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Research Article

New inequalities of hermite-hadamard type for functions whose second derivatives absolute values are exponential trigonometric convex

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ABSTRACT

In this paper, a new class of convex functions, which is called exponential trigonometric convex functions is studied. We obtain some refinements of the Hermite-Hadamard type inequalities via functions whose second derivatives in absolute value at certain power are exponential trigonometric convex functions. The results of the research by the means of Hölder- İşcan and improved-power-mean integral inequalities.

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INTRODUCTION

Convex functions and convexity as a mathematical definition have now played an important and fundamental role in the development of various fields of pure and applied sciences. Convexity theory describes a wide range of significant developments that link the fields of study of mathematics, physics, economics and engineering sciences. Some of these developments have built mutually enriching relationships with other fields. The ideas explaining these concepts have led to the development of new and powerful techniques for solving a wide class of linear and nonlinear problems. For more details see [1-4] and the references cited therein.

In recent years, convexity, directly related to integral inequalities, has been considered as a research area, examining the concept of convexity and its different forms. One of the most studied inequalities regarding the integral mean of a convex is the Hermite–Hadamard inequality. The best-known Hermite and Hadamard inequality in the literature is stated as follows:

$f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \qquad (1)$$

for all $a, b \in I$ with $a < b$. Many researchers have made generalizations to inequality (1). For more results and detail see [5-7] and the references therein.

The concept of trigonometric convexity was defined and studied for the first time by Kadakal (2018). The algebraic

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theoretical knowledge in his work has recently increased attention by researchers in inequality theory involving trigononometric convex functions. Some notable results and literature on the theory of inequalities for trigonometric convexity can be found, see [8-12] and the references therein. With the contribution of the research mentioned above, notable concepts such as exponential trigonometric, Harmonic trigonometric, geometric trigonometric convex functions have advanced as an important new class of convex functions. Kadakal (2021) studied algebraic properties of the concept of exponential trigonometric convex functions and Hermite-Hadamard type inequalities for the newly introduced class of convex functions.

The aim of this study is to present the concepts of exponential convex, trigonometric convex and exponential trigonometric functions and to find some results regarding the newly refined inequalities of the right-hand-side of Hermite-Hadamard inequality for the class of functions whose second derivatives at certain powers are exponential trigonometric convex functions. As well as numerical analysis, the comparison of both Hölder, Hölder-İşcan and power-mean, improved power-mean are carried out subsequently.

Some concepts available in the literature that will be used for the proof of the main results will be given.

In [13], Kadakal defined the following new class of functions:

Definition 1. A non-negative function $f: I \to \mathbb{R}$ is called trigonometrically convex function on interval $[a, b]$, if for each $x, y \in [a, b]$ and $t \in [0,1]$,

$$f(tx+(1-t)y) \leq \sin\left(\frac{\pi t}{2}\right) f(x) + \cos\left(\frac{\pi t}{2}\right) f(y). \quad (2)$$

The class of all trigonometrically convex functions is denoted $TC(l)$ on interval l.

In [13], Kadakal also obtained the following Hermite-Hadamard type inequalities for the trigonometrically convex functions as follows:

Theorem 1. Let $f:[a, b] \to \mathbb{R}$ be a trigonometrically convex function, if $a < b$ and $f \in L[a, b]$, then the following inequality holds:

$$\frac{\sqrt{2}}{2} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{2}{\pi}\big(f(a)+f(b)\big).$$

Recently, the definition of exponential convex functions has been given and studied by Awan et al., [14].

Definition 2. A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is called exponential convex function, if for every $x, y \in I$, $t \in [0,1]$ and $\alpha \in \mathbb{R}$,

$$f\big((1-t)x+ty\big) \leq (1-t)\frac{f(x)}{e^{\alpha x}} + t\frac{f(y)}{e^{\alpha y}}. \quad (3)$$

One can say that f is said to be an exponential concave function, in the case that in (3) reverse inequality holds. For example, the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = -x^2$ is a concave function, so this function is exponential convex for every $a > 0$.

In [14], Awan et al., gave the following results.

