



Research Article

On codes over product of finite chain rings

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ABSTRACT

In this paper, codes over the direct product of two finite commutative chain rings are studied. The standard form of the parity-check matrix is determined. The structure of self-dual codes is described. A distance preserving Gray map from the direct product of chain rings to a finite field is defined. Two upper bounds on minimum distance are obtained.

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INTRODUCTION

In 1997, Rifa and Pujol introduced $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes as subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$; see [27]. The set of coordinates in $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes is partitioned into two parts, the first part in \mathbb{Z}_2 and the last part in \mathbb{Z}_4 . Due to the appearance of $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes, the study of codes over mixed ring alphabets has been widely grown, for example $\mathbb{Z}_p r \times \mathbb{Z}_p s$ -additive codes, $\mathbb{Z}_p\mathbb{Z}_p[u]$ -additive codes, $\mathbb{Z}_p[u', u'']$ -linear codes, etc.; see [1, 2, 3, 6, 7, 18, 4, 11, 30, 29, 25]. Very recently, Dinh et al. and Gao et al. have extensively studied the applications of mixed alphabet codes in constructing new DNA and quantum codes; see [12, 13, 22, 10].

Notice that in all the aforementioned papers, the coordinates of two parts are based on rings that are finite commutative chain rings. Recently, Borges et.al. have defined R_1R_2 -linear codes which are R_2 -submodules of $R_1^\alpha \times R_2^\beta$, where R_1 and R_2 are finite commutative chain rings with the same residue field \mathbb{F}_q . Fundamental results on R_1R_2 -linear

codes including the generator matrix, the duality concept and cyclic codes can be found in [8, 26]. Furthermore, notice that for example in $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes, \mathbb{Z}_2 is a \mathbb{Z}_4 -algebra and in $\mathbb{Z}_2\mathbb{Z}_2[u]$ -additive codes, \mathbb{Z}_2 is a $\mathbb{Z}_2[u]$ -algebra. Based on this fact, Mahmoudi and Samei generalized all the abovementioned papers to SR- additive codes, where S is an R-algebra, see [24].

Motivated by all previous works done on codes over several mixed alphabets and also SR- additive codes, we study the structural properties of R_1R_2 -linear codes.

One of the basic problems in coding theory is to determine the standard form of the parity-check matrix which is used in decoding algorithms efficiently. In this paper, we determine the parity-check matrix of R_1R_2 -linear codes as well as the relation between R_1R_2 -linear codes C and C^\perp .

The homogeneous weight was first discovered by Constantinescu and Heise as a generalization of the

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Hamming weight on the finite fields and the Lee weight on \mathbb{Z}_q ; see [9]. Gereferate and Schmidt used a tensor product to construct a Gray map with a non-linear image on the certain chain ring \mathbb{Z}_3 endowed with the homogeneous metric; see [20]. Jitman generalized the Gray map given in [20]. He presented an algebraic construction for the Gray map on chain rings equipped with the homogeneous metric that is non-linear over special chain rings $\mathbb{Z}_p m$; see [21]. In this paper, we define a weight on $R_1^\alpha \times R_2^\beta$, which is the natural generalization of the homogeneous weight over chain rings. Then, using the definition in [21], we define a distance preserving Gray map from $R_1 R_2$ -linear codes to codes over \mathbb{F}_q equipped with the Hamming weight. Our definition is a natural generalization of the given Gray map on $\mathbb{Z}_p \mathbb{Z}_{p^k}$ -linear codes in [29]. The Gray image of $R_1 R_2$ -linear codes presented in Examples 5.3, 5.4 and 5.5 provides optimal codes which have more simple construction than linear codes with the same parameters in Grassl table; see [19].

The study of several upper bounds on the minimum distance of a code is important in coding theory in view of the fact that codes meeting these bounds have the largest possible minimum distance. In this paper, two upper bounds for the minimum distance are obtained by the Singleton bound for the Gray image and the rank bound for codes over rings. If an $R_1 R_2$ -linear code meets the first bound, it is called MDS with respect to the Singleton bound (MDSS), and if it attains the rank bound, it is called MDS with respect to the rank bound (MDSR); see [5, 28]. We discuss the conditions on the $R_1 R_2$ -linear codes to be MDSS or MDSR.

The link between self-dual codes and many different research areas such as design theory and lattice theory makes studying self-dual codes interesting; see [14, 23]. Some sufficient conditions for constructing self-dual codes over chain rings are presented in [15, 16, 17]. Herein, we use the same two methods to build $R_1 R_2$ -linear self-dual codes.

This paper is organized as follows. In section 2, some basic notations and definitions about chain rings and codes over the products chain rings R_1 and R_2 are given. In section 3, the parity-check matrix of linear codes over $R_1 \times R_2$ in standard form is described and some examples are presented. In section 4, self-dual codes over $R_1 \times R_2$ are studied and self-dual codes over $R_1 \times R_2$ with larger lengths are constructed by two methods. In section 5, a weight for linear codes over $R_1 \times R_2$ is defined and a distance preserving Gray map on $R_1 \times R_2$ corresponding to the homogeneous weight over chain rings is introduced. Moreover, two upper bounds for the minimum distance of linear codes over $R_1 \times R_2$ are obtained.

PRELIMINARIES

Throughout this paper, all rings are assumed to be finite and commutative with identity. A ring R is called a chain ring if its ideals are linearly ordered by inclusion. Obviously, every chain ring has a unique maximal ideal. Consider γ as

the generator of the unique maximal ideal. Since R is finite and its ideals are chain, γ is nilpotent. Denote the nilpotency index of γ by e . We have

$$R = \langle \gamma^0 \rangle \supseteq \langle \gamma^1 \rangle \supseteq \dots \supseteq \langle \gamma^{e-1} \rangle \supseteq \langle \gamma^e \rangle = 0.$$

Note that if R is a finite field, then $\gamma = 0$ and it can be considered that $e = 1$. It is clear that all elements of $\langle \gamma \rangle$ are zero-divisors and all elements of $R \setminus \langle \gamma \rangle$ are units. The residue field $R/\langle \gamma \rangle$ is denoted by \mathbb{F}_q where $q = p^m$, p is a prime number and m is a positive integer. Let $\rho: R \rightarrow \mathbb{F}_q$ be the natural projection map. Let $T = \{r_0, \dots, r_{q-1}\}$ be the Teichmüller set of representatives of R .

Lemma 2.1 [26] *Assume the above notations. Let $V \subseteq R$ be a system of representatives for the equivalence classes of R under congruence modulo γ . Then*

1. For all $r \in R$ there exist unique $a_0(r), a_1(r), \dots, a_{e-1}(r) \in V$ such that $r = \sum_{i=0}^{e-1} a_i(r) \gamma^i$.
2. $|V| = q$.
3. $|\langle \gamma^j \rangle| = q^{e-j}$ for all $j \in \{0, \dots, e-1\}$.

Clearly, $|R| = q^e$ and any elements $r \in R^n$ can be written uniquely as $r = \sum_{i=0}^{e-1} a_i(r) \gamma^i$, where $a_i(r) = (r_{i,0}, r_{i,1}, \dots, r_{i,n-1}) \in V^n$ for all i .

LINEAR CODES OVER CHAIN RINGS

A linear code of length n over R is an R -submodule of R^n . In [20], the homogeneous weight of each element $r \in R$ in the sense of [9], denoted by $w_{\text{hom}}(r)$, is defined as follows:

$$w_{\text{hom}}(r) = \begin{cases} q^{e-1} & r \in \gamma^{e-1} R \setminus \{0\}, \\ q^{(e-2)}(q-1) & r \in R \setminus \gamma^{e-1} R, \\ 0 & \text{o.w.} \end{cases}$$

Naturally, the homogeneous weight can be extended to R^n componentwise. Then, the homogeneous weight of $r = (r_1, \dots, r_n) \in R^n$ becomes $w_{\text{hom}}(r) = \sum_{i=1}^n w_{\text{hom}}(r_i)$. The homogeneous distance $d_{\text{hom}}(r, s)$ between vectors r, s in R^n is defined to be $w_{\text{hom}}(r-s)$.

