpi + \eta$ and $\varpi \times \eta$ for $\varpi, \eta \in \mathbb{R}_I$ are defined by

$$[\eta_*, \eta^*] - [\varpi_*, \varpi^*] = [\eta_* - \varpi_*, \eta^* - \varpi^*],$$
$$[\eta_*, \eta^*] + [\varpi_*, \varpi^*] = [\eta_* + \varpi_*, \eta^* + \varpi^*],$$

and

$$[\eta_*, \eta^*] \times [\varpi_*, \varpi^*] = [min\{\eta_*\varpi_*, \eta^*\varpi_*, \eta_*\varpi^*, \eta^*\varpi^*\}, max\{\eta_*\varpi_*, \eta^*\varpi_*, \eta_*\varpi^*, \eta^*\varpi^*\}].$$

The inclusion " $\subseteq$ " means that

$\eta \subseteq \varpi$ If and only if, $[\eta_*, \eta^*] \subseteq [\varpi_*, \varpi^*]$, if and only if $\varpi_* \leq \eta_*, \eta^* \leq \varpi^*$.

Remark 2.1. [35] (i) The relation " $\leq_p$ " defined on $\mathbb{R}_I$ by $[\eta_*, \eta^*] \leq_p [\varpi_*, \varpi^*]$ if and only if $\eta_* \leq \varpi_*, \eta^* \leq \varpi^*$, for all $[\eta_*, \eta^*]$, $[\varpi_*, \varpi^*] \in \mathbb{R}_I$, it is an pseudo order relation. For given $[\eta_*, \eta^*]$, $[\varpi_*, \varpi^*] \in R_I$, we say that $[\eta_*, \eta^*] \leq_p [\varpi_*, \varpi^*]$ if and only if $\eta_* \leq \varpi_*$, $\eta^* \leq \varpi^*$ or $\eta_* \leq \varpi_*$, $\eta^* < \varpi^*$. The relation $[\eta_*, \eta^*] \leq_p [\varpi_*, \varpi^*]$ coincident to $[\eta_*, \eta^*] \leq [\varpi_*, \varpi^*]$ on $\mathbb{R}_I$.

(ii) It can be easily seen that " $\leq_p$ " looks like "left and right" on the real line $\mathbb{R}$, so we call " $\leq_p$ " is "left and right" (or "LR" order, in short).

For $[\eta_*, \eta^*]$, $[\varpi_*, \varpi^*] \in \mathbb{R}_I$, the Hausdorff–Pompeiu distance between intervals $[\eta_*, \eta^*]$ and $[\varpi_*, \varpi^*]$ is defined by

$$d([\eta_*, \eta^*], [\varpi_*, \varpi^*]) = max\{[\eta_*, \eta^*], [\varpi_*, \varpi^*]\}.$$

It is familiar fact that $(\mathbb{R}_I, d)$ is a complete metric space.

A partition of $[u, v]$ is any finite ordered subset P having the form

$$P = \{u = x_1 < x_2 < x_3 < x_4 < x_5 \dots \dots < x_k = v\}.$$

The mesh of a partition P is the maximum length of the subintervals containing P that is,

$$\text{mesh}(P) = max\{x_j - x_{j-1}: j = 1, 2, 3, \dots \dots k\}.$$

Let $\mathcal{P}(\delta, [u, v])$ be the set of all $P \in \mathcal{P}(\delta, [u, v])$ such that mesh $(P) < \delta$. For each interval $[x_{j-1}, x_j]$, where $1 \leq j \leq k$, choose an arbitrary point η_j and taking the sum

$$S(f, P, \delta) = \Sigma^k f(\eta_j)(x_j - x_{j-1}),$$

Where $f: [u, v] \to \mathbb{R}_I$. We call $S(f, P, \delta)$ a Riemann sum of f corresponding to $P \in \mathcal{P}(\delta, [u, v])$.

Definition 2.2. [37] A function $f: [u, v] \to \mathbb{R}_I$ is called interval Riemann integrable (IR-integrable) on $[u, v]$ if there exists $B \in \mathbb{R}_I$ such that, foe each ϵ, there exists $\delta > 0$ such that

$$d(S(f, P, \delta), B) < \epsilon,$$

for every Riemann sum of f corresponding to $P \in \mathcal{P}(\delta, [u, v])$ and for arbitrary choice of $\eta_j \in [x_{j-1}, x_j]$ for $1 \leq j \leq k$. Then we say that B is the IR-integral of f on $[u, v]$ and is denote by $B = (IR) \int_u^v f(x)dx$.

The concept of Riemann integral for IVF first introduced by Moore [25] is defined as follow:

Theorem 2.3. [25] If $f: [u, v] \subset \mathbb{R} \to \mathbb{R}_I$ is an IVF on such that $f(x) = [f_*, f^*]$. Then f is Riemann integrable over $[u, v]$ if and only if, f_* and f^* both are Riemann integrable over $[u, v]$ such that

$$(IR) \int_u^v f(x)dx = \left[(R) \int_u^v f_*(u)dx \,, (R) \int_u^v f^*(u)dx\right].$$

The collection of all Riemann integrable real valued functions and Riemann integrable IVF is denoted by $\mathcal{R}_{[u,v]}$ and $\mathcal{IR}_{[u,v]}$, respectively.

Definition 2.4. [15] A function $f: [u, v] \to \mathbb{R}^+$ is said to be P convex function if

$$f(\varrho x + (1 - \varrho)y) \leq f(x) + f(y), \tag{2}$$

for all $x, y \in [u, v]$, $\varrho \in [0, 1]$. If (2) is reversed then, f is called P concave.

Definition 2.5. [6] A function $f: K \to \mathbb{R}^+$ is said to be s-convex function in the second sense if

$$f(\varrho x + (1 - \varrho)y) \leq \varrho^s f(x) + (1 - \varrho)^s f(y), \tag{3}$$

for all $x, y \in [u, v]$, $\varrho \in [0, 1]$, where $s \in (0, 1)$. If (3) is reversed then, f is called s-concave in the second sense.

Definition 2.6. [30] A function $f: [u, v] \to \mathbb{R}^+$ is said to be $\hbar$-convex function if for all $x, y \in [u, v]$, $\varrho \in [0, 1]$, we have

$$f(\varrho x + (1 - \varrho)y) \leq \hbar(\varrho) f(x) + \hbar(1 - \varrho) f(y), \tag{4}$$

where $\hbar: \mathcal{L} \to \mathbb{R}^+$ such that $\hbar \not\equiv 0$, $[0, 1] \subseteq \mathcal{L}$. If (4) is reversed then, f is called $\hbar$-concave in the second sense. A function $\hbar: \mathcal{L} \to \mathbb{R}^+$ is called super multiplicative if for all $x, y \in \mathcal{L}$, we have

$$\hbar(xy) \geq \hbar(x)\hbar(y) \tag{5}$$

If (16) is reversed then, $\hbar$ is known as sub multiplicative. If the equality holds in (5) then, $\hbar$ is called multiplicative.

Definition 2.7. [36] Let $p \in \mathbb{R}$ with $p \neq 0$. Then the interval K_p is said to be p-convex if

$$[\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}} \in K_p, \tag{6}$$

for all $x, y \in K_p$, $\varrho \in [0, 1]$, where $p = 2n + 1$ and $n \in N$

Definition 2.8. [36] Let $p \in R$ with $p \neq 0$ and $K_p = [u, v] \subseteq \mathbb{R}$. Then, the function $f: [u, v] \to \mathbb{R}^+$ is said to be p-convex function if

$$f\left([\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}}\right) \leq \varrho f(x) + (1 - \varrho) f(y), \tag{7}$$

for all x, $y \in [u, v]$, $\varrho \in [0, 1]$. If the inequality (7) is reversed then f is called p-concave function.

Definition 2.9. [16] Let K_p be a p-convex set and $\hbar: [0, 1] \subseteq \mathcal{L} \to \mathbb{R}^+$ be a nonnegative real-valued function such that $\hbar \not\equiv 0$, where $\mathcal{L} \subseteq \mathbb{R}$. Then function $f: K_p \to \mathbb{R}$ is said to be $(p, \hbar)$-convex on Kp such that

$$f\left([\varrho x^p + (1 - \varrho)y^p]^{\frac{1}{p}}\right) \leq \hbar(\varrho) f(x) + \hbar(1 - \varrho) f(y), \tag{8}$$

for all $x, y \in K_p = [u, v]$, $\varrho \in [0, 1]$, where $f(x) \geq 0$ and $\hbar: \mathcal{L} \to \mathbb{R}^+$ such that $\hbar \not\equiv 0$ and $[0, 1] \subseteq \mathcal{L}$. If (8) is reversed then, f is called $(p, \hbar)$-concave on $[u, v]$. The set of all $(p, \hbar)$-convex ($(p, \hbar)$-concave, $(p, \hbar)$-affine) functions is denoted by

$$SX([u, v], \mathbb{R}^+, (p, \hbar)) \quad \big(SV([u, v], \mathbb{R}^+, (p, \hbar))\big).$$

Definition 2.10. The IVF f: $[u, v] \to \mathbb{R}_I^+$ is said to be LR-$(p, \hbar)$-convex-IVF if for all $x, y \in [u, v]$ and $\varrho \in [0, 1$ we have

$$f\left([\varrho x^p + (1-\varrho)y^p]^{\frac{1}{p}}\right) \leq_p \hbar(\varrho)\, f(x) + \hbar(1-\varrho)\, f(y), \quad (9)$$

where $\hbar: \mathcal{L} \to \mathbb{R}^+$ such that $\hbar \not\equiv 0, [0,1] \subseteq \mathcal{L}$. If inequality (9) is reversed, then f is said to be LR-$(p, \hbar)$-concave on $[u, v]$. The set of all LR-$(p, \hbar)$-convex (LR-$(p, \hbar)$-concave), is denoted by

$$LRSX([u, v], \mathbb{R}_I^+, (p, \hbar)) \quad \big(LRSV([u, v], \mathbb{R}_I^+, (p, \hbar))\big).$$

Remark 2.11. If $\hbar(\varrho) = \varrho^s$ with $s \in (0, 1)$ then LR-$(p, \hbar)$-convex-IVF becomes LR-(p, s)-convex-IVF in the second sense, that is

$$f\left([\varrho x^p + (1-\varrho)y^p]^{\frac{1}{p}}\right) \leq_p \varrho^s f(x) + (1-\varrho)^s f(y), \forall\, x, y \in K, \varrho \in [0, 1].$$

If $\hbar(\varrho) = \varrho$, then LR-$(p, \hbar)$-convex-IVF becomes LR-p-convex-IVF, that is

$$f\left([\varrho x^p + (1-\varrho)y^p]^{\frac{1}{p}}\right) \leq_p \varrho f(x) + (1-\varrho) f(y), \forall\, x, y \in K, \varrho \in [0, 1].$$

Theorem 2.12. Let $\hbar$: $[0, 1] \to \mathbb{R}^+$ such that $\hbar \not\equiv 0$ and f: $[u, v] \to \mathbb{R}_I^+$ be an IVF defined by $f(x) = [f_*(x), f^*(x)]$, for all $x \in [u, v]$. Then $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$ if and only if, $f_*, f^* \in SX([u, v], \mathbb{R}^+, (p, \hbar))$.