Theorem 2. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be an exponential convex function, if $a < b$ and $f \in L[a, b]$, then the following Hermite-Hadamard type inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \frac{f(u)}{e^{\alpha u}}du \leq \frac{e^{-\alpha a}f(a)+e^{-\alpha b}f(b)}{2}.$$

In [15], Kadakal et al., introduced the concept of exponential trigonometric convex functions defined as follows:

Definition 3. Let $I \subseteq \mathbb{R}$ be an interval. A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is called exponential trigonometric convex function if for every $a, b \in l$, $t \in [0,1]$

$$f(ta+(1-t)b) \leq \frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}} f(a) + \frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}} f(b). \quad (4)$$

Denoted by $EXPTC\ (l)$ the class of all exponential trigonometric convex functions on interval l. We note that every non-negative exponential trigonometric convex function is a trigonometric convex function. Indeed, since $1 \leq e^{1-t} \leq e \Rightarrow 1/e \leq e^{t-1} \leq 1$ we write $e^{t-1}\sin\frac{\pi t}{2} \leq \sin\frac{\pi t}{2}$. In a similar way $1/e \leq e^{-t} \leq 1 \Rightarrow e^{-t}\cos\frac{\pi t}{2} \leq \cos\frac{\pi t}{2}$. So

$$f(ta+(1-t)b) \leq \frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}} f(a) + \frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}} f(b) \leq \sin\left(\frac{\pi t}{2}\right) f(a) + \cos\left(\frac{\pi t}{2}\right) f(b).$$

For example, the mapping $f: I \subset (-\infty,0) \to \mathbb{R}$, defined by $f(x) = x$ is an exponential trigonometric convex function. Since the inequality $t - \frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}} \geq 0$ and $1 - t - \frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}} \geq 0$ hold for $t \in [0,1]$.

In [15], Kadakal et al., gave the following results:

Theorem 3. Let $f:[a, b] \to \mathbb{R}$ be an exponentially convex function, if $a < b$ and $f \in L[a, b]$, then the following Hermite-Hadamard type inequality hold:

$$\sqrt{\frac{e}{2}} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{2\pi+4e}{e(\pi^2+4)}\big(f(a)+f(b)\big).$$

In [16], İşcan gave a refinement of the Hölder integral inequality follows:

Theorem 4. (Hölder-İscan integral inequality) Let $p > 1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|^q$ and $|g|^q$ are integrable functions on $[a, b]$ then

$$i)\ \int_a^b |f(x)g(x)|dx \le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q dx\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)\,|f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q dx\right)^{\frac{1}{q}}\right\} \quad (5)$$

Hölder-İşcan integral inequality offers better outcome approximations than Hölder integral inequality.

A different representation of the Hölder-İşcan inequality in [17] is given belows:

Theorem 5. (Improved power-mean integral inequality) Let q ≥ 1 and $\frac{1}{p}+\frac{1}{q}=1$. If *f* and *g* are real functions defined on interval [*a*, *b*] and if $|f|$ and $|f||g|^q$ are integrable functions on [*a*, *b*] then,

$$i)\ \int_a^b |f(x)g(x)|dx \le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)\,|f(x)||g(x)|^q dx\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)\,|f(x)|dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\,|f(x)||g(x)|^q dx\right)^{\frac{1}{q}}\right\}. \quad (6)$$

Improved power-mean integral inequality offers better approximations than power-mean integral inequality.

Let 0 < *a* < *b* and *q* ≥ 1, throughout this paper,

$$A(|f''(a)|,|f''(b)|) = \frac{|f''(a)|+|f''(b)|}{2}$$

$$A^{\frac{1}{q}}(|f''(a)|^q,|f''(b)|^q) = \left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}$$

will be used for the arithmetic mean.

MAIN RESULTS

Prior to main theorem, following lemma is needed in order to prove the results. This lemma can be easily achieved by taking partial integration into the lemma in [18].