In [21], the Gray map from R^n to $\mathbb{F}_q^{q^e-1}$ is defined as follows.

Let

$$\varepsilon = \xi_0(\varepsilon) + \xi_1(\varepsilon)p + \dots + \xi_{m-1}(\varepsilon)p^{m-1}$$

be the p -adic representation of $\varepsilon \in \mathbb{Z}_{p^m}$ where $\xi_i(\varepsilon) \in \{0, 1, \dots, p-1\}$ for all $i \in \{0, \dots, m-1\}$. Let α be a fixed primitive element of \mathbb{F}_{q^m} . Corresponding to every ε , consider α_ε as

$$\alpha_\varepsilon = \xi_0(\varepsilon) + \xi_1(\varepsilon)\alpha + \dots + \xi_{m-1}(\varepsilon)\alpha^{m-1}.$$

Moreover, let

$$w = \tilde{\xi}_0(w) + \tilde{\xi}_1(w)p^m + \cdots + \tilde{\xi}_{e-2}(w)p^{m(e-2)}$$

be the p -adic representation of $w \in \mathbb{Z}_{p^{m(e-1)}}$, where $\tilde{\xi}_i(w) \in \{0, 1, \dots, p^m - 1\}$ for all $i \in \{0, \dots, e-2\}$. Now, the Gray map $\phi: R^n \rightarrow \mathbb{F}_q^{q^{e-1}n}$ is defined as

$$\phi(r) = (b_0, b_1, \dots, b_{q^{e-1}-1})$$

for all $r = a_0(r) + a_1(r)\gamma + \cdots + a_{e-1}(r)\gamma^{e-1} \in R^n$, where

$$b_{wp^m + \varepsilon} = \alpha_\varepsilon \overline{a_0(r)} + \sum_{l=1}^{e-2} \alpha_{\tilde{\xi}_l(w)} \overline{a_l(r)} + \overline{a_{e-1}(r)}$$

for all $w \in \{0, \dots, p^{m(e-2)} - 1\}$ and $\varepsilon \in \{0, \dots, p^m - 1\}$.

Example 2.2

1. if $R = \mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2$, where $u^3 = 0$, the Gray map $\phi: R \rightarrow \mathbb{F}_2^4$ is

$$\phi(a_0 + a_1u + a_2u^2) = (a_2, a_0 + a_2, a_1 + a_2, a_0 + a_1 + a_2).$$

2. For $R = \mathbb{F}_3 + u\mathbb{F}_3$ where $u^2 = 0$, the Gray map $\phi: R \rightarrow \mathbb{F}_3^5$ is

$$\phi(a_0 + a_1u) = (a_1, a_0 + a_1, 2a_0 + a_1, 3a_0 + a_1, 4a_0 + a_1).$$

Proposition 2.3 [21] *The Gray map ϕ is an isometry from (R^n, d_{hom}) to $(\mathbb{F}_q^{q^{e-1}n}, d_H)$, where d_H denotes the Hamming distance on $\mathbb{F}_q^{q^{e-1}n}$.*

It is well known that Singleton bound for every code C over an alphabet of size q is given by

$$d_H(C) \leq n - \log_q |C| + 1. \quad (2.1)$$

Furthermore, by Theorem 3.7 in [28], if C is a code of length n over chain ring R equipped with the homogeneous distance d_{hom} then the rank bound for C is established as follows:

$$\left\lfloor \frac{d_{\text{hom}}(C) - 1}{q^{e-1}} \right\rfloor \leq n - \text{rank}(C), \quad (2.2)$$

where $\text{rank}(C)$ is the minimum cardinality of the generator set of C .

LINEAR CODES OVER $R_1 \times R_2$

From now on, assume that $R_1 = R_{\gamma_1, e_1, q}$ and $R_2 = R_{\gamma_2, e_2, q}$ denote two finite chain rings where γ_1 and γ_2 are the generators of their maximal ideals with nilpotency indices e_1 and e_2 respectively. Besides, assume that R_1 and R_2 have the same residue field \mathbb{F}_q and $e_1 \leq e_2$. Moreover, suppose that $T_1 = \{r_0, \dots, r'_{q-1}\}$ and $T_2 = \{r'_0, \dots, r'_{q-1}\}$ are the Teichmüller sets of representatives of R_1 and R_2 , respectively.

Define the surjective ring homomorphism π from R_2 to R_1 such that $\pi(\gamma_2) = \gamma_1$ and $\pi(r'_j) = r_j$. It is obvious that $\pi(\gamma_2^i) = 0$ for all $i \geq e_1$. Consider $a \in R_2$ and $u = (u|u') = (u_1, \dots, u_\alpha | u'_1, \dots, u'_\beta) \in R_1^\alpha \times R_2^\beta$. In [8], it is asserted that $R_1^\alpha \times R_2^\beta$ is an R_2 -module with the following scalar multiplication

$$a * u = (\pi(a)u_1, \dots, \pi(a)u_\alpha | au'_1, \dots, au'_\beta).$$

Definition 2.4 [8] *A subset $C \subseteq R_1^\alpha \times R_2^\beta$ is called an R_1 R_2 -linear code if C is a submodule of $R_1^\alpha \times R_2^\beta$.*

Proposition 2.5 [8] *Let C be an $R_1 R_2$ -linear code, then C is permutation equivalent to a code with a standard generator matrix of the form*

$$G = \begin{bmatrix} B & T \\ S & A \end{bmatrix}, \quad (2.3)$$

where

$$B = \begin{bmatrix} I_{k_0} & B_{0,1} & B_{0,2} & B_{0,3} & \cdots & B_{0,e_1-1} & B_{0,e_1} \\ 0 & \gamma_1 I_{k_1} & \gamma_1 B_{1,2} & \gamma_1 B_{1,3} & \cdots & \gamma_1 B_{1,e_1-1} & \gamma_1 B_{1,e_1} \\ 0 & 0 & \gamma_1^2 I_{k_2} & \gamma_1^2 B_{2,3} & \cdots & \gamma_1^2 B_{2,e_1-1} & \gamma_1^2 B_{2,e_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma_1^{e_1-1} I_{k_{e_1-1}} & \gamma_1^{e_1-1} B_{e_1-1,e_1} \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & \cdots & \gamma_2^{e_2-e_1} T_{0,1} & \gamma_2^{e_2-e_1} T_{0,2} & \cdots & \gamma_2^{e_2-e_1} T_{0,e_1} \\ 0 & \cdots & 0 & \gamma_2^{e_2-e_1+1} T_{1,2} & \cdots & \gamma_2^{e_2-e_1+1} T_{1,e_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \gamma_2^{e_2-1} T_{e_1-1,e_1} \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & S_{0,1} & S_{0,2} & \cdots & S_{0,e_1-1} & S_{0,e_1} \\ 0 & S_{1,1} & S_{1,2} & \cdots & S_{1,e_1-1} & S_{1,e_1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & S_{e_2-e_1-1,1} & S_{e_2-e_1-1,2} & \cdots & S_{e_2-e_1-1,e_1-1} & S_{e_2-e_1-1,e_1} \\ 0 & 0 & \gamma_1 S_{e_2-e_1,2} & \cdots & \gamma_1 S_{e_2-e_1,e_1-1} & \gamma_1 S_{e_2-e_1,e_1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_1^{e_1-2} S_{e_2-3,e_1-1} & \gamma_1^{e_1-2} S_{e_2-3,e_1} \\ 0 & 0 & 0 & \cdots & 0 & \gamma_1^{e_1-1} S_{e_2-2,e_1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} I_{l_0} & A_{0,1} & A_{0,2} & A_{0,3} & \cdots & A_{0,e_2-1} & A_{0,e_2} \\ 0 & \gamma_2 I_{l_1} & \gamma_2 A_{1,2} & \gamma_2 A_{1,3} & \cdots & \gamma_2 A_{1,e_2-1} & \gamma_2 A_{1,e_2} \\ 0 & 0 & \gamma_2^2 I_{l_2} & \gamma_2^2 A_{2,3} & \cdots & \gamma_2^2 A_{2,e_2-1} & \gamma_2^2 A_{2,e_2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma_2^{e_2-1} I_{l_{e_2-1}} & \gamma_2^{e_2-1} A_{e_2-1,e_2} \end{bmatrix},$$

such that the entries in $\gamma_1^i B_{ij}$ and $\gamma_1^i S_{ij}$ are in $\langle \gamma_1^i \rangle$ and the entries in $\gamma_2^i A_{ij}$ and $\gamma_2^i T_{ij}$ are in $\langle \gamma_2^i \rangle$.

