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**Research Article**

# Solutions to nonlinear pseudo hyperbolic partial differential equations with nonlocal conditions by using residual power series method

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**ABSTRACT**

In this paper, new solutions to nonlinear pseudo-hyperbolic equations with non-local conditions by residual power series (RPS) method is given. This method is based on the Taylor series formula and the residual error function. A new analytical solution to this equation is given. As an applications we have tested some examples to know the efficiency of current technique. The results obtained showed that the proposed method is effective, accurate and has fast convergent.

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## INTRODUCTION

Firstly, we consider the generalised nonlinear pseudo-hyperbolic equation with nonlocal conditions as follows:

$$u_{tt}(t,x) - u(t,x)u_{txx}(t,x) - u_{xx}(t,x) - f(t,x) = 0,\ x \in (0,X),\ t \in (0,T). \quad (1)$$

With the initial and boundary conditions

$$u(0,x) = W_0(x),\ u_t(0,x) = W_1(x),\ x \in [0,X], \quad (2)$$

$$u(t,0) = \alpha(t) + \int_0^X u(t,x)dx\ , \quad 0 \le t \le T,$$

$$u(t,1) = \beta(t) + \int_0^X u(t,x)dx\ , \quad 0 \le t \le T,$$

Linear and nonlinear partial differential equations describe a range of phenomena in different fields of sciences such as engineering and physics.

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The exact solutions of this type of problems are rare in the literature, because it is not easy to find accurate solutions of nonlinear problems. Therefore, the analytical and numerical methods are used to find the approximation solution. Recently, many authors have been interested in testing linear and nonlinear partial and fractional differential equations using various numerical and analytical methods, among them the residual power series method [1], Adomian decomposition method [2], the finite difference method [3], the Variational iteration method [4], the homotopy analysis method [5], the reproducing kernel Hilbert space method [6] and the multiple Exp-function method [7], has been applied to solve linear and nonlinear partial differential equations.

The residual power series (RPS) method was first proposed by Arqoub [8] and it is an accurate and convenient technique for constructing power series solutions of differential equations and RPSM has been shown to be a suitable and effective method in its applications, and has been widely used to solve different types of problems [9-15].

The Hyperbolic partial differential equations model the longitudinal vibrations of structures, such as beams, buildings and machines [2,16-18] and have been studied to model real engineering problems [19-21].

The pseudo-hyperbolic equations of the form (1) is a type of high order partial differential equation with mixed partial derivatives concerning space and time, which describes many physical phenomena such as, longitudinal vibrations, heat and mass transfer, nerve conduction and reaction-diffusion [18,23,25]. In recent years, some authors have been applied the numerical and analytical methods for solving the pseudo-hyperbolic equation. The authors in [22] obtained the approximate analytical solutions of the pseudo-hyperbolic equation via residual power series method. The numerical approximation scheme based on the H1-Galerkin mixed finite element method to pseu-do-hyperbolic equation was constructed in [23]. The authors in [18] studied pseudo-hyperbolic equations and provided a comprehensive understanding of hyperbolic and pseudo-hyperbolic operators arising in the theory of longitudinal and lateral vibrations of elastic bars. In [24] the authors provided the sufficient condition for the nonexis-tence of weak solutions to the nonlinear pseudo-hyperbolic equation in the Heisenberg theory. In [26] the authors was discussed the splitting positive definite mixed finite ele-ment methods for pseudo-hyperbolic equations and the two least-squares Galerkin finite element schemes was for-mulated to solve pseudo hyperbolic equations in [27]. The aim of this article is to investigate the analytical solution of nonlinear pseudo-hyperbolic partial differential equation that depends on the nonlocal condition by an analytical method namely residual power series (RPS) method.

The rest of this paper organised as follows, in the sec-ond section the analysis of the (RPS) method is given. In the third section applications of (RPS) method to nonlinear Pseudo-Hyperbolic Equation are presented. Finally, in the last section conclusion is presented.

**Analysis of RPSM**

In this section, we give the basic information about the proposed method for solving nonlinear Pseudo-Hyperbolic Equation, the solutions of equations (1) and (2) in (RPS) method are expressed as the power series expansion at the initial point $t = 0$.