Proof. Assume that $f_*, f^* \in SX([u, v], \mathbb{R}^+, (p, \hbar))$. Then, for all $x, y \in [u, v]$, $\varrho \in [0, 1]$, we have

$$f_*\left([\varrho x^p + (1-\varrho)y^p]^{\frac{1}{p}}\right) \leq \hbar(\varrho)\, f_*(x) + \hbar(1-\varrho)\, f_*(y),$$

And

$$f^*\left([\varrho x^p + (1-\varrho)y^p]^{\frac{1}{p}}\right) \leq \hbar(\varrho)\, f^*(x) + \hbar(1-\varrho)\, f^*(y)$$

From inequality (2.8) and order relation $\leq_p$, we have

$$\left[f_*\left([\varrho x^p + (1-\varrho)y^p]^{\frac{1}{p}}\right), f^*\left([\varrho x^p + (1-\varrho)y^p]^{\frac{1}{p}}\right)\right]$$
$$\leq_p [\hbar(\varrho)\, f_*(x) + \hbar(1-\varrho)\, f_*(y), \hbar(\varrho)\, f^*(x) + \hbar(1-\varrho)\, f^*(y)]$$
$$= \hbar(\varrho)\, [f_*(x), f^*(x)] + \hbar(1-\varrho)\, [f_*(y), f^*(y)],$$

that is

$$f\left([\varrho x^p + (1-\varrho)y^p]^{\frac{1}{p}}\right) \leq_p \hbar(\varrho)\, f(x) + \hbar(1-\varrho)\, f(y),$$
$$\forall\, x, y \in [u, v], \varrho \in [0, 1].$$

Hence, $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$.

Conversely, let $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$. Then for all $x, y \in [u, v]$ and $\varrho \in [0, 1]$, we have

$$f\left([\varrho x^p + (1-\varrho)y^p]^{\frac{1}{p}}\right) \leq_p \hbar(\varrho)\, f(x) + \hbar(1-\varrho)\, f(y),$$

That is

$$\left[f_*\left([\varrho x^p + (1-\varrho)y^p]^{\frac{1}{p}}\right), f^*\left([\varrho x^p + (1-\varrho)y^p]^{\frac{1}{p}}\right)\right] \leq_p$$
$$\hbar(\varrho)\, [f_*(x), f^*(x)] + \hbar(1-\varrho)\, [f_*(y), f^*(y)]$$
$$= [\hbar(\varrho)\, f_*(x) + \hbar(1-\varrho)\, f_*(y), \hbar(\varrho)\, f^*(x) + \hbar(1-\varrho)\, f^*(y)].$$

It follows that

$$f_*\left([\varrho x^p + (1-\varrho)y^p]^{\frac{1}{p}}\right) \leq \hbar(\varrho)\, f_*(x) + \hbar(1-\varrho)\, f_*(y),$$

and

$$f^*\left([\varrho x^p + (1-\varrho)y^p]^{\frac{1}{p}}\right) \leq \hbar(\varrho)\, f^*(x) + \hbar(1-\varrho)\, f^*(y),$$

Hence, the result follows.

Remark 2.13. If $f_*(x) = f^*(x)$ then, LR-$(p, \hbar)$-convex-IVF becomes $(p, \hbar)$-convex function, see [16].

If $f_*(x) = f^*(x)$ with $\hbar(\varrho) = \varrho^s$ with $s \in (0, 1)$ then, LR-$(p, \hbar)$-convex-IVF becomes (p, s)-convex function in the second sense, see [16].

If $f_*(x) = f^*(x)$ with $\hbar(\varrho) = \varrho$ then LR-$(p, \hbar)$-convex-IVF becomes classical p-convex function, see [36]. If $f_*(x) = f^*(x)$ with $p = 1$ then, LR-$(p, \hbar)$-convex-IVF becomes $\hbar$-convex function, see [22].

If $f_*(x) = f^*(x)$ with $\hbar(\varrho) = \varrho^s$, $s \in (0, 1)$ and $p = 1$ then, LR-$(p, \hbar)$-convex-IVF becomes s-convex function in the second sense, see [6].

If $f_*(x) = f^*(x)$ with $\hbar(\varrho) \equiv 1$ and $p = 1$ then LR-$(p, \hbar)$-convex-IVF reduces to the P-convex function, see [15]. If $f_*(x) = f^*(x)$ with $\hbar(\varrho) = \varrho$ and $p = 1$ then LR-$(p, \hbar)$-convex-IVF becomes classical convex function.

Example 2.14. We consider $\hbar(\varrho) = \varrho^n$, for $\varrho > 0$, $n \leq 1$ and the IVF $f: (0, \infty) \to \mathbb{R}_I^+$ defined by $f(x) = [3x^p, 5x^p]$, where p is an odd number. Since $f_*, f^* \in SX([u, v], \mathbb{R}^+, (p, \hbar))$ and hence, $f(x)$ is $(p, \hbar)$-convex- IVF.

HERMITE–HADAMARD TYPE INEQUALITIES

In this section, we derive some new HH- and HH-Fejér type inequalities for LR-$(p, \hbar)$-convex-IVF by means of pseudo order relation via the interval Riemann type integrals.

Theorem 3.1. Let $\hbar$: [0, 1] → $\mathbb{R}^+$ such that $\hbar \not\equiv 0$ and and $\hbar\left(\frac{1}{2}\right) \neq 0$, and let f: $[u, v] \to \mathbb{R}_I^+$ be an IVF such that $f(x) = [f_*(x), f^*(x)]$, for all $x \in [u, v]$. If $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$ and $f \in \mathcal{IR}_{([u,\, v])}$, then

$$\frac{1}{2\,\hbar\left(\frac{1}{2}\right)}\, f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{p}{v^p-u^p}\,(IR)\int_u^v x^{p-1}f(x)dx \leq_p [f(u) + f(v)]\int_0^1 \hbar(\varrho)\,d\varrho. \quad (10)$$

If $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$, then

$$\frac{1}{2\,\hbar\left(\frac{1}{2}\right)}\, f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \geq_p \frac{p}{v^p-u^p}\,(IR)\int_u^v x^{p-1}f(x)dx \geq_p [f(u) + f(v)]\int_0^1 \hbar(\varrho)\,d\varrho. \quad (11)$$

Proof. Let If $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$. Then, by hypothesis, we have

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p f\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) + f\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right).$$

Therefore, we have

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} f_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq f_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) + f_*\left((1-\varrho)u^p + \varrho v^p\right),$$
$$\frac{1}{\hbar\left(\frac{1}{2}\right)} f^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq f^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) + f^*\left((1-\varrho)u^p + \varrho v^p\right).$$

Then

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\int_0^1 f_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)d\varrho \leq \int_0^1 f_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)d\varrho + \int_0^1 f_*\left((1-\varrho)u^p + \varrho v^p\right)d\varrho,$$
$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\int_0^1 f^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)d\varrho \leq \int_0^1 f^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)d\varrho + \int_0^1 f^*\left((1-\varrho)u^p + \varrho v^p\right)d\varrho.$$

It follows that

$$\frac{1}{2\,\hbar\left(\frac{1}{2}\right)} f_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{v^p-u^p}\int_u^v x^{p-1}f_*(x)dx\,,$$
$$\frac{1}{2\,\hbar\left(\frac{1}{2}\right)} f^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{v^p-u^p}\int_u^v x^{p-1}f^*(x)dx\,.$$

That is

$$\frac{1}{2\,\hbar\left(\frac{1}{2}\right)}\left[f_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right), f^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)\right] \leq_p \frac{p}{v^p-u^p}\left[\int_u^v x^{p-1}f_*(x)dx\,, \int_u^v x^{p-1}f^*(x)dx\right].$$

Thus,

$$\frac{1}{2\,\hbar\left(\frac{1}{2}\right)}\, f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{p}{v^p-u^p}\,(IR)\int_u^v x^{p-1}f(x)dx. \quad (12)$$

In a similar way as above, we have

$$\frac{p}{v^p-u^p}\,(IR)\int_u^v x^{p-1}f(x)dx \leq_p [f(u) + f(v)]\int_0^1 \hbar(\varrho)\,d\varrho. \quad (13)$$

Combining (12) and (13), we have

$$\frac{1}{2\,\hbar\left(\frac{1}{2}\right)}\, f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{p}{v^p-u^p}\,(IR)\int_u^v x^{p-1}f(x)dx \leq_p [f(u) + f(v)]\int_0^1 \hbar(\varrho)\,d\varrho.$$

Hence, the required result.

Remark 3.2. If $\hbar(\varrho) = \varrho^s$, then Theorem 3.1 reduces to the result for LR-(p, s)-convex-IVF which is also new one:

$$2^{s-1} f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{p}{v^p-u^p}\,(IR)\int_u^v x^{p-1}f(x)dx \leq_p \frac{1}{s+1}[f(u) + f(v)].$$

If $\hbar(\varrho) = \varrho$, then Theorem 3.1 reduces to the result for LR-p-convex-IVF which is also new one:

$$f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{p}{v^p-u^p}\,(IR)\int_u^v x^{p-1}f(x)dx \leq_p \frac{f(u)+f(v)}{2}.$$

If $f_*(x) = f^*(x)$, then Theorem 3.1 reduces to the result for classical $(p, \hbar)$-convex function, see [16]:

$$\frac{1}{2\,\hbar\left(\frac{1}{2}\right)}\, f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{v^p-u^p}\,(R)\int_u^v x^{p-1}f(x)dx \leq [f(u) + f(v)]\int_0^1 \hbar(\varrho)\,d\varrho.$$

If $f_*(x) = f^*(x)$ with $\hbar(\varrho) = \varrho^s$, then Theorem 3.1 reduces to the result for classical (p, s)-convex function, see [16]:

$$2^{s-1} f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{v^p-u^p}\,(R)\int_u^v x^{p-1}f(x)dx \leq \frac{1}{s+1}[f(u) + f(v)].$$

If $f_*(x) = f^*(x)$ with $\hbar(\varrho) = \varrho$, then Theorem 3.1 reduces to the result for classical p-convex function, see [26]:

$$f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{v^p-u^p}\,(R)\int_u^v x^{p-1}f(x)dx \leq \frac{f(u)+f(v)}{2}.$$