It is said that C is of type $(\alpha, \beta; k_0, \dots, l_0, \dots, l_{e_2-1})$. According to Proposition 2.5, it can be concluded that $rank(C) = \sum_{i=0}^{e_1-1} k_i + \sum_{j=0}^{e_2-1} l_j$ and

$$|C| = \left| \langle \gamma_1^0 \rangle \right|^{k_0} \times \left| \langle \gamma_1^1 \rangle \right|^{k_1} \times \dots \times \left| \langle \gamma_1^{e_1-1} \rangle \right|^{k_{e_1-1}} \times \left| \langle \gamma_2^0 \rangle \right|^{l_0} \times \left| \langle \gamma_2^1 \rangle \right|^{l_1} \times \dots \times \left| \langle \gamma_2^{e_2-1} \rangle \right|^{l_{e_2-1}} \\ = q^{\sum_{i=0}^{e_1-1} (e_1-i)k_i + \sum_{j=0}^{e_2-1} (e_2-j)l_j}.$$

Consider injective map $\iota: R_1 \rightarrow R_2$ by definition $\iota(\gamma_1) = \gamma_2$ and $\iota(r_j) = r'_j$. It is obvious that $\pi \iota = Id$.

Definition 2.6 [8] The inner product of vectors $u = (u, u')$ and $v = (v, v')$ in $R_1^\alpha \times R_2^\beta$ is defined by

$$\langle u, v \rangle = \gamma_2^{e_2-e_1} \iota(u.v) + u'.v' \in R_2,$$

where $u.v$ and $u'.v'$ are standard inner product.

The $R_1 R_2$ -dual code of an $R_1 R_2$ -linear code C is defined in the standard way by

$$C^\perp = \{v \in R_1^\alpha \times R_2^\beta : \langle u, v \rangle = 0, \text{ for all } u \in C\},$$

which is an R_2 -submodule of $R_1^\alpha \times R_2^\beta$. We say that an $R_1 R_2$ -linear code C is self-orthogonal if $C \subseteq C^\perp$ and is self-dual if $C \subseteq C^\perp$.

Let C_X be the punctured $R_1 R_2$ -linear code of C by deleting the first α coordinates and C_Y be that of by deleting the last β coordinates. The code C is called separable if $C = C_X \times C_Y$. If C is separable, then its generator matrix is in the form

$$G = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & A \end{array} \right], \tag{2.4}$$

where A and B are matrices in Proposition 2.5. The dual-code of every separable code C is separable and $C^\perp = C_X^\perp \times C_Y^\perp$.

PARITY-CHECK MATRICES OF $R_1 R_2$ -LINEAR CODES

The next theorem generalizes the structure of the parity-check matrices presented in [1, 3] to the case of $R_1 R_2$ -linear codes.

Define $k(B) = k_0 + k_1 + \dots + k_{e_1-1}$ and $l(A) = l_0 + l_1 + \dots + l_{e_2-1}$.

Theorem 3.1 Let C be an $R_1 R_2$ -linear code with the generator matrix G given in Proposition 2.5. Then the standard form for the generator matrix of C^\perp is

$$H = \left[\begin{array}{c|c} \tilde{B} + F & U \\ \hline V & \tilde{A} + E \end{array} \right], \tag{3.1}$$

where

$$\tilde{B} = \left[\begin{array}{cccc} \tilde{B}_{0,e_1} & \tilde{B}_{0,e_1-1} & \tilde{B}_{0,e_1-2} & \dots \\ \gamma_1 \tilde{B}_{1,e_1} & \gamma_1 \tilde{B}_{1,e_1-1} & \gamma_1 \tilde{B}_{1,e_1-2} & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{e_1-2} \tilde{B}_{e_1-2,e_1} & \gamma_1^{e_1-2} \tilde{B}_{e_1-2,e_1-1} & \gamma_1^{e_1-2} I_{k_2} & \dots \\ \gamma_1^{e_1-1} \tilde{B}_{e_1-1,e_1} & \gamma_1^{e_1-2} I_{k_1} & 0 & \dots \\ \tilde{B}_{0,3} & \tilde{B}_{0,2} & \tilde{B}_{0,1} & I_{\alpha-k(B)} \\ \gamma_1 \tilde{B}_{1,3} & \gamma_1 \tilde{B}_{1,2} & \gamma_1 I_{k_{e_1-1}} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & 0 & 0 \\ \dots & 0 & 0 & 0 \end{array} \right]$$

$$F = \left[\begin{array}{cccccc} F_{0,e_1-2} & F_{0,e_1-3} & \dots & F_{0,2} & F_{0,1} & 0 & 0 & 0 \\ \gamma_1 F_{1,e_1-2} & \gamma_1 F_{1,e_1-3} & \dots & \gamma_1 F_{1,2} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{e_1-4} F_{e_1-4,e_1-2} & \gamma_1^{e_1-4} F_{e_1-4,e_1-3} & \dots & 0 & 0 & 0 & 0 & 0 \\ \gamma_1^{e_1-3} F_{e_1-3,e_1-2} & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$U = \left[\begin{array}{cccc} U_{0,e_2-1} & U_{0,e_2-2} & \dots & U_{0,3} & U_{0,2} \\ \gamma_2 U_{1,e_2-1} & \gamma_2 U_{1,e_2-2} & \dots & \gamma_2 U_{1,3} & \gamma_2 U_{1,2} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \gamma_2^{e_1-2} U_{e_1-2,e_1-1} & \gamma_2^{e_1-2} U_{e_1-2,e_2-2} & \dots & 0 & 0 \\ \gamma_2^{e_1-1} U_{e_1-1,e_2-1} & \gamma_2^{e_1-1} U_{e_1-1,e_2-2} & \dots & 0 & 0 \\ U_{0,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$V = \left[\begin{array}{cccc} V_{0,e_1} & V_{0,e_1-1} & V_{0,e_1-2} & \dots & V_{0,3} \\ \gamma_1 V_{1,e_1} & \gamma_1 V_{1,e_1-1} & \gamma_1 V_{1,e_1-2} & \dots & \gamma_1 V_{1,3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{e_1-1} V_{e_1-1,e_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ V_{0,2} & V_{0,1} & 0 & 0 & 0 \\ \gamma_1 V_{1,2} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{0,e_2} & \tilde{A}_{0,e_2-1} & \tilde{A}_{0,e_2-2} & \dots \\ \gamma_2 \tilde{A}_{1,e_2} & \gamma_2 \tilde{A}_{1,e_2-1} & \gamma_2 \tilde{A}_{1,e_2-2} & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_2^{e_2-2} \tilde{A}_{e_2-2,e_2} & \gamma_2^{e_2-2} \tilde{A}_{e_2-2,e_2-1} & \gamma_2^{e_2-2} I_{l_2} & \dots \\ \gamma_2^{e_2-1} \tilde{A}_{e_2-1,e_2} & \gamma_2^{e_2-1} I_{l_1} & 0 & \dots \\ \tilde{A}_{0,3} & \tilde{A}_{0,2} & \tilde{A}_{0,1} & I_{\beta-1(A)} \\ \gamma_2 \tilde{A}_{1,3} & \gamma_2 \tilde{A}_{1,2} & \gamma_2 I_{l_{e_2-1}} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & 0 & 0 \\ \dots & 0 & 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} E_{0,e_2-2} & E_{0,e_2-3} & E_{0,e_2-4} & \dots \\ \gamma_2 E_{1,e_2-2} & \gamma_2 E_{1,e_2-3} & \gamma_2 E_{1,e_2-4} & \dots \\ \vdots & \vdots & \vdots & \dots \\ \gamma_2^{e_1-2} E_{e_1-2,e_2-2} & \gamma_2^{e_1-2} E_{e_1-2,e_2-3} & \gamma_2^{e_1-2} E_{e_1-2,e_2-4} & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & \dots \end{bmatrix}$$