First, we suppose that the solutions take expansion

$$u(t,x) = \sum_{m=0}^{\infty} f_m(x)\, t^m, \mathrm{m} = 0,1,2,\ldots, \tag{3}$$

now, we determine $u_k(x,t)$ to indicate the $k^{th}$ truncated series of the u(t,x) as follows

$$u_k(t,x) = \sum_{m=0}^{k} f_m(x)\, t^m, \mathrm{k} = 1,2,3,\ldots. \tag{4}$$

It is clear that u(t,x) verify the initial condition (2), therefore the approximate solutions to zeroth via (RPS) method for u(t,x) is

$$u(0,x) = W_0(x) = f_0(x), \quad u_t(0,x) = W_1(x) = f_1(x). \tag{5}$$

However, the initial condition expressed by approximate solution from equation (4), then the first approximate solutions of (RPS) method must be

$$u_1(t,x) = f_0(x) + f_1(x)\, t. \tag{6}$$

Consequently, the expanding power series of Equation (4) can be expressed as

$$u_k(t,x) = f_0(x) + f_1(x)\, t + \sum_{m=2}^{k} f_m(x)\, t^m, \ \ \mathrm{k} = 2,3,4,\ldots. \tag{7}$$

To obtain the values of coefficients $f_m(x)$, for m = 2,3,4, ... k by the residual power series method, in the series expanding of equation (7), the residual functions could be defined as

$$Res_u(t,x) = u_{tt}(x,t) - u(t,x)u_{txx}(t,x) - u_{xx}(t,x) - f(t,x). \tag{8}$$

Consequently, the $k^{th}$ residual functions $Res_u(x,t)$, are of the following form

$$Res_{u,k}(t,x) = (u_k)_{tt} - u_k(u_k)_{txx} - (u_k)_{xx} - f(t,x), \mathrm{k} = 1,2,3,\ldots, \tag{9}$$

from the equation (9), we write $\lim_{x\to\infty} Res_k(t,x) = Res(t,x)$, where $x \in [0, X]$ and $t \geq 0$ Thus, $\frac{\partial^s}{\partial t^s} Res(t,x) = 0$, when $t = 0$ and $s = 0, \ldots, k$ as shown in [12].

For obtaining the coefficients $f_m(x)$, where $m$ = 2, 3, 4, ... k, we use the following procedures; substituting k[th] truncated series of $u(t, x)$ in equation (9), and applying the formula $\frac{\partial^s}{\partial t^s} Res_k(t,x) = 0$, for $m$ = 2, 3, 4, ... k, substitute $t$ = 0, then the following equation is equalised it to zero. Finally to obtain the form of other coefficients, we need to solve the following equation , for more details see [22]:

$$\frac{\partial^s}{\partial t^s} Res_k(t,x) = 0\,, \quad s = 2,3,4,\ldots,k. \tag{10}$$

In this way, we can get all the required coefficient of equations (1) and (2) for a series of multiple exponents.

**Application of the (RPS) Method to Pseudo-Hyperbolic Equations**

This section presents the applications of (RPS) method for solving nonlinear pseudo-hyperbolic equations depends on nonlocal conditions. for this, We test some examples to show the accuracy and efficacy of the proposed method.

**Example 1.** Consider the nonlinear pseudo-hyperbolic equation

$$\begin{aligned} &u_{tt}(t,x) - u(t,x)u_{txx}(t,x) - u_{xx}(t,x) - [t^3 + 6t + 1 \\ &+ 3t^2(t^3+1)sinx]sinx = 0,\ x \in (0,\pi),\ t \in (0,1), \end{aligned} \tag{11}$$