Lemma 1. Let $I \subseteq \mathbb{R}$, f: $I \to \mathbb{R}$ be a twice differentiable function on I^0 with $f'' \in L[a, b]$ where $a, b \in l$ with $a < b$, then

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \le \frac{(b-a)^2}{2}\int_0^1 (t-t^2)f''(ta+(1-t)b)dt.$$

Theorem 6. f: $I \subset [0,\infty) \to \mathbb{R}$ be twice differentiable mapping on I^0 such that $f'' \in L[a, b]$, where $a, b \in l$ with $a < b$. If $|f''|$ is exponential trigonometric convex on [*a*, *b*]. Then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{2}\left(\frac{-256\pi e^{-1}+96\pi^2-4\pi^4-64}{(\pi^2+4)^3}\right)A(|f''(a)|,|f''(b)|). \quad (7)$$

Proof: If the function $|f''|$ is exponential trigonometric convex on [*a*, *b*], using lemma 1 and

$$|f''(ta+(1-t)b)| \le \frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}|f''(a)| + \frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}|f''(b)|,$$

we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{2}\int_0^1 |t-t^2||f''(ta+(1-t)b)|dt$$
$$\le \frac{(b-a)^2}{2}\int_0^1 t(1-t)|f''(ta+(1-t)b)|dt$$
$$\le \frac{(b-a)^2}{2}\int_0^1 t(1-t)\left[\left(\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}\right)|f''(a)| + \left(\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}\right)|f''(b)|\right]dt$$
$$= \frac{(b-a)^2}{2}\left[|f''(a)|\left(\frac{-256\pi e^{-1}+96\pi^2-4\pi^4-64}{(\pi^2+4)^3}\right) + |f''(a)|\left(\frac{-256\pi e^{-1}+96\pi^2-4\pi^4-64}{(\pi^2+4)^3}\right)\right]$$
$$= \frac{(b-a)^2}{2}\left(\frac{-256\pi e^{-1}+96\pi^2-4\pi^4-64}{(\pi^2+4)^3}\right)A(|f''(a)|,|f''(b)|)$$

where

$$\int_0^1 t(1-t)\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}dt = \int_0^1 t(1-t)\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}dt = \frac{-256\pi e^{-1}+96\pi^2-4\pi^4-64}{(\pi^2+4)^3}.$$

The proof is completed.

Theorem 7. f: $I \subset [0,\infty) \to \mathbb{R}$ be twice differentiable mapping on I^0 such that $f'' \in L[a, b]$, where $a, b \in l$ with $a < b$. If $|f''|^q$ is exponential trigonometric convex on [*a*, *b*], *p, q > 1 and* $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+1)\left(\frac{4\pi e^{-1}+8}{\pi^2+4}\right)^{\frac{1}{q}}A^{\frac{1}{q}}(|f''(a)|^q,|f''(b)|^q). \quad (8)$$

Proof: Suppose that *p > 1*, from lemma 1 and by the Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{2}\int_0^1 |t-t^2||f''(ta+(1-t)b)|dt$$
$$\le \frac{(b-a)^2}{2}\int_0^1 t(1-t)|f''(ta+(1-t)b)|dt$$
$$\le \frac{(b-a)^2}{2}\left(\int_0^1 t^p(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}.$$

Since $|f''|^q$ is exponential trigonometric convex, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{(b-a)^2}{2}\left(\int_0^1 t^p(1-t)^p dt\right)^{\frac{1}{p}}\left[\int_0^1\left(\left(\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}\right)|f''(a)|^q + \left(\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}\right)|f''(b)|^q\right)dt\right]^{\frac{1}{q}}$$
$$= \frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+1)\left(|f''(a)|^q\left(\frac{2\pi e^{-1}+4}{\pi^2+4}\right) + |f''(b)|^q\left(\frac{2\pi e^{-1}+4}{\pi^2+4}\right)\right)^{\frac{1}{q}}$$
$$= \frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+1)\left(\frac{4\pi e^{-1}+8}{\pi^2+4}\right)^{\frac{1}{q}}A^{\frac{1}{q}}(|f''(a)|^q,|f''(b)|^q).$$

Where

$$\int_0^1 \frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}dt = \int_0^1 \frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}dt = \frac{2\pi e^{-1}+4}{\pi^2+4}$$

$$\int_0^1 t^p(1-t)^p dt = \beta(p+1,p+1)$$

which is required.