Example 3.3. Let p be an odd number and $\hbar(\varrho) = \varrho$, for $\varrho \in [0, 1]$, and the IVF $f: [u, v] = [-1, 1] \to \mathbb{R}_I^+$ defined by, $f(x) = [x^p,\ e^{xp}]$, where p be an odd number. Since end point functions $f_*(x) = x^p$ and $f^*(x) = e^{xp}$ both are $(p, \hbar)$-convex functions. Hence $f(x)$ is LR-$(p, \hbar)$-convex-IVF. We now computing the following

$$\frac{1}{2\,\hbar\left(\frac{1}{2}\right)}\, f_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{v^p-u^p}\int_u^v x^{p-1}f_*(x)dx \leq \left[f_*(u) + f_*(v)\right]\int_0^1 \hbar(\varrho)\,d\varrho.$$

$$\frac{1}{2\,\hbar\left(\frac{1}{2}\right)}\, f_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) = f_*(0) = 0,$$

$$\frac{p}{v^p-u^p}\int_u^v x^{p-1}f_*(x)dx = \frac{1}{2}\int_{-1}^1 x^{2p-1}dx = 0,$$

$$\left[f_*(u) + f_*(v)\right]\int_0^1 \hbar(\varrho)\,d\varrho = 0,$$

that means

$$0 \leq 0 \leq 0.$$

Similarly, it can be easily show that

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} f^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{v^p-u^p}\int_u^v x^{p-1} f^*(x)dx \leq [f^*(u)+\ f^*(v)]\int_0^1 \hbar(\varrho)\,d\varrho,$$

such that

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} f^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) = f^*(0) = 1,$$

$$\frac{p}{v^p-u^p}\int_u^v x^{p-1} f^*(x)dx = \frac{1}{2}\int_{-1}^1 x^{p-1}e^{x^p}dx = \frac{e-e^{-1}}{2},$$

$$[f^*(u)+\ f^*(v)]\int_0^1 \hbar(\varrho)\,d\varrho = \frac{e+e^{-1}}{2},$$

From which, it follows that

$$1 \leq \frac{e-e^{-1}}{2} \leq \frac{e+e^{-1}}{2},$$

that is

$$[0,1] \leq_p \left[0, \frac{e-e^{-1}}{2}\right] \leq_p \left[0, \frac{e+e^{-1}}{2}\right].$$

Hence,

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{p}{v^p-u^p}\,(R)\int_u^v x^{p-1} f(x)dx \leq_p [f(u)+\ f(v)]\int_0^1 \hbar(\varrho)\,d\varrho.$$

Theorem 3.4. Let $\hbar$: $[0, 1] \to \mathbb{R}^+$ such that $\hbar \not\equiv 0$ and and $\hbar\left(\frac{1}{2}\right) \neq 0$, and let f: $[u, v] \to \mathbb{R}_I^+$ be an IVF such that $f(x) = [f_*(x), f^*(x)]$, for all $x \in [u, v]$. If $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$ and $f \in \mathcal{IR}_{([u,v])}$, then

$$\frac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2} f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \triangleright_2 \leq_p \frac{p}{v^p-u^p}\,(IR)\int_u^v x^{p-1} f(x)dx \leq_p \triangleright_1 \leq_p [f(u)+f(v)]\left[\frac{1}{2}+\hbar\left(\frac{1}{2}\right)\right]\int_0^1 \hbar(\varrho)\,d\varrho,$$

where

$$\triangleright_1 = \left[\frac{f(u)+f(v)}{2} + f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)\right]\int_0^1 \hbar(\varrho)\,d\varrho,$$

$$\triangleright_2 = \frac{1}{4\hbar\left(\frac{1}{2}\right)}\left[f\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}}\right) + f\left(\left[\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}}\right)\right],$$

and $\triangleright_1 = [\triangleright_{1*}, \triangleright_1{}^*]$, $\triangleright_2 = [\triangleright_{2*}, \triangleright_2{}^*]$.

Proof. Take $\left[u^p, \frac{u^p+v^p}{2}\right]$, we have

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} f\left(\left[\frac{\varrho u^p+(1-\varrho)\frac{u^p+v^p}{2}}{2} + \frac{(1-\varrho)u^p+\varrho\frac{u^p+v^p}{2}}{2}\right]^{\frac{1}{p}}\right) \leq_p f\left(\left[\varrho u^p + (1-\varrho)\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) + f\left(\left[(1-\varrho)u^p + \varrho\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right).$$

Therefore, we have

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} f_*\left(\left[\frac{\varrho u^p+(1-\varrho)\frac{u^p+v^p}{2}}{2} + \frac{(1-\varrho)u^p+\varrho\frac{u^p+v^p}{2}}{2}\right]^{\frac{1}{p}}\right)$$
$$\leq f_*\left(\left[\varrho u^p + (1-\varrho)\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) + f_*\left(\left[(1-\varrho)u^p + \varrho\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right),$$

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} f^*\left(\left[\frac{\varrho u^p+(1-\varrho)\frac{u^p+v^p}{2}}{2} + \frac{(1-\varrho)u^p+\varrho\frac{u^p+v^p}{2}}{2}\right]^{\frac{1}{p}}\right)$$
$$\leq f^*\left(\left[\varrho u^p + (1-\varrho)\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) + f^*\left(\left[(1-\varrho)u^p + \varrho\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right).$$

In consequence, we obtain

$$\frac{1}{4\hbar\left(\frac{1}{2}\right)} f_*\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}}\right) \leq \frac{p}{v^p-u^p}\int_u^{\frac{u^p+v^p}{2}} x^{p-1} f_*(x)dx\,,$$
$$\frac{1}{4\hbar\left(\frac{1}{2}\right)} f^*\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}}\right) \leq \frac{p}{v^p-u^p}\int_u^{\frac{u^p+v^p}{2}} x^{p-1} f^*(x)dx\,.$$

That is

$$\frac{1}{4\hbar\left(\frac{1}{2}\right)}\left[f_*\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}}\right), f^*\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}}\right)\right] \leq_p \frac{p}{v^p-u^p}\left[\int_u^{\frac{u^p+v^p}{2}} x^{p-1} f_*(x)dx\,, \int_u^{\frac{u^p+v^p}{2}} x^{p-1} f^*(x)dx\right].$$

It follows that

$$\frac{1}{4\hbar\left(\frac{1}{2}\right)} f\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}}\right) \leq_p \frac{p}{v^p-u^p}\int_u^{\frac{u^p+v^p}{2}} x^{p-1} f(x)dx. \quad (14)$$

In a similar way as above, we have

$$\frac{1}{4\hbar\left(\frac{1}{2}\right)} f\left(\left[\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}}\right) \leq_p \frac{p}{v^p-u^p}\int_{\frac{u^p+v^p}{2}}^{v} x^{p-1} f(x)dx. \quad (15)$$

Combining (14) and (15), we have

$$\frac{1}{4\hbar\left(\frac{1}{2}\right)}\left[f\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}}\right) + f\left(\left[\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}}\right)\right] \leq_p \frac{p}{v^p-u^p}\int_u^v x^{p-1} f(x)dx.$$

By using Theorem 3.1, we have

$$\frac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2} f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) = \frac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2} f\left(\left[\frac{1}{2}.\frac{3u^p+v^p}{4} + \frac{1}{2}.\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}}\right).$$

Therefore, we have

$$\frac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2} f_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) = \frac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2} f_*\left(\left[\frac{1}{2}.\frac{3u^p+v^p}{4} + \frac{1}{2}.\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}}\right),$$
$$\frac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2} f^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) = \frac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2} f^*\left(\left[\frac{1}{2}.\frac{3u^p+v^p}{4} + \frac{1}{2}.\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}}\right),$$

$$\leq \frac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2}\left[\hbar\left(\frac{1}{2}\right) f_*\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}}\right)+\hbar\left(\frac{1}{2}\right) f_*\left(\left[\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}}\right)\right],$$
$$\leq \frac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2}\left[\hbar\left(\frac{1}{2}\right) f^*\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}}\right)+\hbar\left(\frac{1}{2}\right) f^*\left(\left[\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}}\right)\right],$$

$$=\rhd_{2*},$$
$$=\rhd_2{}^*,$$

$$\leq \frac{p}{v^p-u^p}\int_u^v x^{p-1} f_*(x)dx,$$
$$\leq \frac{p}{v^p-u^p}\int_u^v x^{p-1} f^*(x)dx,$$

$$\leq\left[\frac{f_*(u)+f_*(v)}{2}+f_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)\right]\int_0^1 \hbar(\varrho)\,d\varrho,$$
$$\leq\left[\frac{f^*(u)+f^*(v)}{2}+f^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)\right]\int_0^1 \hbar(\varrho)\,d\varrho,$$

$$=\rhd_{1*},$$
$$=\rhd_1{}^*,$$

$$\leq\left[\frac{f_*(u)+f_*(v)}{2}+\hbar(\varrho)\left(f_*(u)+f_*(v)\right)\right]\int_0^1 \hbar(\varrho)\,d\varrho,$$
$$\leq\left[\frac{f^*(u)+f^*(v)}{2}+\hbar(\varrho)\left(f^*(u)+f^*(v)\right)\right]\int_0^1 \hbar(\varrho)\,d\varrho,$$

$$=[f_*(u)+f_*(v)]\left[\frac{1}{2}+\hbar\left(\frac{1}{2}\right)\right]\int_0^1 \hbar(\varrho)\,d\varrho,$$
$$=[f^*(u)+f^*(v)]\left[\frac{1}{2}+\hbar\left(\frac{1}{2}\right)\right]\int_0^1 \hbar(\varrho)\,d\varrho,$$

that is

$$\frac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2} f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)\leq_p \rhd_2 \leq_p \frac{p}{v^p-u^p}\,(IR)\int_u^v x^{p-1} f(x)dx \leq_p \rhd_1 \leq_p [f(u)+f(v)]\left[\frac{1}{2}+\hbar\left(\frac{1}{2}\right)\right]\int_0^1 \hbar(\varrho)\,d\varrho,$$

hence, the result follows.

Example 3.5. Let p be an odd number and $\hbar(\varrho)=\varrho$, for $\varrho\in[0,1]$, and the IVF $f:[u,v]=[-1,1]\to\mathbb{R}_I^+$ defined by, $f(x)=[x^p,\ e^{xp}]$, as in Example 3.3, then $f(x)$ is $(p, \hbar)$-convex IVF and satisfying (4.1). We have $f_*(x)=x^p$ and $f^*(x)=e^{xp}$. We now computing the following

$$[f_*(u)+f_*(\vartheta)]\left[\frac{1}{2}+\hbar\left(\frac{1}{2}\right)\right]\int_0^1 \hbar(\varrho)\,d\varrho=0,$$
$$[f^*(u)+f^*(\vartheta)]\left[\frac{1}{2}+\hbar\left(\frac{1}{2}\right)\right]\int_0^1 \hbar(\varrho)\,d\varrho=\frac{e+e^{-1}}{2},$$

$$\rhd_{1*}=\left[\frac{f_*(u)+f_*(\vartheta)}{2}+f_*\left(\left[\frac{u^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right)\right]\int_0^1 \hbar(\varrho)\,d\varrho=0,$$
$$\rhd_1{}^*=\left[\frac{f^*(u)+f^*(\vartheta)}{2}+f^*\left(\left[\frac{u^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right)\right]\int_0^1 \hbar(\varrho)\,d\varrho=\frac{e+e^{-1}+2}{4},$$

$$\rhd_{2*}=\frac{1}{4\,\hbar\left(\frac{1}{2}\right)}\left[f_*\left(\left[\frac{3u^p+\vartheta^p}{4}\right]^{\frac{1}{p}}\right)+f_*\left(\left[\frac{u^p+3\vartheta^p}{4}\right]^{\frac{1}{p}}\right)\right]=0$$
$$\rhd_2{}^*=\frac{1}{4\,\hbar\left(\frac{1}{2}\right)}\left[f^*\left(\left[\frac{3u^p+\vartheta^p}{4}\right]^{\frac{1}{p}}\right)+f^*\left(\left[\frac{u^p+3\vartheta^p}{4}\right]^{\frac{1}{p}}\right)\right]=\frac{e^{-\frac{1}{2}}+e^{\frac{1}{2}}}{2},$$

$$\frac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2} f_*\left(\left[\frac{u^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right)=0,$$
$$\frac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2} f^*\left(\left[\frac{u^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right)=1.$$

Then we obtain that

$$0\leq 0\leq 0\leq 0\leq 0,$$
$$1\leq\frac{e^{-\frac{1}{2}}+e^{\frac{1}{2}}}{2}\leq\frac{e-e^{-1}}{2}\leq\frac{e+e^{-1}+2}{4}\leq\frac{e+e^{-1}}{2},$$

Hence, Theorem 3.4 **is** verified.