$$\begin{bmatrix} E_{0,2} & E_{0,1} \\ \gamma_2 E_{1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

such that

$$\tilde{B}_{i,j} = -\sum_{k=i+1}^{j-1} \tilde{B}_{i,k} \tilde{B}_{e_1-j,e_1-k}^t - B_{e_1-j,e_1-i}^t, \quad 0 \leq i \leq j \leq e_1,$$

$$\tilde{A}_{i,j} = -\sum_{k=i+1}^{j-1} \tilde{A}_{i,k} A_{e_2-j,e_2-k}^t - A_{e_2-j,e_2-i}^t, \quad 0 \leq i \leq j \leq e_2,$$

$$U_{i,j} = -\sum_{k=i+1}^{j-1} U_{i,k} A_{e_2-j,e_2-k-1}^t - \gamma_2^{j-e_1+1-i} \left(\left(\left(\gamma_1^a \left(\sum_{l=i+1}^{j-1} \tilde{B}_{i,l} S_{e_2-j-1,e_1-l}^t + \sum_{m=i+1}^{j-3} F_{i,m} S_{e_2-j-1,e_1-m-2}^t \right) \right) \right) + S_{e_2-j-1,e_1-i}^t \right)$$

$$a = \begin{cases} i & j \geq e_1, \\ e_1 - j + i & j \geq e_1, \end{cases} \quad m \leq e_1 - 3, \quad l \leq e_1 - 1,$$

$$E_{i,j} = -\sum_{k=i+1}^{j-1} E_{i,k} A_{e_2-j-2,e_2-k-2}^t - \gamma_2^{j+2-i-e_1} \left(\left(\left(\gamma_1^a \sum_{l=i+1}^j V_{i,l} S_{e_2-j-2,e_1-l}^t \right) \right) \right)$$

$$a = \begin{cases} i & j \geq e_1 - 1, \\ e_1 - j - 1 + i & j \geq e_1 - 1, \end{cases} \quad l \leq e_1 - 1,$$

$$\iota(\gamma_1^{e_1-2-j+i}(F_{i,j})) = -\iota \left(\gamma_1^{e_1-2-j+i} \sum_{k=i+1}^{j-1} F_{i,k} B_{e_1-j-2,e_1-k-2}^t - \gamma_2^{e_1-2-j+i} \left(\sum_{k=i+1}^j U_{i,k} T_{e_1-j-2,e_1-k-1}^t \right) \right),$$

$$\iota(\gamma_1^{e_1-j+i} V_{i,j}) = -\iota \left(\gamma_1^{e_1-j+i} \left(\sum_{k=i+1}^{j-1} V_{i,k} B_{e_1-j,e_1-k}^t \right) \right) - \gamma_2^{e_1-j+i} \left(\sum_{k=i+1}^{j-1} \tilde{A}_{i,k} T_{e_1-j,e_1-k}^t + \sum_{k=i+1}^{j-3} E_{i,i+j-k-2} \right) \left(T_{e_1-j,e_1+k-i-j}^t + T_{e_1-j,e_1-i}^t \right).$$

Proof. It is time-consuming but easy to check that $HG_t = 0$. Hence, we conclude that the rows of H are orthogonal to the rows of G , i.e. $C' \subset C^\perp$, where C' is the code generated by H . Moreover,

$$|C'| = |R_1|^{\alpha-k(B)} \times |\gamma_1 R_1|^{k_{e_1-1}} \times \dots \times |\gamma_1^{e_1-1} R_1|^{k_1} \times |R_2|^{\beta-1(A)} \times |\gamma_2 R_2|^{l_{e_2-1}} \times \dots \times |\gamma_2^{e_2-1} R_2|^{l_1},$$

which implies

$$|C||C'| = q^{e_1\alpha + e_1\beta} = |R_1^\alpha \times R_2^\beta|.$$

Therefore, H generates the dual space of C .

Corollary 3.2 For every $R_1 R_2$ -linear code C we have

1. $|C||C^\perp| = |R_1^\alpha \times R_2^\beta|$.
2. $(C^\perp)^\perp = C$.

Proof. The statement was shown throughout the proof of Theorem 3.1. To prove we note that $C \subset (C^\perp)^\perp$. Since C^\perp is of type

$$(\alpha, \beta; \alpha - k(B), k_{e_1-1}, \dots, k_1; \beta - 1(A), l_{e_2-1}, \dots, l_1),$$

then $(C^\perp)^\perp$ is of type $(\alpha, \beta; k_0, \dots, k_{e_1-1}; l_0, \dots, l_{e_2-1})$ and hence C and $(C^\perp)^\perp$ have the same size, completing (2).

Example 3.3 Let $R_1 = R_{\gamma_1,2,2}$, $R_2 = R_{\gamma_2,3,2}$ and C be an $R_1 R_2$ -linear code of type $(3,4; 1,1; 1,1,1)$ generated by

$$G = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & \gamma_2 & \gamma_2 \\ 0 & \gamma_1 & \gamma_1 & 0 & 0 & 0 & \gamma_2^2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \gamma_1 & 0 & \gamma_2 & \gamma_2 & \gamma_2 \\ 0 & 0 & 0 & 0 & 0 & \gamma_2^2 & \gamma_2^2 \end{bmatrix}.$$

We calculate

$$\begin{aligned} 3V_{0,1} &= 1, & V_{0,2} &= 1, & \gamma_1 V_{1,2} &= \gamma_1, \\ U_{0,1} &= \gamma_2, & U_{0,2} &= \gamma_2, & \gamma_2 U_{1,2} &= \gamma_2^2, \\ \tilde{B}_{0,1} &= 1, & \tilde{B}_{0,2} &= 0, & \gamma_1 \tilde{B}_{1,2} &= \gamma_1, \\ \tilde{A}_{0,1} &= 1, & \tilde{A}_{0,2} &= \tilde{A}_{0,3} = 0, & \gamma_2 \tilde{A}_{1,2} &= \gamma_2, \\ \gamma_2 \tilde{A}_{1,3} &= 0, & \gamma_2^2 \tilde{A}_{2,3} &= \gamma_2^2, & E_{0,1} &= \gamma_2. \end{aligned}$$

Therefore, the parity-check matrix is in the form

$$H = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & \gamma_2 & \gamma_2 & 0 & 0 \\ \gamma_1 & \gamma_1 & 0 & \gamma_2^2 & 0 & 0 & 0 \\ 1 & 1 & 0 & \gamma_2 & 0 & 1 & 1 \\ \gamma_1 & 0 & 0 & 0 & \gamma_2 & \gamma_2 & 0 \\ 0 & 0 & 0 & \gamma_2^2 & \gamma_2^2 & 0 & 0 \end{array} \right].$$

Example 3.4 [2] Suppose C is a $\mathbb{Z}_2\mathbb{Z}_2[u]$ -linear code generated by

$$G = \left[\begin{array}{cc|ccc} 1 & 1 & u & u & u \\ 0 & 1 & 1 & 0 & 1+u \\ 1 & 0 & 1 & u & u \end{array} \right].$$

G is permutation equivalent with

$$G' = \left[\begin{array}{cc|ccc} 1 & 1 & 0 & 0 & u \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array} \right].$$

Consider the natural injective map $\iota: \mathbb{Z}_2 \rightarrow \mathbb{Z}_2[u]$ where $\iota(0) = 0$ and $\iota(1) = 1$. We have

$$F = E = 0, \quad V_{0,1} = 1, \quad \tilde{B}_{0,1} = 1, \quad \tilde{A}_{0,2} = (0 \ 0), \quad U_{0,1} = (u \ 0).$$

Thus, the parity-check matrix is

$$H = \left[\begin{array}{cc|cc} 1 & 1 & u & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array} \right].$$