Theorem 8. f: $I \subset [0,\infty) \to \mathbb{R}$ be twice differentiable mapping on I^0 such that $f'' \in L[a, b]$, where $a, b \in l$ with $a < b$. If $|f''|^q$ is exponential trigonometric convex on $[a, b]$, $q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds:

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\le \frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+2)\left(|f''(a)|^q\left(\frac{2\pi^3e^{-1}-4\pi^2+24\pi e^{-1}+16}{(\pi^2+4)^2}\right)+|f''(b)|^q\left(\frac{8\pi^2-16\pi e^{-1}}{(\pi^2+4)^2}\right)\right)^{\frac{1}{q}}\\ &+\frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+2,p+1)\left(|f''(a)|^q\left(\frac{8\pi^2-16\pi e^{-1}}{(\pi^2+4)^2}\right)+|f''(b)|^q\left(\frac{2\pi^3e^{-1}-4\pi^2+24\pi e^{-1}+16}{(\pi^2+4)^2}\right)\right)^{\frac{1}{q}}.\end{aligned} \tag{9}$$

Proof: If the function $|f''|^q$ for $q > 1$ is exponential trigonometric convex on $[a, b]$, from lemma 1 and by the Hölder-İşcan inequality and

$$|f''(ta+(1-t)b)|^q \le \frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}|f''(a)|^q+\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}|f''(b)|^q$$

we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\le \frac{(b-a)^2}{2}\int_0^1 |t-t^2||f''(ta+(1-t)b)|dt\\ &\le \frac{(b-a)^2}{2}\left(\int_0^1 t^p(1-t)^{p+1}dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)|f''(ta+(1-t)b)|dt\right)^{\frac{1}{q}}\\ &+\frac{(b-a)^2}{2}\left(\int_0^1 t^{p+1}(1-t)^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1 t|f''(ta+(1-t)b)|dt\right)^{\frac{1}{q}}\\ &\le \frac{(b-a)^2}{2}\left(\int_0^1 t^p(1-t)^{p+1}dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left(\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}|f''(a)|^q+\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}|f''(b)|^q\right)dt\right)^{\frac{1}{q}}\\ &+\frac{(b-a)^2}{2}\left(\int_0^1 t^{p+1}(1-t)^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left(\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}|f''(a)|^q+\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}|f''(b)|^q\right)dt\right)^{\frac{1}{q}}\\ &\le \frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+2)\left(|f''(a)|^q\left(\frac{2\pi^3e^{-1}-4\pi^2+24\pi e^{-1}+16}{(\pi^2+4)^2}\right)+|f''(b)|^q\left(\frac{8\pi^2-16\pi e^{-1}}{(\pi^2+4)^2}\right)\right)^{\frac{1}{q}}\\ &+\frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+2,p+1)\left(|f''(a)|^q\left(\frac{8\pi^2-16\pi e^{-1}}{(\pi^2+4)^2}\right)+|f''(b)|^q\left(\frac{2\pi^3e^{-1}-4\pi^2+24\pi e^{-1}+16}{(\pi^2+4)^2}\right)\right)^{\frac{1}{q}},\end{aligned}$$

where

$$\int_0^1 t^p(1-t)^{p+1}dt = \beta(p+1,p+2),$$

$$\int_0^1 t^{p+1}(1-t)^{p}dt = \beta(p+2,p+1),$$

$$\int_0^1 (1-t)\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}dt = \int_0^1 t\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}dt = \frac{2\pi^3e^{-1}-4\pi^2+24\pi e^{-1}+16}{(\pi^2+4)^2},$$

$$\int_0^1 (1-t)\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}dt = \int_0^1 t\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}} = \frac{8\pi^2-16\pi e^{-1}}{(\pi^2+4)^2},$$

the proof is completed.