subject to the initial and boundary conditions

$$u(0,x) = sinx,\ u_t(0,x) = 0,\ 0 \le x \le \pi, \tag{12}$$

$$u(t,0) = 2(t^3+1) = u(t,\pi), \qquad 0 \le t \le 1.$$

Using the iterative formula of (RPS) method for $m$ = 0, 1, 2, ... we get

$$\begin{aligned} Res_k(t,x) = &\left(\sum_{m=2}^{k} m(m-1)f_m(x)t^{m-2}\right) - \left(\sum_{m=0}^{k} f_m(x)t^m\right) \times \left(\sum_{m=1}^{k}\left(\frac{d^2}{dx^2}f_m(x)\right)mt^{m-1}\right) \\ &-\left(\sum_{m=0}^{k}\left(\frac{d^2}{dx^2}f_m(x)\right)t^m\right) - [t^3 + 6t + 1 + 2t^2(t^3+1)sinx]sinx. \end{aligned} \tag{13}$$

By using (6) and from the initial condition (12), we get

$u_1(t, x) = sinx,$

substitute $u_1(t, x)$ in formula(13), we obtain

$u_2(t, x) = 0,$

in the same way, we can obtain $u_3(t, x)$, $u_4(t, x)$, ..., and substitute in formula (7), then we get the exact solution:

$$u(t,x) = (t^3+1)sinx. \tag{14}$$

**Example 2.** Consider the nonlinear pseudo-hyperbolic equation

$$\begin{aligned} &u_{tt}(t,x) - u_{txx}(t,x) - u(t,x)u_{xx}(t,x) \\ &- e^{-2t}sin^2x = 0,\ x \in (0,\pi),\ t \in (0,1), \end{aligned} \tag{15}$$