In Theorem 3.6 and theorem 3.7, we obtain some interval integral inequalities for the product of two LR-$(p, \hbar)$-convex-IVFs.

Theorem 3.6. Let $\hbar_1$, $\hbar_2$: $[0, 1]\to\mathbb{R}^+$ such that $\hbar_1$, $\hbar_2\not\equiv 0$, and let f, g: $[u, v]\to\mathbb{R}_I^+$ be two IVFs, respectively, defined by $f(x)=[f_*(x), f^*(x)]$ and $g(x)=[g_*(x), g^*(x)]$ for all $x\in[u, v]$, respectively. If $f\in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar_1))$, $g\in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar_2))$ and $fg\in\mathcal{IR}_{([u,v])}$, then

$$\frac{p}{v^p-u^p}\,(IR)\int_u^v x^{p-1} f(x)g(x)dx\leq_p \mathcal{M}(u,v)\int_0^1 \hbar_1(\varrho)\,\hbar_2(\varrho)\,d\varrho+\mathcal{N}(u,v)\int_0^1 \hbar_1(\varrho)\,\hbar_2(1-\varrho)\,d\varrho,$$

where $\mathcal{M}(u, v)=f(u)g(u)+f(v)g(v)$, $\mathcal{N}(u, v)=f(u)g(v)+f(v)g(u)$, and $\mathcal{M}(u, v)=[\mathcal{M}_*((u, v)), \mathcal{M}^*((u, v))]$ and $\mathcal{N}(u, v)=[\mathcal{N}_*((u, v)), \mathcal{N}^*((u, v))]$.

Proof. Since $f\in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar_1))$ and $g\in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar_2))$then, we have

$$f_*\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)\leq\hbar_1(\varrho)f_*(u)+\hbar_1(1-\varrho)f_*(v),$$
$$f^*\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)\leq\hbar_1(\varrho)f^*(u)+\hbar_1(1-\varrho)f^*(v).$$

And

$$g_*\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)\leq\hbar_2(\varrho)g_*(u)+\hbar_2(1-\varrho)g_*(v),$$
$$g^*\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)\leq\hbar_2(\varrho)g^*(u)+\hbar_2(1-\varrho)g^*(v).$$

From the definition of LR-$(p, \hbar)$-convex-IVFs it follows that $f(x)\geq_p 0$ and $g(x)\geq_p 0$, so

$$\begin{aligned}
&f_*\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)g_*\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)\\
&\leq\begin{pmatrix}\hbar_1(\varrho)f_*(u)\\+\hbar_1(1-\varrho)f_*(v)\end{pmatrix}\begin{pmatrix}\hbar_2(\varrho)g_*(u)\\+\hbar_2(1-\varrho)g_*(v)\end{pmatrix}\\
&=f_*(u)g_*(u)[\hbar_1(\varrho)\,\hbar_2(\varrho)]+f_*(v)g_*(v)[\hbar_1(1-\varrho)\,\hbar_2(1-\varrho)]\\
&\quad+f_*(u)g_*(v)\,\hbar_1(\varrho)\,\hbar_2(1-\varrho)\\
&\quad+f_*(v)g_*(u)\,\hbar_1(1-\varrho)\,\hbar_2(\varrho),\\
&f^*\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)g^*\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)\\
&\leq\begin{pmatrix}\hbar_1(\varrho)f^*(u)\\+\hbar_1(1-\varrho)f^*(v)\end{pmatrix}\begin{pmatrix}\hbar_2(\varrho)\,g^*(u)\\+\hbar_2(1-\varrho)\,g^*(v)\end{pmatrix}\\
&=f^*(u)g^*(u)[\hbar_1(\varrho)\,\hbar_2(\varrho)]+f^*(v)g^*(v)[\hbar_1(1-\varrho)\,\hbar_2(1-\varrho)]\\
&\quad+f^*(u)g^*(v)\,\hbar_1(\varrho)\,\hbar_2(1-\varrho)\\
&\quad+f^*(v)g^*(u)\,\hbar_1(1-\varrho)\,\hbar_2(\varrho),
\end{aligned}$$

Integrating both sides of above inequality over [0, 1] we get

$$\begin{aligned}\int_0^1 f_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) g_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) &= \frac{p}{v^p-u^p}\int_u^v x^{p-1} f_*(x) g_*(x)\, dx \\ &\le \left(f_*(u)g_*(u) + f_*(v)g_*(v)\right)\int_0^1 \hbar_1(\varrho)\,\hbar_2(\varrho)\, d\varrho \\ &\quad +\left(f_*(u)g_*(v) + f_*(v)g_*(u)\right)\int_0^1 \hbar_1(\varrho)\,\hbar_2(1-\varrho)\, d\varrho, \\ \int_0^1 f^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) g^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) &= \frac{p}{v^p-u^p}\int_u^v x^{p-1} f^*(x) g^*(x)\, dx \\ &\le \left(f^*(u)g^*(u) + f^*(v)g^*(v)\right)\int_0^1 \hbar_1(\varrho)\,\hbar_2(\varrho)\, d\varrho \\ &\quad +\left(f^*(u)g^*(v) + f^*(v)g^*(u)\right)\int_0^1 \hbar_1(\varrho)\,\hbar_2(1-\varrho)\, d\varrho.\end{aligned}$$

It follows that,

$$\begin{aligned}\frac{p}{v^p-u^p}\int_u^v x^{p-1} f_*(x) g_*(x)\, dx &\le \mathcal{M}_*\left((u,v)\right)\int_0^1 \hbar_1(\varrho)\,\hbar_2(\varrho)\, d\varrho \\ &\quad +\mathcal{N}_*\left((u,v)\right)\int_0^1 \hbar_1(\varrho)\,\hbar_2(1-\varrho)\, d\varrho, \\ \frac{p}{v^p-u^p}\int_u^v x^{p-1} f^*(x) g^*(x)\, dx &\le \mathcal{M}^*\left((u,v)\right)\int_0^1 \hbar_1(\varrho)\,\hbar_2(\varrho)\, d\varrho \\ &\quad +\mathcal{N}^*\left((u,v)\right)\int_0^1 \hbar_1(\varrho)\,\hbar_2(1-\varrho)\, d\varrho,\end{aligned}$$

that is

$$\begin{aligned}\frac{p}{v^p-u^p}&\left[\int_u^v x^{p-1} f_*(x) g_*(x) dx\, , \int_u^v x^{p-1} f^*(x) g^*(x)\, dx\right] \\ &\le_p \left[\mathcal{M}_*\left((u,v)\right), \mathcal{M}^*\left((u,v)\right)\right]\int_0^1 \hbar_1(\varrho)\,\hbar_2(\varrho)\, d\varrho \\ &\quad +\left[\mathcal{N}_*\left((u,v)\right), \mathcal{N}^*\left((u,v)\right)\right]\int_0^1 \hbar_1(\varrho)\,\hbar_2(1-\varrho)\, d\varrho.\end{aligned}$$

Thus,

$$\frac{p}{v^p-u^p}\ (IR)\int_u^v x^{p-1} f(x) g(x) dx \le_p \mathcal{M}(u,v)\int_0^1 \hbar_1(\varrho)\,\hbar_2(\varrho)\, d\varrho\ +\ \mathcal{N}(u,v)\int_0^1 \hbar_1(\varrho)\,\hbar_2(1-\varrho)\, d\varrho.$$

Hence, this concludes the proof.

Theorem 3.7. Let $\hbar_1$, $\hbar_2$: $[0, 1] \to \mathbb{R}^+$ such that $\hbar_1$, $\hbar_2 \not\equiv 0$ and $\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right) \neq 0$, and let f, g: $[u, v] \to \mathbb{R}_I^+$ be two IVFs, respectively, defined by $f(x) = [f_*(x), f^*(x)]$ and $g(x) = [g_*(x), g^*(x)]$ for all $x \in [u, v]$. If $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar_1))$, $g \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar_2))$ and $fg \in \mathcal{IR}_{([u,v]})$, then

$$\begin{aligned}\frac{1}{2\,\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\, f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) &\le_p \frac{p}{v^p-u^p}\ (IR)\int_u^v x^{p-1} f(x) g(x) dx \\ &\quad + \mathcal{M}(u,v)\int_0^1 \hbar_1(\varrho)\,\hbar_2(1-\varrho)\, d\varrho \\ &\quad +\mathcal{N}(u,v)\int_0^1 \hbar_1(\varrho)\,\hbar_2(\varrho)\, d\varrho,\end{aligned}$$

where $\mathcal{M}(u, v) = f(u)g(u) + f(v)g(v)$, $\mathcal{N}(u, v) = f(u)g(v) + f(v)g(u)$, and $\mathcal{M}(u, v) = [\mathcal{M}_*((u, v)), \mathcal{M}^*((u, v))]$ and $\mathcal{N}(u, v) = [\mathcal{N}_*((u, v)), \mathcal{N}^*((u, v))]$