Remark 1. Since h: $[0,\infty) \to \mathbb{R}$, $h(x) = x^t$, $0 < t \le 1$ is a concave function, for all $u, v \ge 0$, we have

$$\begin{aligned}\frac{u^t+v^t}{2} &= \frac{h(u)+h(v)}{2}\\ &\le h\left(\frac{u+v}{2}\right)\\ &= \left(\frac{u+v}{2}\right)^t\end{aligned}$$

and by using the following properties

$$\beta(p+1,p+2) = \beta(p+2,p+1)$$

$$\beta(p+1,p+2) = \beta(p+1,p+1)\frac{p+1}{2(p+1)}$$

we can write the right hand-side of the inequality (9) as follow:

$$\begin{aligned}&\frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+2)\left(|f''(a)|^q\left(\frac{2\pi^3e^{-1}-4\pi^2+24\pi e^{-1}+16}{(\pi^2+4)^2}\right)+|f''(b)|^q\left(\frac{8\pi^2-16\pi e^{-1}}{(\pi^2+4)^2}\right)\right)^{\frac{1}{q}}\\ &+\frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+2,p+1)\left(|f''(a)|^q\left(\frac{8\pi^2-16\pi e^{-1}}{(\pi^2+4)^2}\right)+|f''(b)|^q\left(\frac{2\pi^3e^{-1}-4\pi^2+24\pi e^{-1}+16}{(\pi^2+4)^2}\right)\right)^{\frac{1}{q}}\\ &\le 2\frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+2)\left(\frac{|f''(a)|^q\left(\frac{2\pi e^{-1}+4}{\pi^2+4}\right)+|f''(b)|^q\left(\frac{2\pi e^{-1}+4}{\pi^2+4}\right)}{2}\right)^{\frac{1}{q}}\\ &= 2\frac{(b-a)^2}{2}\left[\beta(p+1,p+1)\frac{p+1}{2(p+1)}\right]^{\frac{1}{p}}\left(\frac{|f''(a)|^q\left(\frac{2\pi e^{-1}+4}{\pi^2+4}\right)+|f''(b)|^q\left(\frac{2\pi e^{-1}+4}{\pi^2+4}\right)}{2}\right)^{\frac{1}{q}}\\ &= \frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+1)\left(\frac{2\pi e^{-1}+4}{\pi^2+4}\right)^{\frac{1}{q}}A^{\frac{1}{q}}(|f''(a)|^q,|f''(b)|^q).\end{aligned}$$

The required result is completed which indicates that the inequality (9) gives a better approximation than the inequality (8).

Theorem 9. f: $I \subset [0,\infty) \to \mathbb{R}$ be twice differentiable mapping on I^0 such that $f'' \in L[a, b]$, where $a, b \in l$ with $a < b$. If $|f''|^q$ is exponential trigonometric convex on $[a, b]$ and $q \ge 1$. Then the following inequality holds:

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\le (b-a)^2\left(\frac{1}{12}\right)^{\frac{1}{p}}\left(\frac{-128\pi(2e^{-1}-\pi)-4(\pi^2+4)^2}{(\pi^2+4)^3}\right)A^{\frac{1}{q}}(|f''(a)|^q,|f''(b)|^q).\end{aligned} \tag{10}$$

Proof: Suppose that $q \ge 1$. From lemma 1, using the well known power-mean inequality and exponential trigonometric convexity of $|f''|^q$, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\le \frac{(b-a)^2}{2}\int_0^1 |t-t^2||f''(ta+(1-t)b)|dt\\ &\le \frac{(b-a)^2}{2}\left(\int_0^1 t(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)|f''(ta+(1-t)b)|dt\right)^{\frac{1}{q}}\\ &\le \frac{(b-a)^2}{2}\left(\int_0^1 t(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left(\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}|f''(a)|^q+\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}|f''(b)|^q\right)dt\right)^{\frac{1}{q}}\\ &= \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\\ &\text{x}\left(|f''(a)|^q\left(\frac{-128\pi(2e^{-1}-\pi)-4(\pi^2+4)^2}{(\pi^2+4)^3}\right)+|f''(b)|^q\left(\frac{-128\pi(2e^{-1}-\pi)-4(\pi^2+4)^2}{(\pi^2+4)^3}\right)\right)^{\frac{1}{q}}\\ &= (b-a)^2\left(\frac{1}{12}\right)^{\frac{1}{p}}\left(\frac{-128\pi(2e^{-1}-\pi)-4(\pi^2+4)^2}{(\pi^2+4)^3}\right)A^{\frac{1}{q}}(|f''(a)|^q,|f''(b)|^q)\end{aligned}$$

where

$$\int_0^1 t(1-t)dt = \frac{1}{6},$$

$$\int_0^1 t(1-t)\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}dt = \int_0^1 t(1-t)\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}dt = \frac{-128\pi(2e^{-1}-\pi)-4(\pi^2+4)^2}{(\pi^2+4)^3},$$

this completes the proof of the theorem.