Example 3.5 [8] Let $R_1 = \mathbb{Z}_2, R_2 = R_{u,3,2}$ and C be an R_1R_2 -linear code generated by

$$G = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & u & u+u^2 & 1+u & 1+u \\ 0 & 1 & 0 & 1 & u & u^2 & 0 \\ 0 & 1 & 1 & 0 & u^2 & 0 & u^2 \\ 1 & 1 & 1 & u^2 & u & u+u^2 & 0 \end{array} \right].$$

It is easy to show that G is permutation equivalent with

$$G' = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & u & u+u^2 \end{array} \right].$$

Hence, C is of type $(3,4; 1; 2,1,0)$ and

$$\begin{aligned} 3E = F = 0, & \quad V_{0,1} = 0, \\ U_{0,1} &= \begin{bmatrix} u \\ u \end{bmatrix}, & \quad U_{0,2} &= \begin{bmatrix} 0 & u^2 \\ u^2 & 0 \end{bmatrix}, & \quad \tilde{B}_{0,1} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \tilde{A}_{0,2} &= 1+u, & \quad \tilde{A}_{0,3} &= [u \ 1], & \quad \tilde{A}_{2,3} &= [0 \ 0]. \end{aligned}$$

Therefore,

$$H = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & u^2 & u & 0 \\ 1 & 0 & 1 & u^2 & 0 & u & 0 \\ 0 & 0 & 0 & u & 1 & 1+u & 1 \\ 0 & 0 & 0 & 0 & 0 & u^2 & 0 \end{array} \right].$$

R_1R_2 -LINEAR SELF-DUAL CODES

Definition 4.1 Let $u = (u|u') \in R_1^\alpha \times R_2^\beta$. The weight function $wt^*(u)$ is defined by

$$wt^*(u) = w_{\text{hom}_{R_1}}(u) + w_{\text{hom}_{R_2}}(u'),$$

where $w_{\text{hom}_{R_1}}$ and $w_{\text{hom}_{R_2}}$ are the homogeneous weights over R_1^α and R_2^β , respectively.

The distance between two elements $u, v \in R_1^\alpha \times R_2^\beta$, denoted by $d^*(u, v)$, is $wt^*(u-v)$. The minimum distance of an R_1R_2 -linear code C , denoted by $d^*(C)$ or d^* , is the minimum value of $d^*(u, v)$ for all $u, v \in C$ such that $u \neq v$. Consider two notations $k_{e_1} = \alpha - k(B)$ and $l_{e_2} = \beta - l(A)$.

Lemma 4.2 Let C be an R_1R_2 -linear code of type $(\alpha, \beta; k_0, \dots, k_{e_1-1}; l_0, \dots, l_{e_2-1})$. If C is self-dual, then $k_i = k_{e_1-i}$ and $l_j = l_{e_2-j}$ for all $i \in \{0, \dots, e_1\}$ and $j \in \{0, \dots, e_2\}$.

Proof. By Theorem 3.1, C^\perp is of type $(\alpha, \beta; k_{e_1}, k_{e_1-1}, \dots, k_1; l_{e_2}, l_{e_2-1}, \dots, l_0)$. Since $C = C^\perp$, their types are equal and the result follows.

Theorem 4.3 An R_1R_2 -linear self-orthogonal code C is self-dual if and only if $|C| = q^{\frac{e_1\alpha + e_2\beta}{2}}$.

Proof. By Corollary 3.2, we have $|C||C^\perp| = q^{e_1\alpha + e_2\beta}$ which gives the result.

Example 4.4 Assume $R_1 = R_{\gamma_1, 2, 5}, R_2 = R_{\gamma_2, 3, 5}$ and C is an R_1R_2 -linear code with the generator matrix

$$G = \left[\begin{array}{cc|cc} \gamma_1 I_3 & T_1 & T_2 & \\ 0 & 2\gamma_1 I_4 & A & \\ 0 & 0 & \gamma_2^2 I_4 & \end{array} \right].$$

where T_1 and T_2 are arbitrary matrices over $\gamma_2^2 R_2$, I_3 and I_4 are identity matrices and

$$A = \begin{bmatrix} 3\gamma_2 & 3\gamma_2 & 2\gamma_2 & 3\gamma_2 \\ 3\gamma_2 & 3\gamma_2 & 3\gamma_2 & 2\gamma_2 \\ 3\gamma_2 & 2\gamma_2 & 3\gamma_2 & 3\gamma_2 \\ 2\gamma_2 & 3\gamma_2 & 3\gamma_2 & 3\gamma_2 \end{bmatrix}$$

It can be easily seen that C is self-orthogonal and $|C| = q_{15}$ and therefore C is self-dual.

Theorem 4.5 Let $q = 2$ and C be an $R_1 R_2$ -linear self-dual code. Then $d^*(C^\perp) \leq q^{e_2-1} \beta$.

Proof. For any arbitrary element $u = (u_1, \dots, u_\alpha | u'_1, \dots, u'_\beta) \in C$, we have

$$\gamma_2^{e_2-e_1} \sum_{i=1}^\alpha \iota(u_i^2) + \sum_{j=1}^\beta (u'_j)^2 = 0.$$

Consider the natural homomorphism $\rho: R_2 \rightarrow \frac{R_2}{\langle \gamma_2 \rangle} = \mathbb{F}_2$. Thus

$$\rho\left(\gamma_2^{e_2-e_1} \sum_{i=1}^\alpha \iota(u_i^2)\right) + \rho\left(\sum_{j=1}^\beta (u'_j)^2\right) = 0.$$

So $\sum_{j=1}^\beta (\rho(u'_j))^2 = 0$ and since \mathbb{F}^2 has characteristic 2, $\sum_{j=1}^\beta \rho(u'_j) = 0$, which implies $\sum_{j=1}^\beta u'_j \in \langle \gamma_2 \rangle$. Take $v = (0, \dots, 0 | \gamma_2^{e_2-1}, \dots, \gamma_2^{e_2-1})$. We have $\langle u, v \rangle = \gamma_2^{e_2-1} \sum_{j=1}^\beta u'_j = 0$. As a result, C^\perp contains the element v and hence $d^*(C^\perp) \leq q^{e_2-1} \beta$.

If C_X and C_Y are self-dual codes over chain rings R_1 and R_2 , respectively, then the separable code $C = C_X \times C_Y$ is a separable self-dual code over $R_1 \times R_2$. Some sufficient conditions for the existence of self-dual codes over chain rings are presented in [15, 16, 17]. In the following, we present some conditions for a self-dual code to be non-separable.

Theorem 4.6 Let C and C' be self-dual linear codes in $R_1^\alpha \times R_2^\beta$ and $R_1^{\alpha'} \times R_2^{\beta'}$ with generator matrices $G = (G_1 | G_2)$ and $G' = (G'_1 | G'_2)$, respectively. Then

$$G = \begin{bmatrix} G_1 & 0 & G_2 & 0 \\ 0 & G'_1 & 0 & G'_2 \end{bmatrix}$$

generates the self-dual linear code D in $R_1^{\alpha+\alpha'} \times R_2^{\beta+\beta'}$. Moreover, if either G'_1 or G'_2 is non-zero, then D is a non-separable self-dual code

Proof. Since the inner product of any two rows of G is zero, D is self-orthogonal. In addition, we have

$$|D| = |C||C'| = q^{\frac{e_1(\alpha+\alpha') + e_2(\beta+\beta')}{2}},$$

which implies D is self-dual.

The following theorem describes a technique for constructing $R_1 R_2$ -linear self-dual codes with larger lengths, which is a generalization of the presented technique in [17].