Proof. By hypothesis, we have

$$f_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) g_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)$$
$$f^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) g^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)$$

$$\begin{aligned}&\le \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} f_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) g_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) \\ +f_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) g_*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\end{bmatrix} \\ &\quad + \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} f_*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) g_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) \\ +f_*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) g_*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\end{bmatrix}, \\ &\le \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} f^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) g^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) \\ +f^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) g^*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\end{bmatrix} \\ &\quad + \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} f^*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) g^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) \\ +f^*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) g^*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\end{bmatrix},\end{aligned}$$

$$\begin{aligned}&\le \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} f_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) g_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) \\ +f_*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) g_*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\end{bmatrix} \\ &\quad + \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \left(\hbar_1(\varrho)\, f_*(u) + \hbar_1(1-\varrho)\, f_*(v)\right) \\ \left(\hbar_2(1-\varrho)\, g_*(u) + \hbar_2(\varrho)\, g_*(v)\right) \\ +\left(\hbar_1(1-\varrho)\, f_*(u) + \hbar_1(\varrho)\, f_*(v)\right) \\ \left(\hbar_2(\varrho)\, g_*(u) + \hbar_2(1-\varrho)\, g_*(v)\right)\end{bmatrix}, \\ &\le \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} f^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) g^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) \\ +f^*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) g^*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\end{bmatrix} \\ &\quad + \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \left(\hbar_1(\varrho)\, f^*(u) + \hbar_1(1-\varrho)\, f^*(v)\right) \\ \left(\hbar_2(1-\varrho)\, g^*(u) + \hbar_2(\varrho)\, g^*(v)\right) \\ +\left(\hbar_1(1-\varrho)\, f^*(u) + \hbar_1(\varrho)\, f^*(v)\right) \\ \left(\hbar_2(\varrho)\, g^*(u) + \hbar_2(1-\varrho)\, g^*(v)\right)\end{bmatrix},\end{aligned}$$

$$\begin{aligned}&= \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} f_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) g_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) \\ +f_*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) g_*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\end{bmatrix} \\ &\quad + 2\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \{\hbar_1(\varrho)\,\hbar_2(\varrho) + \hbar_1(1-\varrho)\,\hbar_2(1-\varrho)\}\mathcal{N}_*\left((u,v)\right) \\ +\{\hbar_1(\varrho)\,\hbar_2(1-\varrho) + \hbar_1(1-\varrho)\,\hbar_2(\varrho)\}\mathcal{M}_*\left((u,v)\right)\end{bmatrix}, \\ &= \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} f^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) g^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) \\ +f^*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) g^*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\end{bmatrix} \\ &\quad + 2\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \{\hbar_1(\varrho)\,\hbar_2(\varrho) + \hbar_1(1-\varrho)\,\hbar_2(1-\varrho)\}\mathcal{N}^*\left((u,v)\right) \\ +\{\hbar_1(\varrho)\,\hbar_2(1-\varrho) + \hbar_1(1-\varrho)\,\hbar_2(\varrho)\}\mathcal{M}^*\left((u,v)\right)\end{bmatrix},\end{aligned}$$

Integrating over [0, 1], we have

$$\begin{aligned}\frac{1}{2\,\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\, f_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) g_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) &\le \frac{p}{v^p-u^p}\ (R)\int_u^v x^{p-1} f_*(x) g_*(x) dx \\ &\quad + \mathcal{M}_*\left((u,v)\right)\int_0^1 \hbar_1(\varrho)\,\hbar_2(1-\varrho)\, d\varrho \\ &\quad +\mathcal{N}_*\left((u,v)\right)\int_0^1 \hbar_1(\varrho)\,\hbar_2(\varrho)\, d\varrho, \\ \frac{1}{2\,\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\, f^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) g^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) &\le \frac{p}{v^p-u^p}\ (R)\int_u^v x^{p-1} f^*(x) g^*(x) dx \\ &\quad + \mathcal{M}^*\left((u,v)\right)\int_0^1 \hbar_1(\varrho)\,\hbar_2(1-\varrho)\, d\varrho \\ &\quad +\mathcal{N}^*\left((u,v)\right)\int_0^1 \hbar_1(\varrho)\,\hbar_2(\varrho)\, d\varrho\,,\end{aligned}$$

that is

$$\begin{aligned}\frac{1}{2\,\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\, f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) &\le_p \frac{p}{v^p-u^p}\ (IR)\int_u^v x^{p-1} f(x) g(x) dx \\ &\quad + \mathcal{M}(u,v)\int_0^1 \hbar_1(\varrho)\,\hbar_2(1-\varrho)\, d\varrho \\ &\quad +\mathcal{N}(u,v)\int_0^1 \hbar_1(\varrho)\,\hbar_2(\varrho)\, d\varrho,\end{aligned}$$

Hence, the required result.

Example 3.8. Let p be an odd number and $\hbar_1(\varrho) = \varrho$, $\hbar_2(\varrho) = 1$, for $\varrho \in [0, 1]$, and the LR-$(p, \hbar_1)$-convex

f: $[u, \vartheta] = [2, 3] \to \mathbb{R}_I^+$ and LR-$(p, \hbar_2)$-convex IVFs g: $[u, \vartheta] = [2, 3] \to \mathbb{R}_I^+$ are, respectively defined by, $f(x) = \left[2 - x^{\frac{p}{2}}, 2\left(2 - x^{\frac{p}{2}}\right)\right]$, and $g(x) = [x^p, 2x^p]$. Since $f_*(x) = 2 - x^{\frac{p}{2}}$, $f^*(x) = 2\,(2 - x^{\frac{p}{2}})$, and $g_*(x) = x^p$, $g^*(x) = 2x^p$, then we computing the following

$$\frac{p}{\vartheta^p - u^p} \int_u^{\vartheta} x^{p-1} f_*(x) \times g_*(x) dx = 1,$$
$$\frac{p}{\vartheta^p - u^p} \int_u^{\vartheta} x^{p-1} f^*(x) \times g^*(x) dx = 4,$$

$$\mathcal{M}_*(u, \vartheta) \int_0^1 \hbar_1(\varrho)\, \hbar_2(\varrho)\, d\varrho = \left(10 - 2\sqrt{2} - 3\sqrt{3}\right)\frac{1}{2},$$
$$\mathcal{M}^*(u, \vartheta) \int_0^1 \hbar_1(\varrho)\, \hbar_2(\varrho)\, d\varrho = \left(10 - 2\sqrt{2} - 3\sqrt{3}\right)\frac{4}{2},$$

$$\mathcal{N}_*(u, \vartheta) \int_0^1 \hbar_1(\varrho)\, \hbar_2(1-\varrho)\, d\varrho = \left(10 - 3\sqrt{2} - 2\sqrt{3}\right)\frac{1}{2},$$
$$\mathcal{N}^*(u, \vartheta) \int_0^1 \hbar_1(\varrho)\, \hbar_2(1-\varrho)\, d\varrho = \left(10 - 3\sqrt{2} - 2\sqrt{3}\right)\frac{4}{2},$$

that means

$$1 \le \left(20 - 5\sqrt{2} - 5\sqrt{3}\right)\frac{1}{2},$$
$$4 \le \left(20 - 5\sqrt{2} - 5\sqrt{3}\right)\frac{4}{2}.$$

Hence, Theorem 3.6 has been demonstrated.

For Theorem 3.7, we have

$$\frac{1}{2\,\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\; f_*\left(\left[\frac{u^p + \vartheta^p}{2}\right]^{\frac{1}{p}}\right) \times g_*\left(\left[\frac{u^p + \vartheta^p}{2}\right]^{\frac{1}{p}}\right) = \frac{20 - 5\sqrt{10}}{4},$$
$$\frac{1}{2\,\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\; f^*\left(\left[\frac{u^p + \vartheta^p}{2}\right]^{\frac{1}{p}}\right) \times g^*\left(\left[\frac{u^p + \vartheta^p}{2}\right]^{\frac{1}{p}}\right) = 20 - 5\sqrt{10},$$

$$\mathcal{M}_*(u, \vartheta) \int_0^1 \hbar_1(\varrho)\, \hbar_2(1-\varrho)\, d\varrho = \frac{1}{2}\left(10 - 2\sqrt{2} - 3\sqrt{3}\right),$$
$$\mathcal{M}^*(u, \vartheta) \int_0^1 \hbar_1(\varrho)\, \hbar_2(1-\varrho)\, d\varrho = \frac{4}{2}\left(10 - 2\sqrt{2} - 3\sqrt{3}\right),$$

$$\mathcal{N}_*(u, \vartheta) \int_0^1 \hbar_1(\varrho)\, \hbar_2(\varrho)\, d\varrho = \frac{1}{2}\left(10 - 3\sqrt{2} - 2\sqrt{3}\right),$$
$$\mathcal{N}^*(u, \vartheta) \int_0^1 \hbar_1(\varrho)\, \hbar_2(\varrho)\, d\varrho = \frac{4}{2}\left(10 - 3\sqrt{2} - 2\sqrt{3}\right),$$

that means

$$\frac{20 - 5\sqrt{10}}{4} \le \left(1 + \frac{20 - 5\sqrt{2} - 5\sqrt{3}}{2}\right),$$
$$20 - 5\sqrt{10} \le \left(4 + \frac{20 - 5\sqrt{2} - 5\sqrt{3}}{2}\right),$$

hence, Theorem 3.7 is verified.

Next we derive HH-Fejér type inequality for LR-$(p, \hbar)$-convex-IVF by means of pseudo order relation.

Theorem 3.9. (Second HH-Fejér type inequality for LR-$(p, \hbar)$-convex-IVF) Let $\hbar$: $[0, 1] \to \mathbb{R}^+$ be a nonnegative real valued function and f: $[u, v] \to \mathbb{R}_I^+$ be an IVF with $u < v$, such that $f(x) = [f_*(x), f^*(x)]$ for all $x \in [u, v]$ and $f \in \mathcal{IR}_{([u,v])}$. If $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$, then $\mathcal{W}$: $[u, v] \to \mathbb{R}$, $\mathcal{W}(x) \ge 0$, p-symmetric with respect to $\left[\frac{u^p + v^p}{2}\right]^{\frac{1}{p}}$, then

$$\frac{p}{v^p - u^p}\,(IR)\int_u^v x^{p-1} f(x)\mathcal{W}(x)dx \le_p [f(u) + f(v)] \int_0^1 \hbar(\varrho)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right) d\varrho. \quad (16)$$

If $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$ then, inequality (16) is reversed.

Proof. Let $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$. Then we have

$$\begin{aligned}
&f_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)\\
&\qquad \le \left(\hbar(\varrho)f_*(u) + \hbar(1-\varrho)f_*(v)\right)\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right),\\
&f^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)\\
&\qquad \le \left(\hbar(\varrho)f^*(u) + \hbar(1-\varrho)f^*(v)\right)\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right).
\end{aligned} \quad (17)$$

And

$$\begin{aligned}
&f_*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\\
&\qquad \le \left(\hbar(1-\varrho)f_*(u) + \hbar(\varrho)f_*(v)\right)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right),\\
&f^*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\\
&\qquad \le \left(\hbar(1-\varrho)f^*(u) + \hbar(\varrho)f^*(v)\right)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right).
\end{aligned} \quad (18)$$

After adding (17) and (18), and integrating over [0, 1]. we get

$$\begin{aligned}
&\int_0^1 f_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)d\varrho\\
&\qquad + \int_0^1 f_*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)d\varrho\\
&\quad \le \int_0^1 \begin{bmatrix} f_*(u)\left\{\hbar(\varrho)\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) + \hbar(1-\varrho)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\right\}\\ + f_*(v)\left\{\hbar(1-\varrho)\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) + \hbar(\varrho)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\right\}\end{bmatrix} d\varrho,\\
&\int_0^1 f^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)d\varrho\\
&\qquad + \int_0^1 f^*\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)d\varrho\\
&\quad \le \int_0^1 \begin{bmatrix} f^*(u)\left\{\hbar(\varrho)\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) + \hbar(1-\varrho)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\right\}\\ + f^*(v)\left\{\hbar(1-\varrho)\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right) + \hbar(\varrho)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)\right\}\end{bmatrix} d\varrho.\\
&\quad = 2f_*(u)\int_0^1 \hbar(\varrho)\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)d\varrho + 2f_*(v)\int_0^1 \hbar(\varrho)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)d\varrho,\\
&\quad = 2f^*(u)\int_0^1 \hbar(\varrho)\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)d\varrho + 2f^*(v)\int_0^1 \hbar(\varrho)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)d\varrho.
\end{aligned}$$