Theorem 10. f: $I \subset [0,\infty) \to \mathbb{R}$ be twice differentiable mapping on I^0 such that $f'' \in L[a, b]$, where $a, b \in l$ with $a < b$. If $|f''|^q$ is exponential trigonometric convex on $[a, b]$ and $q \geq 1$. Then the following inequality holds:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left[|f''(a)|^q\left(\frac{-16}{(\pi^2+4)^4}(304\pi e^{-1}-\pi^5e^{-1}-24\pi^3e^{-1}-128\pi^2+12\pi^4+64)\right)\right. \\
&\left.+|f''(b)|^q\left(\frac{-4}{(\pi^2+4)^4}(160\pi^3e^{-1}+4\pi^5e^{-1}-960\pi e^{-1}+432\pi^2-68\pi^4+\pi^6-192)\right)\right]^{\frac{1}{q}} \\
&+\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left[|f''(a)|^q\left(\frac{-4}{(\pi^2+4)^4}(160\pi^3e^{-1}+4\pi^5e^{-1}-960\pi e^{-1}+432\pi^2-68\pi^4+\pi^6-192)\right)\right. \\
&\left.+|f''(b)|^q\left(\frac{-16}{(\pi^2+4)^4}(304\pi e^{-1}-\pi^5e^{-1}-24\pi^3e^{-1}-128\pi^2+12\pi^4+64)\right)\right]^{\frac{1}{q}}.
\end{aligned} \tag{11}$$

Proof: Suppose that $q \geq 1$. From lemma 1, using improved power-mean inequality and exponential trigonometric convexity of $|f''|^q$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{2}\int_0^1 |t-t^2||f''(ta+(1-t)b)|dt \\
&\leq \frac{(b-a)^2}{2}\left(\int_0^1 t(1-t)^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^2|f''(ta+(1-t)b)|^qdt\right)^{\frac{1}{q}} \\
&+\frac{(b-a)^2}{2}\left(\int_0^1 t^2(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2(1-t)|f''(ta+(1-t)b)|^qdt\right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^2\left(\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}|f''(a)|^q+\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}|f''(b)|^q\right)dt\right)^{\frac{1}{q}} \\
&+\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2(1-t)\left(\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}|f''(a)|^q+\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}|f''(b)|^q\right)dt\right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(|f''(a)|^q\int_0^1 t(1-t)^2\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}dt+|f''(b)|^q\int_0^1 t(1-t)^2\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}dt\right)^{\frac{1}{q}} \\
&+\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(|f''(a)|^q\int_0^1 t^2(1-t)\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}dt+|f''(b)|^q\int_0^1 t^2(1-t)\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}dt\right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left[|f''(a)|^q\left(\frac{-16}{(\pi^2+4)^4}(304\pi e^{-1}-\pi^5e^{-1}-24\pi^3e^{-1}-128\pi^2+12\pi^4+64)\right)\right. \\
&\left.+|f''(b)|^q\left(\frac{-4}{(\pi^2+4)^4}(160\pi^3e^{-1}+4\pi^5e^{-1}-960\pi e^{-1}+432\pi^2-68\pi^4+\pi^6-192)\right)\right]^{\frac{1}{q}} \\
&+\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left[|f''(a)|^q\left(\frac{-4}{(\pi^2+4)^4}(160\pi^3e^{-1}+4\pi^5e^{-1}-960\pi e^{-1}+432\pi^2-68\pi^4+\pi^6-192)\right)\right. \\
&\left.+|f''(b)|^q\left(\frac{-16}{(\pi^2+4)^4}(304\pi e^{-1}-\pi^5e^{-1}-24\pi^3e^{-1}-128\pi^2+12\pi^4+64)\right)\right]^{\frac{1}{q}}
\end{aligned}$$