Theorem 4.7 Let B_p, T_p, S_j and A_j be the rows of B, T, S and A in Matrix (2.3). Assume that there are $c_1 \in R_1$ and $c_2 \in R_2$ such that $c_1^2 = -1$ and $c_2^2 = -1$. Let C be an $R_1 R_2$ -linear self-dual code of length n generated by the matrix (2.3). Consider $b = (b_1, \dots, b_\alpha) \in R_1^\alpha$ and $a = (a_1, \dots, a_\beta) \in R_2^\beta$ satisfying $b.b = -1$ and $a.a = -1$. Put $u_i = b.B_i^t, v_i = a.T_i^t, z_j = b.S_j^t$ and $w_j = a.A_j^t$. Then

$$g = \begin{bmatrix} 1 & 0 & b_1 & \dots & b_\alpha & 0 & 0 & 0 & \dots & 0 \\ -u_1 & c_1 u_1 & & & & -v_1 & c_2 v_1 & & & \\ \vdots & \vdots & & & B & \vdots & \vdots & & & T \\ -u_{k(B)} & c_1 u_{k(B)} & & & & -v_{k(B)} & c_2 u_{k(B)} & & & \\ & & & & & & & & & \\ 0 & 0 & 0 & 0 & & 1 & 0 & a_1 & \dots & a_\beta \\ -z_1 & c_1 z_1 & & & & -w_1 & c_2 w_1 & & & \\ \vdots & \vdots & & & S & \vdots & \vdots & & & A \\ -z_{l(A)} & c_1 z_{l(A)} & & & & -w_{l(A)} & c_2 w_{l(A)} & & & \end{bmatrix}$$

generates the $R_1 R_2$ -linear self-dual code D of length $n + 4$.

Proof. To simplify, we display the first $k(B)$ rows of G by $g_i, i \in \{1, \dots, k(B)\}$, and the last $l(A)$ rows of G by $h_j, j \in \{1, \dots, l(A)\}$. We have $\langle g_i, g_i \rangle = 1 + b.b = 0$ and $\langle h_j, h_j \rangle = 1 + a.a = 0$. Besides, since C is self-dual, for all $i \neq 1$ and $j \neq 1$, we obtain

$$\begin{aligned} \langle g_i, g_i \rangle &= \gamma_2^{e_2-e_1} \iota(u_i^2 + c_1^2 u_i^2 + B_i.B_i) \\ &\quad + (v_i^2 + c_2^2 v_i^2 + T_i.T_i) \\ &= \gamma_2^{e_2-e_1} \iota(B_i.B_i) + T_i.T_i = 0, \\ \langle h_j, h_j \rangle &= \gamma_2^{e_2-e_1} \iota(z_j^2 + c_1^2 z_j^2 + S_j.S_j) \\ &\quad + (w_j^2 + c_2^2 w_j^2 + A_j.A_j) \\ &= \gamma_2^{e_2-e_1} \iota(S_j.S_j) + A_j.A_j = 0. \end{aligned}$$

Thus, the rows of matrix G are orthogonal to themselves. Moreover, we have

$$\begin{aligned} 3\langle g_1, h_1 \rangle &= 0, \\ \langle g_1, h_j \rangle &= \gamma_2^{e_2-e_1} \iota(-z_j + b.S_j) + 0 = 0 \text{ for all } j \neq 1, \\ \langle g_i, h_1 \rangle &= \gamma_2^{e_2-e_1} \iota(0) + (-v_i + a.T_i) = 0 \text{ for all } i \neq 1. \end{aligned}$$

Furthermore, since C is self-dual, for $i \neq 1$ and $j \neq 1$ we get

$$\begin{aligned} \langle g_i, h_j \rangle &= \gamma_2^{e_2-e_1} \iota(u_i z_j + c_1^2 u_i z_j + B_i.S_j) \\ &\quad + (v_i w_j + c_2^2 v_i w_j + T_i.A_j) \\ &= \gamma_2^{e_2-e_1} \iota(B_i.S_j) + (T_i.A_j) = 0. \end{aligned}$$

Besides, for $i_1, i_2 \in \{1, \dots, k(B)\}$ and $j_1, j_2 \in \{1, \dots, l(A)\}$ such that $i_1 \neq i_2$ and $j_1 \neq j_2$ we have

$$\begin{aligned} \langle g_{i_1}, g_{i_2} \rangle &= \gamma_2^{e_2 - e_1} \iota(u_{i_1} u_{i_2} + c_1^2 u_{i_1} u_{i_2} + B_{i_1} \cdot B_{i_2}) \\ &\quad + (v_{i_1} v_{i_2} + c_2^2 v_{i_1} v_{i_2} + T_{i_1} \cdot T_{i_2}) \\ &= \gamma_2^{e_2 - e_1} \iota(B_{i_1} \cdot B_{i_2}) + T_{i_1} \cdot T_{i_2} = 0, \\ \langle h_{j_1}, h_{j_2} \rangle &= \gamma_2^{e_2 - e_1} \iota(z_{j_1} z_{j_2} + c_1^2 z_{j_1} z_{j_2} + S_{j_1} \cdot S_{j_2}) + \\ &\quad (w_{j_1} w_{j_2} + c_2^2 w_{j_1} w_{j_2} + A_{j_1} \cdot A_{j_2}) \\ &= \gamma_2^{e_2 - e_1} \iota(S_{j_1} \cdot S_{j_2}) + A_{j_1} \cdot A_{j_2} = 0. \end{aligned}$$

Thus, any two distinct rows of G are orthogonal and therefore D is self-orthogonal. To complete the proof, we need the size of D . By the elementary row and column operations, we conclude that G is of type

$$(\alpha+2, \beta+2; k_0+1, k_1, \dots, k_{e_1-1}; l_0+1, l_1, \dots, l_{e_2-1}).$$

So

$$\begin{aligned} |D| &= q^{e_1(k_0+1) + \sum_{i=1}^{e_1-1} (e_1-i)k_i + e_2(l_0+1) + \sum_{j=1}^{e_2-1} (e_2-j)l_j} \\ &= q^{e_1+e_2} |C| = q^{\frac{e_1(\alpha+2) + e_2(\beta+2)}{2}}. \end{aligned}$$

Consequently, D is self-dual.

Example 4.8 Let $R_1 = \mathbb{F}_5 + u\mathbb{F}_5$, where $u^2 = 0$, and $R_2 = \mathbb{F}_5 + u\mathbb{F}_5 + u^2\mathbb{F}_5$, where $u^3 = 0$. Take $c_1 = c_2 = 3$, $b = (3, 2 + u, 4 + 2u, 3u) \in R_1^4$ and $a = (1 + 4u + 3u^2, 0, 2 + 3u + u^2, 3) \in R_2^4$. Then $R_1 R_2$ -linear code C with the generator matrix $\mathcal{G} = [\mathcal{G}_1 | \mathcal{G}_2]$ where

$$\mathcal{G}_2 = \begin{bmatrix} 1 & 0 & 3 & 2+u & 3u & 0 \\ 1+2u & 2+4u & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1+3u & 2+3u & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{G}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2+3u=u^2 & 4+u=2u^2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1+4u=3u^2 & 0 & 2+3u+u^2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1+u+2u^2 & 2+2u+4u^2 & 0 & 0 & 3 & 1 \end{bmatrix},$$

is a non-separable self-dual code.

BOUNDS ON MINIMUM DISTANCE

In this section, a Gray map is introduced, which is a generalization of the given Gray map in [29].

Definition 5.1 Define the Gray map Φ from $R_1^\alpha \times R_2^\beta$ to $F_q^{\alpha+1-1\alpha+q\alpha-1\beta}$ as $\Phi(u|u') = (\phi_1(u)|\phi_2(u'))$, where ϕ_1 and ϕ_2 are the Gray maps over R_1 and R_2 , respectively.

The following theorem can be easily verified according to Definition 5.1 and Proposition 2.3.

Theorem 5.2 The Gray map Φ is an isometry from $(R_1^\alpha \times R_2^\beta, d^*)$ to $F_q^{\alpha+1-1\alpha+q\alpha-1\beta}, d_H$, where d_H denotes the Hamming distance on $F_q^{\alpha+1-1\alpha+q\alpha-1\beta}$.

The following examples provide optimal codes which are obtained directly in spite of the indirect construction presented in [19].

Example 5.3 Let $R_1 = R_{\gamma_1, 2, 2}, R_2 = R_{\gamma_2, 3, 2}$ and C be a linear code over $R_1 \times R_2$ generated by

$$G = \begin{bmatrix} 1 & 0 & 1+\gamma_1 & \gamma_1 & 0 \\ 0 & 1 & \gamma_1 & 1+\gamma_1 & 0 \\ 0 & \gamma_1 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & 0 & \gamma_2^2 \end{bmatrix}$$

Then $\Phi(C)$ is a binary linear code with parameters [12,5,4].