Since $\mathcal{W}$ is symmetric, then

$$\begin{aligned}
&= 2[f_*(u) + f_*(v)]\int_0^1 \hbar(\varrho)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)d\varrho,\\
&= 2[f^*(u) + f^*(v)]\int_0^1 \hbar(\varrho)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)d\varrho.
\end{aligned} \quad (19)$$

Since

$$\begin{aligned}
&\int_0^1 f_*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)d\varrho\\
&\quad = \int_0^1 f_*((1-\varrho)u + \varrho v)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)d\varrho = \frac{p}{v^p - u^p}\int_u^v x^{p-1} f_*(x)\mathcal{W}(x)dx,\\
&\int_0^1 f^*\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\varrho u^p + (1-\varrho)v^p]^{\frac{1}{p}}\right)d\varrho\\
&\quad = \int_0^1 f^*((1-\varrho)u + \varrho v)\mathcal{W}\left([(1-\varrho)u^p + \varrho v^p]^{\frac{1}{p}}\right)d\varrho = \frac{p}{v^p - u^p}\int_u^v x^{p-1} f^*(x)\mathcal{W}(x)dx.
\end{aligned} \quad (20)$$

From (20), we have

$$\frac{p}{v^p-u^p}\int_u^v x^{p-1}f_*(x)\mathcal{W}(x)dx \le [f_*(u)+f_*(v)]\int_0^1 \hbar(\varrho)\mathcal{W}\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)d\varrho,$$
$$\frac{p}{v^p-u^p}\int_u^v x^{p-1}f^*(x)\mathcal{W}(x)dx \le [f^*(u)+f^*(v)]\int_0^1 \hbar(\varrho)\mathcal{W}\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)d\varrho,$$

that is

$$\frac{p}{v^p-u^p}\left[\int_u^v x^{p-1}f_*(x)\mathcal{W}(x)dx,\ \int_u^v x^{p-1}f^*(x)\mathcal{W}(x)dx\right]$$
$$\le_p [f_*(u)+f_*(v), f^*(u)+f^*(v)]\int_0^1 \hbar(\varrho)\mathcal{W}\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)d\varrho,$$

hence

$$\frac{p}{v^p-u^p}(IR)\int_u^v x^{p-1}f(x)\mathcal{W}(x)dx \le_p [f(u)+f(v)]\int_0^1 \hbar(\varrho)\mathcal{W}\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)d\varrho.$$

By the same technique used in the proof of Theorem 3.9 we get result following first HH-Fejér inequality for LR-(p, $\hbar_2$)-convex-IVF.

Theorem 3.10. (First HH-Fejér inequality for LR-(p, $\hbar$)-convex-IVF) Let $\hbar$: $[0, 1] \to \mathbb{R}^+$ such that $\hbar\left(\frac{1}{2}\right) \not\equiv 0$ and f: $[u, v] \to \mathbb{R}_I^+$ be an IVF with $u < v$, such that $f(x) = [f_*(x), f^*(x)]$ for all $x \in [u, v]$ and $f \in \mathcal{IR}_{([u,v])}$. If $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$ and $\mathcal{W}$: $[u,v] \to \mathbb{R}$, $\mathcal{W}(x) \ge 0$, p-symmetric with respect to $\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}$, and $\int_u^v x^{p-1}\mathcal{W}(x)dx > 0$, then

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \le_p \frac{1}{\int_u^v x^{p-1}\mathcal{W}(x)dx}(IR)\int_u^v x^{p-1}f(x)\mathcal{W}(x)dx. \quad (21)$$

If $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$ then, inequality (21) is reversed.

Proof. Since $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$ then, we have

$$\begin{aligned} f_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) &\le \hbar\left(\tfrac{1}{2}\right)\left(f_*\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)+f_*\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)\right),\\ f^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) &\le \hbar\left(\tfrac{1}{2}\right)\left(f^*\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)+f^*\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)\right), \end{aligned} \quad (22)$$

By multiplying (22) by $\mathcal{W}\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right) = \mathcal{W}\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)$ and integrate it by ϱ over [0, 1], we obtain

$$\begin{aligned} &f_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)\int_0^1 \mathcal{W}\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)d\varrho\\ &\le \hbar\left(\tfrac{1}{2}\right)\left(\begin{array}{c}\int_0^1 f_*\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)d\varrho\\ +\int_0^1 f_*\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)d\varrho\end{array}\right),\\ &f^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)\int_0^1 \mathcal{W}\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)d\varrho\\ &\le \hbar\left(\tfrac{1}{2}\right)\left(\begin{array}{c}\int_0^1 f^*\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)d\varrho\\ +\int_0^1 f^*\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)d\varrho\end{array}\right), \end{aligned} \quad (23)$$

Since

$$\begin{aligned} &\int_0^1 f_*\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)d\varrho\\ &\quad=\int_0^1 f_*\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)d\varrho\\ &\quad=\frac{p}{v^p-u^p}\int_u^v x^{p-1}f_*(x)\mathcal{W}(x)dx,\\ &\int_0^1 f^*\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\varrho u^p+(1-\varrho)v^p]^{\frac{1}{p}}\right)d\varrho\\ &\quad=\int_0^1 f^*\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\varrho)u^p+\varrho v^p]^{\frac{1}{p}}\right)d\varrho\\ &\quad=\frac{p}{v^p-u^p}\int_u^v x^{p-1}f^*(x)\mathcal{W}(x)dx, \end{aligned} \quad (24)$$

From (24), we have

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} f_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{1}{\int_u^v x^{p-1}\mathcal{W}(x)dx}\int_u^v x^{p-1}f_*(x)\mathcal{W}(x)dx,$$
$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} f^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{1}{\int_u^v x^{p-1}\mathcal{W}(x)dx}\int_u^v x^{p-1}f^*(x)\mathcal{W}(x)dx,$$

From which, we have

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\left[f_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right),\ f^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)\right] \le_p \frac{1}{\int_u^v x^{p-1}\mathcal{W}(x)dx}\left[\int_u^v x^{p-1}f_*(x)\mathcal{W}(x)dx,\ \int_u^v x^{p-1}f^*(x)\mathcal{W}(x)dx\right],$$

that is

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} f\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right) \le_p \frac{1}{\int_u^v x^{p-1}\mathcal{W}(x)dx}(IR)\int_u^v x^{p-1}f(x)\mathcal{W}(x)dx.$$

this completes the proof.

Remark 3.11. If in the Theorems 3.9 and 3.10 $\hbar(\varrho) = \varrho^s$ then, we obtain the appropriate theorems for LR-s-convex-IVFs on the second sense which are also new one.

If in the Theorems 3.9 and 3.10 $\hbar(\varrho) = \varrho$, then we obtain the appropriate theorems for LR-convex-IVFs which are also new one.

If $f_*(x) = f^*(x)$ then Theorems 3.9 and 3.10 reduce to classical first and second HH-Fejér inequality for $\hbar$-convex function, see [16].

If $\mathcal{W}(x) = 1$ then by combining Theorems 3.9 and 3.10, we get Theorem 3.1.

Example 3.12. We consider $\hbar(\varrho) = \varrho$ for $\varrho \in [0, 1]$, and the IVF f: $[1, 4] \to \mathbb{R}_I^+$ defined by,

$$f(x) = \left[e^{x^p}, 2e^{x^p}\right]. \quad (25)$$

Since end point functions $f_*(x)$, $f^*(x)$ both are (p, $\hbar$)-convex functions, then $\mathcal{T}(x)$ is LR-(p, $\hbar$)-convex-IVF. If

$$\mathcal{W}(x) = \begin{cases} x^p - 1, & \sigma \in \left[1, \frac{5}{2}\right],\\ 4 - x^p, & \sigma \in \left(\frac{5}{2}, 4\right], \end{cases} \quad (26)$$

where $p = 1$. Then, we have

$$\begin{aligned}
\frac{p}{\vartheta^p-u^p}\int_1^4 x^{p-1}f_*(x)\mathcal{W}(x)dx &= \frac{1}{3}\int_1^4 x^{p-1}f_*(x)\mathcal{W}(x)dx\\
&= \frac{1}{3}\int_1^{\frac{5}{2}} x^{p-1}f_*(x)\mathcal{W}(x)dx + \frac{1}{3}\int_{\frac{5}{2}}^4 x^{p-1}f_*(x)\mathcal{W}(x)dx\,,\\
&= \frac{1}{3}\int_1^{\frac{5}{2}} e^x(x-1)dx + \frac{1}{3}\int_{\frac{5}{2}}^4 e^x(4-x)dx \approx 11,\\
\frac{p}{\vartheta^p-u^p}\int_1^4 x^{p-1}f^*(x)\mathcal{W}(x)dx &= \frac{1}{3}\int_1^4 x^{p-1}f^*(x)\mathcal{W}(x)dx\\
&= \frac{1}{3}\int_1^{\frac{5}{2}} x^{p-1}f^*(x)\mathcal{W}(x)dx + \frac{1}{3}\int_{\frac{5}{2}}^4 x^{p-1}f^*(x)\mathcal{W}(x)dx,\\
&= \frac{2}{3}\int_1^{\frac{5}{2}} e^x(x-1)dx + \frac{2}{3}\int_{\frac{5}{2}}^4 e^x(4-x)dx \approx 22,
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&[f_*(u)+f_*(\vartheta)]\int_0^1 \hbar(\varrho)\mathcal{W}\left(\left[(1-\varrho)u^p+\varrho\vartheta^p\right]^{\frac{1}{p}}\right)d\varrho\\
&[f^*(u)+f^*(\vartheta)]\int_0^1 \hbar(\varrho)\mathcal{W}\left(\left[(1-\varrho)u^p+\varrho\vartheta^p\right]^{\frac{1}{p}}\right)d\varrho\\
&\qquad= [e+e^4]\left[\int_0^{\frac{1}{2}} 3\varrho^2 dx + \int_{\frac{1}{2}}^1 \varrho(3-3\varrho)d\varrho\right] \approx \frac{43}{2}.\\
&\qquad= 2[e+e^4]\left[\int_0^{\frac{1}{2}} 3\varrho^2 dx + \int_{\frac{1}{2}}^1 \varrho(3-3\varrho)d\varrho\right] \approx 43.
\end{aligned} \tag{28}$$

From (27) and (28), we have

$$[11, 22] \leq_p \left[\frac{43}{2}, 43\right].$$

Hence, Theorem 3.9 is verified.