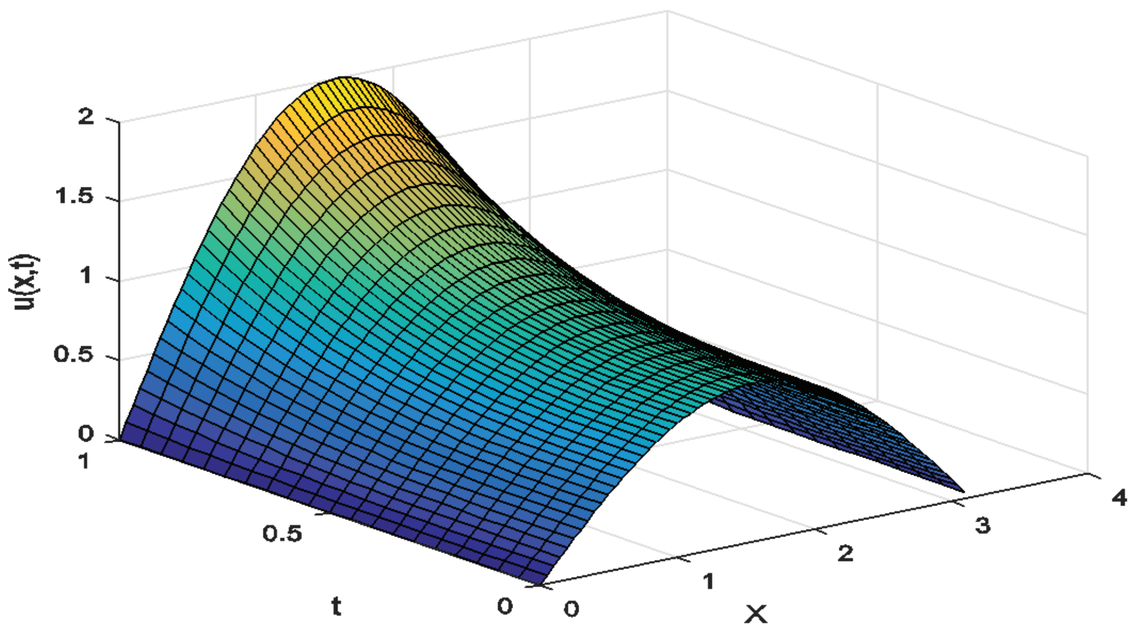


**Figure 1.** Gives the exact solution of u(t,x) for **example 1.**, at 0 ≤ t ≤ 1 and 0 ≤ x ≤ π.

Subject to the initial and boundary conditions

$$u(0,x) = sinx,\ u_t(x,0) = 0,\ 0 \le x \le \pi, \tag{16}$$

$$u(t,0) = \int_0^{\pi} u(t,x)dx - 2e^{-t} = u(t,\pi), \quad 0 \le t \le 1.$$

Using the iterative formula of (RPS) method for $m = 0, 1, 2, \ldots$ we get

$$Res_k(t,x) = \left(\sum_{m=2}^{k} m(m-1) f_m(x) t^{m-2}\right) - \left(\sum_{m=1}^{k} \left(\frac{d^2}{dx^2} f_m(x)\right) m t^{m-1}\right) - \left(\sum_{m=0}^{k} f_m(x) t^m\right) \times \left(\sum_{m=0}^{k} \left(\frac{d^2}{dx^2} f_m(x)\right) t^m\right) - e^{-2t} sinx. \tag{17}$$

From the initial condition (16) and formula (6), we get

$$u_1(t,x) = -tsinx,$$

substitute $u_1(t, x)$ in formula(17), we obtain

$$u_2(t,x) = \frac{t^2}{2} sinx,$$

by the same way for $u_3(t, x)$, $u_4(t, x)$, ..., and using the Taylor series formula, then it gives exact solution

$$u(t,x) = e^{-t} sinx. \tag{18}$$

## CONCLUSION

In this paper, a new results for nonlinear pseudo-hyperbolic equations depend non-local conditions by using (RPS) method are given, this method based on the Taylor series formula with residual error function. The method has excellent and accurate results for this equations. Simulations are shown for the obtained results. Therefore, the RPSM method is notables because it gives the exact solution to nonlinear pseudo-hyperbolic equations. The obtained result reveal that the accuracy and fast convergence for the proposed method.


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## AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

## DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

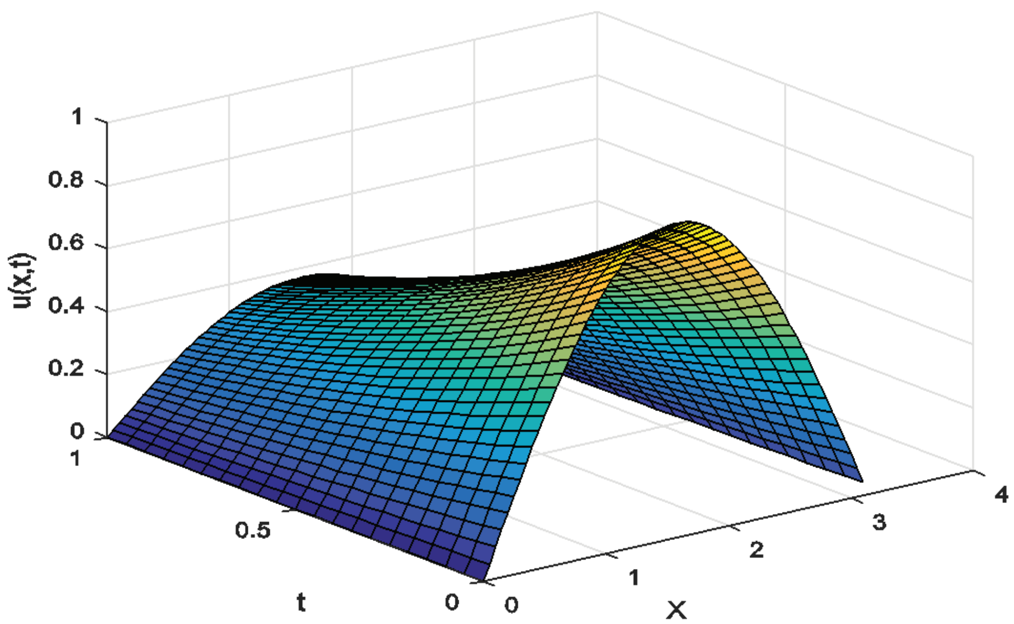


**Figure 2.** Illustrates the exact solution of u(t,x) for **example 2.**, at 0 ≤ t ≤ 1 and 0 ≤ x ≤ π.

## CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

## ETHICS

There are no ethical issues with the publication of this manuscript.

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