Example 5.4 Let $R_1 = R_{\gamma_1, 2, 2}, R_2 = R_{\gamma_2, 3, 2}$ and C be a linear code over generated by

$$G = \begin{bmatrix} \gamma_1 | 0 & \gamma_2^2 & \gamma_2^2 & 0 & \gamma_2^2 & 0 & \gamma_2^2 & 0 \\ 0 | 1 & 0 & 1 & \gamma_2 & \gamma_2 & 1+\gamma_2 & 1+\gamma_2 & \gamma_2^2 \\ \gamma_2^2 & 0 & \gamma_2^2 & 0 & \gamma_2^2 & & & \\ \gamma_2^2 & 1+\gamma_2^2 & 1+\gamma_2^2 & \gamma_2 + \gamma_2^2 & \gamma_2 + \gamma_2^2 & & & \\ 0 & & \gamma_2^2 & & & & & \\ 1+\gamma_2 + \gamma_2^2 & 1+\gamma_2 + \gamma_2^2 & & & & & & \end{bmatrix}.$$

Then $\Phi(C)$ is a binary linear code with parameters [62,4,32].

Example 5.5 Let $R_1 = R_{\gamma_1, 2, 2}, R_2 = R_{\gamma_2, 3, 2}$ and C be a linear code over $R_1 \times R_2$ generated by

$$G = \begin{bmatrix} 1 | 1 & 0 & 2 & \gamma_2 & 1+\gamma_2 & 1 & 2 & 1+\gamma_2 \\ 0 | 0 & \gamma_2 & \gamma_2 & 0 & 0 & \gamma_2 & 2\gamma_2 & \gamma_2 \\ 2 & \gamma_2 & 2+\gamma_2 & 1+2\gamma_2 & 2\gamma_2 & & & \\ \gamma_2 & \gamma_2 & \gamma_2 & 2\gamma_2 & & & & \gamma_2 \end{bmatrix}.$$

Then $\Phi(C)$ is a ternary linear code with parameters [40,3,27].

The next theorem presents two bounds on the minimum distance of $R_1 R_2$ -linear codes.

Theorem 5.6 Let C be an $R_1 R_2$ -linear code of type $(\alpha, \beta; k_0, \dots, k_{e_1-1}; l_0, \dots, l_{e_2-1})$. Then

$$3 \frac{d^*(C)-1}{q^{e_2-1}} \leq \frac{\alpha}{q^{e_2-e_1}} + \beta - \frac{\sum_{i=0}^{e_1-1} (e_1-i)k_i + \sum_{j=0}^{e_2-1} (e_2-j)l_j}{q^{e_2-1}}, \quad (5.1)$$

$$\left| \frac{d^*(C)-1}{q^{e_2-1}} \right| \leq \alpha + \beta - (\sum_{i=0}^{e_1-1} k_i + \sum_{j=0}^{e_2-1} l_j). \quad (5.2)$$

Proof. Note that $\Phi(C)$ is a code over $F_q^{q^{e_1-1}\alpha+q^{e_1-1}\beta}$ with size $|C|$. Applying Inequality (2.1) on $\Phi(C)$, we obtain

$$d_H(\Phi(C)) \leq q^{e_1-1}\alpha + q^{e_1-1}\beta - (\sum_{i=0}^{e_1-1} (e_1-i)k_i + \sum_{j=0}^{e_2-1} (e_2-j)l_j) + 1,$$

which implies Inequality (5.1).

Next, let $\chi: R_1 \rightarrow R_2$ be a map such that $\chi(x) = \gamma_2^{e_2-e_1} \iota(x)$. Extend χ to the map (χ, Id) from $R_1^\alpha \times R_2^\beta$ to $R_2^\alpha \times R_2^\beta$, where Id is the identity map over R_2^β . Obviously $d^*(C) \leq d_{hom_{R_2}}((\chi, Id)(C))$. In addition, it is clear that $\text{rank}((\chi, Id)(C)) = \text{rank}(C)$. Applying Inequality (2.2) on $(\chi, Id)(C)$, we obtain

$$\left| \frac{d^*(C)-1}{q^{e_2-1}} \right| \leq \alpha + \beta - \text{rank}(C) = \alpha + \beta - (\sum_{i=0}^{e_1-1} k_i + \sum_{j=0}^{e_2-1} l_j).$$

We say that an R_1R_2 -linear code is a maximum distance separable (MDS) code if $d^*(C)$ meets the bound given in Inequality (5.1) or (5.2). In the first case, we say that C is MDS with respect to the Singleton bound (MDSS). In the second case, C is MDS with respect to the rank bound (MDSR); see [5].

Lemma 5.7 Let C be an R_1R_2 -linear code of type $(\alpha, \beta; k_0, \dots, k_{e_1-1}; l_0, \dots, l_{e_2-1})$.

1. If $\alpha + \beta = \text{rank}(C)$, then C is MDSR and $1 \leq d^*(C) \leq q^{e_2-1}$.
2. If $k_0 + l_0 = 0$, then C^\perp is MDSR and $1 \leq d^*(C^\perp) \leq q^{e_2-1}$.

Proof.

1. We know that $\text{rank}(C) = \sum_{i=0}^{e_1-1} k_i + \sum_{j=0}^{e_2-1} l_j$. Now use the second inequality in Theorem 5.6.

2. Using Theorem 3.1, C^\perp is of type $(\alpha, \beta; k_{e_1}, k_{e_1-1}, \dots, k_1; l_{e_2}, l_{e_2-1}, \dots, l_1)$, where $k_{e_1} = \alpha - k(B)$ and $l_{e_2} = \beta - l(A)$. So $\text{rank } C^\perp = \alpha - k_0 + \beta - l_0$. Now the proof is similar to the first part.

Example 5.8 Every R_1R_2 -linear code of type $(\alpha, \beta; \alpha, 0, \dots, 0; \beta, 0, \dots, 0)$ is MDSR. Furthermore, the dual code of every R_1R_2 -linear code of type $(\alpha, \beta; 0, k_1, \dots, k_{e_1-1}; 0, l_1, \dots, l_{e_2-1})$ is MDSR. Moreover, if C is an R_1R_2 -linear code of type $(\alpha, \beta; 0, \alpha, 0, \dots, 0; 0, \beta, 0, \dots, 0)$, then C and C^\perp are MDSR.

Example 5.9 The code C given in Example 4.4 is an MDSR self-dual code.

Example 5.10 Let C be an R_1R_2 -linear code of type $(\alpha, \beta; 0, \dots, 0, 1; 0, \dots, 0)$ generated by

$$G = \langle (\gamma_1^{e_1-1}, \dots, \gamma_1^{e_1-1} | \gamma_2^{e_2-1}, \dots, \gamma_2^{e_2-1}) \rangle.$$

Clearly, $d^*(C) = \alpha q^{e_1-1} + \beta q^{e_2-1}$ and so C is an MDSS code.

Choose α such that $\alpha \leq q^{e_2-e_1}$. We have $\left| \frac{d^*(C)-1}{q^{e_2-1}} \right| = \beta$. Now

if $\alpha = 1$, then C is MDSR and if $\alpha > 1$, then C is not MDSR.

Example 5.11 Assume $R_1 = R_{\gamma_{1,2,5}}$ and $R_2 = R_{\gamma_{2,3,5}}$. The R_1R_2 -linear code C with the generator matrix

$$G = \left[\begin{array}{c|ccc} y_1 & 0 & \gamma_2^2 & 3\gamma_2^2 \\ \hline 0 & \gamma_2^2 & 3\gamma_2^2 & 2\gamma_2^2 \end{array} \right]$$

is of type $(1,3; 0,1; 0,0,1)$ with the minimum distance $d^*(C) = 54$. Applying the bound (5.1) and (5.2), we obtain that C is MDSR and is not MDSS.