For Theorem 3.10, we have

$$\begin{aligned}
\frac{1}{2\hbar\left(\frac{1}{2}\right)} f_*\left(\left[\frac{u^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right) &\approx \frac{61}{5},\\
\frac{1}{2\hbar\left(\frac{1}{2}\right)} f^*\left(\left[\frac{u^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right) &\approx \frac{122}{5}\ ,
\end{aligned} \tag{29}$$

$$\int_u^\vartheta x^{p-1}\mathcal{W}(x)dx = \int_1^{\frac{5}{2}} x^{p-1}(x-1)dx + \int_{\frac{5}{2}}^4 x^{p-1}(4-x)dx = \frac{9}{4},$$

$$\begin{aligned}
\frac{1}{\int_u^\vartheta x^{p-1}\mathcal{W}(x)dx}\int_1^4 x^{p-1}f_*(x)\mathcal{W}(x)dx &\approx \frac{73}{5},\\
\frac{1}{\int_u^\vartheta x^{p-1}\mathcal{W}(x)dx}\int_1^4 x^{p-1}f^*(x)\mathcal{W}(x)dx &\approx \frac{146}{5}.
\end{aligned} \tag{30}$$

From (29) and (30), we have

$$\left[\frac{61}{5}, \frac{122}{5}\right] \leq_p \left[\frac{73}{5}, \frac{293}{10}\right].$$

Hence, Theorem 3.10 is demonstrated.

DISCRETE JENSEN AND SCHUR-TYPE INEQUALITIES

This section proposes the discrete Jensen and Schur type inequalities for LR-(p, $\hbar$)-convex-IVF and proves some refinements of both inequalities. First of all, discrete Jensen type inequality for LR-(p, $\hbar$)-convex-IVF is proved in the following result.

Theorem 4.1. (Discrete Jensen type inequality for LR-(p, $\hbar$)-convex-IVF) Let $w_j \in \mathbb{R}^+$, $u_j \in [u, v]$, $(j = 1, 2, 3, \dots\dots\dots\, k, k \geq 2)$ and let $f\colon [u, v] \to \mathbb{R}_I^+$ be an IVF such that $f(x) = [f_*(x), f^*(x)]$ for all $x \in [u, v]$. If If $f \in LRSX([u, v], \mathbb{R}_I^+$ and $\hbar$ is a nonnegative supermultiplicative function on $\mathcal{L}$ then

$$f\left(\left[\frac{1}{W_k}\sum_{j=1}^k w_j\, u_j{}^p\right]^{\frac{1}{p}}\right) \leq_p \sum_j^k \hbar\left(\frac{w_j}{W_k}\right) f(u_j), \tag{31}$$

where $W_k = \sum_{j=1}^k w_j$. If the a nonnegative function $\hbar$ is sub-multiplicative function and $f \in LRSX([u, v], \mathbb{R}_I^+$, ($p$, $\hbar$) then, eq. (31) is reversed.

Proof. When $k = 2$ then, eq. (30) is true. Consider eq. (2.8) is true for $k = n - 1$, then

$$f\left(\left[\frac{1}{W_{n-1}}\sum_{j=1}^{n-1} w_j\, u_j{}^p\right]^{\frac{1}{p}}\right) \leq_p \sum_{j=1}^{n-1} \hbar\left(\frac{w_j}{W_{n-1}}\right) f(u_j),$$

Now, let us prove that eq. (31) holds for $k =$ n.

$$\begin{aligned}
&f\left(\left[\frac{1}{W_n}\sum_{j=1}^n w_j\, u_j{}^p\right]^{\frac{1}{p}}\right)\\
&\qquad= f\left(\left[\frac{1}{W_n}\sum_{j=1}^{n-2} w_j\, u_j{}^p + \frac{w_{n-1}+w_n}{W_n}\left(\frac{w_{n-1}}{w_{n-1}+w_n}u_{n-1}{}^p + \frac{w_n}{w_{n-1}+w_n}u_n{}^p\right]^{\frac{1}{p}}\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&f_*\left(\left[\frac{1}{W_n}\sum_{j=1}^n w_j\, u_j{}^p\right]^{\frac{1}{p}}\right)\\
&f^*\left(\left[\frac{1}{W_n}\sum_{j=1}^n w_j\, u_j{}^p\right]^{\frac{1}{p}}\right)\\
&\quad= f_*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n-2} w_j\, u_j{}^p + \frac{w_{n-1}+w_n}{W_n}\left(\frac{w_{n-1}}{w_{n-1}+w_n}u_{n-1}{}^p + \frac{w_n}{w_{n-1}+w_n}u_n{}^p\right]^{\frac{1}{p}}\right),\\
&\quad= f^*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n-2} w_j\, u_j{}^p + \frac{w_{n-1}+w_n}{W_n}\left(\frac{w_{n-1}}{w_{n-1}+w_n}u_{n-1}{}^p + \frac{w_n}{w_{n-1}+w_n}u_n{}^p\right]^{\frac{1}{p}}\right),\\
&\quad\leq \sum_{j=1}^{n-2} \hbar\left(\frac{w_j}{W_n}\right) f_*(u_j) + \hbar\left(\frac{w_{n-1}+w_n}{W_n}\right) f_*\left(\left[\frac{w_{n-1}}{w_{n-1}+w_n}u_{n-1}{}^p + \frac{w_n}{w_{n-1}+w_n}u_n{}^p\right]^{\frac{1}{p}}\right),\\
&\quad\leq \sum_{j=1}^{n-2} \hbar\left(\frac{w_j}{W_n}\right) f^*(u_j) + \hbar\left(\frac{w_{n-1}+w_n}{W_n}\right) f^*\left(\left[\frac{w_{n-1}}{w_{n-1}+w_n}u_{n-1}{}^p + \frac{w_n}{w_{n-1}+w_n}u_n{}^p\right]^{\frac{1}{p}}\right),\\
&\quad\leq \sum_{j=1}^{n-2} \hbar\left(\frac{w_j}{W_n}\right) f_*(u_j) + \hbar\left(\frac{w_{n-1}+w_n}{W_n}\right)\left[\hbar\left(\frac{w_{n-1}}{w_{n-1}+w_n}\right) f_*(u_{n-1}) + \hbar\left(\frac{w_n}{w_{n-1}+w_n}\right) f_*(u_n)\right],\\
&\quad\leq \sum_{j=1}^{n-2} \hbar\left(\frac{w_j}{W_n}\right) f^*(u_j) + \hbar\left(\frac{w_{n-1}+w_n}{W_n}\right)\left[\hbar\left(\frac{w_{n-1}}{w_{n-1}+w_n}\right) f^*(u_{n-1}) + \hbar\left(\frac{w_n}{w_{n-1}+w_n}\right) f^*(u_n)\right],\\
&\quad\leq \sum_{j=1}^{n-2} \hbar\left(\frac{w_j}{W_n}\right) f_*(u_j) + \left[\hbar\left(\frac{w_{n-1}}{W_n}\right) f_*(u_{n-1}) + \hbar\left(\frac{w_n}{W_n}\right) f_*(u_n)\right],\\
&\quad\leq \sum_{j=1}^{n-2} \hbar\left(\frac{w_j}{W_n}\right) f^*(u_j) + \left[\hbar\left(\frac{w_{n-1}}{W_n}\right) f^*(u_{n-1}) + \hbar\left(\frac{w_n}{W_n}\right) f^*(u_n)\right],\\
&\quad= \sum_{j=1}^n \hbar\left(\frac{w_j}{W_n}\right) f_*(u_j)\,,\\
&\quad= \sum_{j=1}^n \hbar\left(\frac{w_j}{W_n}\right) f^*(u_j)\,.
\end{aligned}$$

From which, we have

$$\left[f_*\left(\left[\frac{1}{W_n}\sum_{j=1}^n w_j\, u_j{}^p\right]^{\frac{1}{p}}\right), f^*\left(\left[\frac{1}{W_n}\sum_{j=1}^n w_j\, u_j{}^p\right]^{\frac{1}{p}}\right)\right] \leq_p \left[\sum_{j=1}^n \hbar\left(\frac{w_j}{W_n}\right) f_*(u_j)\,, \sum_{j=1}^n \hbar\left(\frac{w_j}{W_n}\right) f^*(u_j)\right],$$

that is,

$$f\left(\left[\frac{1}{W_n}\sum_{j=1}^n w_j\, u_j{}^p\right]^{\frac{1}{p}}\right) \leq_p \sum_{j=1}^n \hbar\left(\frac{w_j}{W_n}\right) f(u_j),$$

and the result follows.

If $w_1 = w_2 = w_3 = \ldots\ldots\ldots\ldots = w_k = 1$, then Theorem 4.1 reduces to the following result:

Corollary 4.2. Let $u_j \in [u, v]$, $(j = 1, 2, 3, \ldots\ldots\ldots\ldots k, k \geq 2)$ and let $f\colon [u, v] \to \mathbb{R}_I^+$ be an IVF such that $f(x) = [f_*(x), f^*(x)]$ for all $x \in [u, v]$. If $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$ and $\hbar$ is a nonnegative super-multiplicative function on $\mathcal{L}$, then

$$f\left(\left[\frac{1}{W_k}\sum_{j=1}^{k} w_j\, u_j{}^p\right]^{\frac{1}{p}}\right) \leq_p \sum_{j=1}^{k} \hbar\left(\frac{1}{k}\right) f(u_j), \tag{32}$$

If function $\hbar$ is sub-multiplicative and $f \in LRSV([u, v], \mathbb{R}_I^+, (p, \hbar))$ then, inequality (32) is reversed.

To obtain a refinement of Jensen inequality for LR-$(p, \hbar)$-convex-IVFs firstly, we prove the following the result:

Theorem 4.3. Let $\hbar\colon \mathcal{L} \to \mathbb{R}^+$ be a nonnegative super-multiplicative function and $f\colon [u, v] \to \mathbb{R}_I^+$ be an IVF such that $f(x) = [f_*(x), f^*(x)]$ for all $x \in [u, v]$. If $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$, then for $u_1, u_2, u_3 \in [u, v]$, such that $u_1 < u_2 < u_3$ and $u_3{}^p - u_1{}^p, u_3{}^p - u_2{}^p, u_2{}^p - u_1{}^p \in \mathcal{L}$, we have

$$\hbar(u_3{}^p - u_1{}^p) f(u_2) \leq_p \hbar(u_3{}^p - u_2{}^p) f(u_1) + \hbar(u_2{}^p - u_1{}^p) f(u_3) \tag{33}$$

If the function $\hbar$ is a nonnegative sub-multiplicative function and $f \in LRSV([u, v], \mathbb{R}_I^+, (p, \hbar))$ then, inequalit (33) is reversed.