where

$$\begin{aligned}
&\int_0^1 t(1-t)^2dt = \int_0^1 t^2(1-t)dt = \frac{1}{12} \\
&\int_0^1 t(1-t)^2\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}dt = \int_0^1 t^2(1-t)\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}dt \\
&= \frac{-16}{(\pi^2+4)^4}(304\pi e^{-1}-\pi^5e^{-1}-24\pi^3e^{-1}-128\pi^2+12\pi^4+64) \\
&\int_0^1 t^2(1-t)\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}dt = \int_0^1 t(1-t)^2\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}dt \\
&= \left(\frac{-4}{(\pi^2+4)^4}(160\pi^3e^{-1}+4\pi^5e^{-1}-960\pi e^{-1}+432\pi^2-68\pi^4+\pi^6-192)\right)
\end{aligned}$$

the proof is completed.

Remark 2. The inequality (11) gives a better result than the inequality (10).

Proof: The proof is make by giving the numerical results of integrals.

$$\begin{aligned}
&\int_0^1 t(1-t)^2\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}dt = \int_0^1 t^2(1-t)\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}dt \\
&= \frac{-16}{(\pi^2+4)^4}(304\pi e^{-1}-\pi^5e^{-1}-24\pi^3e^{-1}-128\pi^2+12\pi^4+64) = 0.028 \\
&\int_0^1 t^2(1-t)\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}dt = \int_0^1 t(1-t)^2\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}dt \\
&= \left(\frac{-4}{(\pi^2+4)^4}(160\pi^3e^{-1}+4\pi^5e^{-1}-960\pi e^{-1}+432\pi^2-68\pi^4+\pi^6-192)\right) = 0.045, \\
&\int_0^1 t(1-t)\frac{\sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}dt = \int_0^1 t(1-t)\frac{\cos\left(\frac{\pi t}{2}\right)}{e^{t}}dt = \frac{-128\pi(2e^{-1}-\pi)-4(\pi^2+4)^2}{(\pi^2+4)^3} = 0.074
\end{aligned}$$

We can write the inequality (10) and the inequality (11) with numerical results.

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)^2\left(\frac{1}{12}\right)^{\frac{1}{p}}(0.074)A^{\frac{1}{q}}(|f''(a)|^q,|f''(b)|^q).$$

Similarly, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\right. \\
&\left.-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}(|f''(a)|^q(0.028)+|f''(b)|^q(0.045))^{\frac{1}{q}} \\
&+\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}(|f''(a)|^q(0.045)+|f''(b)|^q(0.028))^{\frac{1}{q}}.
\end{aligned} \tag{12}$$

By using concavity of the function h: $[0,\infty) \to \mathbb{R}$, $h(x) = x^t$, $0 < t \leq 1$, the write right- hand-side of the inequality (12) can be written as follows:

$$\begin{aligned}
&\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}(|f''(a)|^q(0.028)+|f''(b)|^q(0.045))^{\frac{1}{q}} \\
&+\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}(|f''(a)|^q(0.045)+|f''(b)|^q(0.028))^{\frac{1}{q}} \\
&\leq 2\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\frac{|f''(a)|^q(0.073)+|f''(b)|^q(0.073)}{2}\right)^{\frac{1}{q}} \\
&= (b-a)^2\left(\frac{1}{12}\right)^{1-\frac{1}{q}}(0.073)^{\frac{1}{q}}A^{\frac{1}{q}}(|f''(a)|^q,|f''(b)|^q).
\end{aligned}$$

The required result is completed which indicates that the inequality (11) gives a better result than the inequality (10).

CONCLUSION

In this article, some new versions of Hermite-Hadamard type inequalities for differential exponential trigonometric convex functions were obtained. Additionally, some special cases are given in detail. Finally, it was obtained a few interesting inequalities from our main results using special tools and comparison as an application. This demonstrated the efficiency of the results. It is expected that this article will trigger new ideas and techniques in some different integral equations for researchers working in functional analysis, information theory and statistical theory.

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