Example 5.12 Let $R_1 = Z_{49}$ and $R_2 = R_{\gamma_{2,3,7}}$. The R_1R_2 -linear code C with the generator matrix

$$\left[\begin{array}{c|cccc} 7 & 0 & 0 & \gamma_2^2 & 4\gamma_2^2 & \gamma_2^2 \\ \hline 0 & \gamma_2^2 & \gamma_2^2 & 0 & 4\gamma_2^2 & 0 \\ 0 & 0 & \gamma_2^2 & \gamma_2^2 & 0 & 4\gamma_2^2 \\ 0 & 0 & 0 & \gamma_2^2 & 5\gamma_2^2 & 6\gamma_2^2 \end{array} \right]$$

is of type $(1,5; 0,1; 0,0,1)$ with the minimum distance $d^*(C) = 105$. Clearly, C is MDSR and is not MDSS.

CONCLUSION

In this paper, we study R_1R_2 -linear codes of length $n = \alpha + \beta$. We first determine the parity-check matrix of R_1R_2 -linear codes as well as the relation between R_1R_2 -linear codes C and C^\perp . Also, we provide some examples to show that our results on duality and parity-check matrix recover that of on several mixed alphabet codes. As an application of the results on dual codes, we construct some separable and non-separable self-dual R_1R_2 -linear codes. After that, we define a weight function on $R_1^\alpha \times R_2^\beta$ which is the natural generalization of the homogeneous weight over chain rings. Then, we define a distance preserving Gray map from R_1R_2 -linear codes to codes over \mathbb{F}_q equipped with the Hamming weight. The Gray image of R_1R_2 -linear codes presented in Examples 5.3, 5.4 and 5.5 provide optimal codes which have more simple construction than linear codes with the same parameters in Grassl table. Moreover, two upper bounds for the minimum distance are obtained by the Singleton bound for the Gray image and the rank bound for codes over rings. Finally, we discuss the conditions on the R_1R_2 -linear codes to be MDSS or MDSR.

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The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

- [1] Aydogdu I. Codes over $Z_p[u]/\langle u^r \rangle \times Z_p[u]/\langle u^s \rangle$. *J Algebra Comb Discrete Struct Appl* 2019;32:39–51. [\[CrossRef\]](#)
- [2] Aydogdu I, Abualrub T, Siap, I. On Z_2Z_2 -additive codes. *Int J Comput Math* 2014;92:1806–1814. [\[CrossRef\]](#)
- [3] Aydogdu I, Siap I. On $Z_pZ_{p^s}$ -additive codes. *Linear Multilinear Algebra* 2014;63:2089–2102. [\[CrossRef\]](#)
- [4] Aydogdu I, Siap I, Ten-Valls R. On the structure of $Z_2Z_2[u^3]$ -linear and cyclic codes. *Finite Fields Their Appl* 2017;48:241–260. [\[CrossRef\]](#)
- [5] Bilal M, Borges J, Dougherty ST, Fernández-Córdoba C. Maximum distance separable codes over Z_4 and $Z_4 \times Z_2$. *Des Codes Cryptogr* 2010;61:31–40. [\[CrossRef\]](#)
- [6] Borges J, Fernández-Córdoba C. A characterization of $Z_2Z_2[u]$ -linear codes. *Des Codes Cryptogr* 2017; 86:1377–1389. [\[CrossRef\]](#)
- [7] Borges J, Fernandez-Cordoba C, Ten-Valls R. Z_2Z_4 -additive cyclic codes, generator polynomials, and dual codes. *IEEE Trans Inf Theory* 2016;62:6348–6354. [\[CrossRef\]](#)
- [8] Borges J, Fernández-Córdoba C, Ten-Valls R. Linear and cyclic codes over direct product of finite chain rings. *Math Methods Appl Sci* 2017;41:6519–6529. [\[CrossRef\]](#)
- [9] Constantinescu I, Heise W. A metric for codes over residue class rings of integers. *Probl Inf Transm* 1997;33:22–28.
- [10] Dinh HQ, Bag T, Kewat PK, Pathak S, Upadhyay AK, Chinnakum W. Constacyclic codes of length (p^r, p^s) over mixed alphabets. *J Appl Math Comput* 2021;67:807–832. [\[CrossRef\]](#)
- [11] Diao L, Gao J, Lu J. Some results on $Z_pZ_p[v]$ -additive cyclic codes. *Adv Math Commun* 2020;14:555–572. [\[CrossRef\]](#)
- [12] Dinh HQ, Pathak S, Bag T, Upadhyay AK, Chinnakum W. A study of F_qR -cyclic codes and their applications in constructing quantum codes. *IEEE Access* 2020;8:190049–190063. [\[CrossRef\]](#)
- [13] Dinh HQ, Pathak S, Upadhyay AK, Yamaka W. New DNA codes from cyclic codes over mixed alphabets. *Mathematics* 2020;8:1977. [\[CrossRef\]](#)
- [14] Dougherty ST. *Algebraic Coding Theory Over Finite Commutative Rings*. 1st ed. Berlin: Springer; 2017. [\[CrossRef\]](#)
- [15] Dougherty ST, Kim JL, Kulosman H. MDS codes over finite principal ideal rings. *Des Codes Cryptogr* 2008;50:77–92. [\[CrossRef\]](#)
- [16] Dougherty ST, Kim JL, Kulosman H, Liu H. Self-dual codes over commutative Frobenius rings. *Finite Fields Their Appl* 2010;16:14–26. [\[CrossRef\]](#)
- [17] Dougherty ST, Kim JL, Liu H. Constructions of self-dual codes over finite commutative chain rings. *Int J Inf Cod Theory* 2010;1:171. [\[CrossRef\]](#)
- [18] Gao J, Diao L. $Z_pZ_p[u]$ -additive cyclic codes. *Int J Inf Cod Theory* 2018;5:1–17. [\[CrossRef\]](#)
- [19] Grassl M. Bounds on the minimum distance of linear codes and quantum codes. Available at: <http://www.codetables.de>. Accessed on May 22, 2023. [\[CrossRef\]](#)
- [20] Greferath M, Schmidt SE. Gray isometries for finite chain rings and a nonlinear ternary code. *IEEE Trans Inf Theory* 1999;45:2522–2524. [\[CrossRef\]](#)
- [21] Jitman J, Udomkavanich P. The gray image of codes over finite chain rings. *Int J Contemp Math Sci* 2010;5:449–458.
- [22] Li J, Gao J, Fu FW, Ma F. F_qR -linear skew constacyclic codes and their application of constructing quantum codes. *Quantum Inf Process* 2020;19:193. [\[CrossRef\]](#)
- [23] MacWilliams F, Sloane N. *The Theory of Error-Correcting Codes*. 1st ed. Amsterdam: North-Holland; 1977.
- [24] Mahmoudi S, Samei K. SR -additive codes. *Bull Korean Math Soc* 2019;56:1235–1255. [\[CrossRef\]](#)
- [25] Melakhessou A, Aydin N, Hebbache Z, Guenda K. $Z_q(Z_q + uZ_q)$ -linear skew constacyclic codes. *J Algebra Comb Discrete Struct Appl* 2020;7:85–101. [\[CrossRef\]](#)
- [26] Norton GH, Sălăgean A. On the structure of linear and cyclic codes over a finite chain ring. *Appl*

-
- Algebra Eng Commun Comput 2000;10:489–506. [\[CrossRef\]](#)
- [27] Rifa J, Pujol J. Translation-invariant propelinear codes. *IEEE Trans Inf Theory* 1997;43:590–598. [\[CrossRef\]](#)
- [28] Samei K, Mahmoudi S. Singleton bounds for R -additive codes. *Adv Math Commun* 2018;12:107–114. [\[CrossRef\]](#)
- [29] Shi M, Wu R, Krotov DS. On $\mathbb{Z}_p\mathbb{Z}_{p^k}$ -additive codes and their duality. *IEEE Trans Inf Theory* 2019;65:3841–3847. [\[CrossRef\]](#)
- [30] Wu R, Shi M. Some classes of mixed alphabet codes with few weights. *IEEE Commun Lett* 2021;25:1431–1434. [\[CrossRef\]](#)