Proof. Let $u_1, u_2, u_3 \in [u, v]$ and $\hbar(u_3{}^p - u_1{}^p) > 0$. Then by hypothesis, we have

$$\hbar\left(\frac{u_3{}^p - u_2{}^p}{u_3{}^p - u_1{}^p}\right) = \frac{\hbar(u_3{}^p - u_2{}^p)}{\hbar(u_3{}^p - u_1{}^p)} \text{ and } \hbar\left(\frac{u_2{}^p - u_1{}^p}{u_3{}^p - u_1{}^p}\right) = \frac{\hbar(u_2{}^p - u_1{}^p)}{\hbar(u_3{}^p - u_1{}^p)}.$$

Consider $\varrho = \frac{u_3{}^p - u_2{}^p}{u_3{}^p - u_1{}^p}$, then $u_2{}^p = \varrho u_1{}^p + (1-\varrho) u_3{}^p$. Since $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$ then, by hypothesis, we have

$$\begin{aligned} f_*(u_2) &\leq \hbar\left(\frac{u_3{}^p - u_2{}^p}{u_3{}^p - u_1{}^p}\right) f_*(u_1) + \hbar\left(\frac{u_2{}^p - u_1{}^p}{u_3{}^p - u_1{}^p}\right) f_*(u_3), \\ f^*(u_2) &\leq \hbar\left(\frac{u_3{}^p - u_2{}^p}{u_3{}^p - u_1{}^p}\right) f^*(u_1) + \hbar\left(\frac{u_2{}^p - u_1{}^p}{u_3{}^p - u_1{}^p}\right) f^*(u_3), \end{aligned} \tag{34}$$

$$\begin{aligned} &= \frac{\hbar(u_3{}^p - u_2{}^p)}{\hbar(u_3{}^p - u_1{}^p)} f_*(u_1) + \frac{\hbar(u_2{}^p - u_1{}^p)}{\hbar(u_3{}^p - u_1{}^p)} f_*(u_3), \\ &= \frac{\hbar(u_3{}^p - u_2{}^p)}{\hbar(u_3{}^p - u_1{}^p)} f^*(u_1) + \frac{\hbar(u_2{}^p - u_1{}^p)}{\hbar(u_3{}^p - u_1{}^p)} f^*(u_3). \end{aligned} \tag{35}$$

From (35), we have

$$\begin{aligned} \hbar(u_3{}^p - u_1{}^p) f_*(u_2) &\leq \hbar(u_3{}^p - u_2{}^p) f_*(u_1) + \hbar(u_2{}^p - u_1{}^p) f_*(u_3), \\ \hbar(u_3{}^p - u_1{}^p) f^*(u_2) &\leq \hbar(u_3{}^p - u_2{}^p) f^*(u_1) + \hbar(u_2{}^p - u_1{}^p) f^*(u_3), \end{aligned}$$

That is

$$\begin{aligned} &[\hbar(u_3{}^p - u_1{}^p) f_*(u_2),\ \hbar(u_3{}^p - u_1{}^p) f^*(u_2)] \\ &\quad \leq_p [\hbar(u_3{}^p - u_2{}^p) f_*(u_1) + \hbar(u_2{}^p - u_1{}^p) f_*(u_3), \\ &\quad \hbar(u_3{}^p - u_2{}^p) f^*(u_1) + \hbar(u_2{}^p - u_1{}^p) f^*(u_3)], \end{aligned}$$

Hence

$$\hbar(u_3{}^p - u_1{}^p) f(u_2) \leq_p \hbar(u_3{}^p - u_2{}^p) f(u_1) + \hbar(u_2{}^p - u_1{}^p) f(u_3).$$

Following result find a refinement of Schur type inequality for LR-$(p, \hbar)$-convex-IVF such that:.

Theorem 4.4. Let $w_j \in \mathbb{R}^+$, $u_j \in [u, v]$, $(j = 1, 2, 3, \ldots\ldots\ldots\ldots k, k \geq 2)$, $\hbar$ be a nonnegative super-multiplicative function on $\mathcal{L}$ and let $f\colon [u, v] \to \mathbb{R}_I^+$ be an IVF such that $f(x) = [f_*(x), f^*(x)]$ for all $x \in [u, v]$.. If $f \in LRSX([u, v], \mathbb{R}_I^+, (p, \hbar))$ and $u_1, u_2, \ldots\ldots, u_j \in (L, U) \subseteq [u, v]$ then,

$$\sum_{j=1}^{k} \hbar\left(\frac{w_j}{W_k}\right) f(u_j) \leq_p \sum_{j=1}^{k}\left(\hbar\left(\frac{U^p - u_j{}^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f(L) + \hbar\left(\frac{u_j{}^p - L^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f(U)\right), \tag{36}$$

where $W_k = \sum_{j=1}^{k} w_j$. If $\hbar$ is sub-multiplicative function and $f \in LRSV([u, v], \mathbb{R}_I^+, (p, \hbar))$ then, eq. (36) is reversed.

Proof. Consider $= u_1, u_j = u_2$, $(j = 1, 2, 3, \ldots\ldots\ldots\ldots k)$, $U = u_3$. Then, by hypothesis and eq. (36), we have

$$\begin{aligned} f_*(u_j) &\leq \hbar\left(\frac{U^p - u_j{}^p}{U^p - L^p}\right) f_*(L) + \hbar\left(\frac{u_j{}^p - L^p}{U^p - L^p}\right) f_*(U), \\ f^*(u_j) &\leq \hbar\left(\frac{U - u_j{}^p}{U^p - L^p}\right) f^*(L) + \hbar\left(\frac{u_j{}^p - L^p}{U^p - L^p}\right) f^*(U). \end{aligned}$$

Above inequality can be written as,

$$\begin{aligned} \hbar\left(\frac{w_j}{W_k}\right) f_*(u_j) &\leq \hbar\left(\frac{U^p - u_j{}^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f_*(L) + \hbar\left(\frac{u_j{}^p - L^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f_*(U), \\ \hbar\left(\frac{w_j}{W_k}\right) f^*(u_j) &\leq \hbar\left(\frac{U^p - u_j{}^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f^*(L) + \hbar\left(\frac{u_j{}^p - L^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f^*(U). \end{aligned} \tag{37}$$

Taking sum of all inequalities (37) for $j = 1, 2, 3, \ldots\ldots\ldots\ldots k$, we have

$$\begin{aligned} \sum_{j=1}^{k} \hbar\left(\frac{w_j}{W_k}\right) f_*(u_j) &\leq \sum_{j=1}^{k}\left(\hbar\left(\frac{U^p - u_j{}^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f_*(L) + \hbar\left(\frac{u_j{}^p - L^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f_*(U)\right), \\ \sum_{j=1}^{k} \hbar\left(\frac{w_j}{W_k}\right) f^*(u_j) &\leq \sum_{j=1}^{k}\left(\hbar\left(\frac{U^p - u_j{}^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f^*(L) + \hbar\left(\frac{u_j{}^p - L^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f^*(U)\right). \end{aligned}$$

that is

$$\begin{aligned} \sum_{j=1}^{k} \hbar\left(\frac{w_j}{W_k}\right) f(u_j) &= \left[\sum_{j=1}^{k} \hbar\left(\frac{w_j}{W_k}\right) f_*(u_j),\ \sum_{j=1}^{k} \hbar\left(\frac{w_j}{W_k}\right) f^*(u_j)\right] \\ &\leq_p \left[\sum_{j=1}^{k}\left(\hbar\left(\frac{U^p - u_j{}^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f_*(L) + \hbar\left(\frac{u_j{}^p - L^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f_*(U)\right),\ \sum_{j=1}^{k}\left(\hbar\left(\frac{U^p - u_j{}^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f^*(L) + \hbar\left(\frac{u_j{}^p - L^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f^*(U)\right)\right], \\ &\leq_p \sum_{j=1}^{k} \hbar\left(\frac{U^p - u_j{}^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right)[f_*(L),\ f^*(L)] + \sum_{j=1}^{k} \hbar\left(\frac{u_j{}^p - L^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right)[f_*(U),\ f^*(U)]. \\ &= \sum_{j=1}^{k} \hbar\left(\frac{U^p - u_j{}^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f(L) + \sum_{j=1}^{k} \hbar\left(\frac{u_j{}^p - L^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f(U). \end{aligned}$$

Thus,

$$\sum_{j=1}^{k} \hbar\left(\frac{w_j}{W_k}\right) f(u_j) \leq_p \sum_{j=1}^{k}\left(\hbar\left(\frac{U^p - u_j{}^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f(L) + \hbar\left(\frac{u_j{}^p - L^p}{U^p - L^p}\right)\hbar\left(\frac{w_j}{W_k}\right) f(U)\right)$$

this completes the proof.

We now consider some special cases of Theorem 4.1 and 4.4.

If $f_*(x) = f_*(x)$, then Theorem 4.1 and 4.4 reduce to the following results :

Corollary 4.5. [16] (Jensen inequality for LR-(p, $\hbar$)-convex function) Let $w_j \in \mathbb{R}^+$, $u_j \in [u, v]$, $(j = 1, 2, 3, \dots\dots\dots k, k \geq 2)$ and let $f : [u, v] \to \mathbb{R}^+$ be a non-negative real-valued function. If $f \in SX([u, v], \mathbb{R}_I^+, (p, \hbar))$ and $\hbar$ is a nonnegative supermultiplicative function on $\mathcal{L}$ then

$$f\left(\left[\frac{1}{W_k}\sum_{j=1}^{k} w_j\, {u_j}^p\right]^{\frac{1}{p}}\right) \leq \sum_{j=1}^{k} \hbar\left(\frac{w_j}{W_k}\right) f(u_j), \quad (38)$$

where $W_k = \sum_{j=1}^{k} w_j$. If $\hbar$ is sub-multiplicative function and $f \in SV([u, v], \mathbb{R}^+, (p, \hbar))$ then, eq. (38) is reversed.

Corollary 4.6. 16] Let $w_j \in \mathbb{R}^+$, $u_j \in [u, v]$, $(j = 1, 2, 3, \dots\dots\dots k, k \geq 2)$ $\hbar$ be a nonnegative super-multiplicative function on $\mathcal{L}$ and let $f : [u, v] \to \mathbb{R}^+$ be an non-negative real-valued function. If $f \in SX([u, v], \mathbb{R}_I^+, (p, \hbar))$ and $u_1, u_2, \dots\dots, u_j \in (L, U) \subseteq [u, v]$ then,

$$\sum_{j=1}^{k} \hbar\left(\frac{w_j}{W_k}\right) f(u_j) \leq \sum_{j=1}^{k}\left(\hbar\left(\frac{U^p - {u_j}^p}{U^p - L^p}\right) \hbar\left(\frac{w_j}{W_k}\right) f(L) + \hbar\left(\frac{{u_j}^p - L^p}{U^p - L^p}\right) \hbar\left(\frac{w_j}{W_k}\right) f(U)\right), \quad (39)$$

where $W_k = \sum_{j=1}^{k} w_j$. If $\hbar$ is sub-multiplicative function and $f \in SV([u, v], \mathbb{R}^+, (p, \hbar))$ then, eq. (39) is reversed.

CONCLUSION

We introduced the class of LR-(p, $\hbar$)-convex interval-valued functions by means of pseudo order relation and investigated some properties. Some novel Inequalities for LR-(p, $\hbar$)-convex interval-valued functions were proved. The results of this study can be applied in optimization, uncertainty analysis and also different areas of applied and pure sciences. We intend to use various types of LR-convex interval-valued functions to construct interval inequalities of interval-valued functions by means of pseudo order relation.

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The authors declare that they have no competing interests.

AUTHORS' CONTRIBUTIONS

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