Sigma J Eng Nat Sci, Vol. 40, No. 3, pp. 585–592, September, 2022

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2022.00063

Research Article

On (*s*, *P*)-functions and related inequalities

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ARTICLE INFO

Article history
Received: 01 February 2021
Accepted: 27 April 2021

Keywords:
s-convex function;
(*s*,*P*)-function;
Hermite-Hadamard inequality

ABSTRACT

In this paper, we introduce and study the concept of (s,*P*)-function and establish Hermite-Hadamard's inequalities for this type of functions. Also, we obtain some new Hermite-Hadamard type inequalities for functions whose first derivative in absolute value is (s,*P*)-function by using Hölder and power-mean integral inequalities. We also extend our initial results to functions of several variables. Next, we point out some applications of our results to give estimates for the approximation error of the integral function in the trapezoidal formula and for some inequalities related to special means of real numbers.

Cite this article as: Numan S, İşçan İ. On (s, P)–functions and related inequalities. Sigma J Eng Nat Sci 2022;40(3):585–592.

INTRODUCTION

Let $f:I\to\mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2} \qquad (1.1)$$

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function *f* is concave. This double inequality is well known as the Hermite-Hadamard inequality [8]. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping *f*.

In [7], S. S. Dragomir, et al. gave the following definition and related Hermite-Hadamard integral inequalities as follow:

Definition 1. A nonnegative function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to be *P*-function if the inequality

$$f(tx+(1-t)y)\le f(x)+f(y),$$

holds for all $x, y \in I$ and $t\in(0,1)$.

Theorem 1. Let $f\in P(I)$, $a, b \in I$ with $a < b$ and $f\in L[a,b]$. Then

$$f\left(\frac{a+b}{2}\right)\le\frac{2}{b-a}\int_a^b f(x)dx\le\left[f(a)+f(b)\right]. \qquad (1.2)$$

Definition 2. [13] Let $h:J\to\mathbb{R}$ be a non-negative function, $h\neq 0$ We say that $f:I\to\mathbb{R}$ is an *h*-convex function,

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This paper was recommended for publication in revised form by Regional Editor Vildan Çetkin

Published by Yıldız Technical University Press, İstanbul, Turkey

or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x,y \in I$, $\alpha \in (0,1)$ we have

$$f(\alpha x + (1 - \alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y)$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV(h,I)$. It is clear that, if we choose $h(\alpha) = \alpha$ and $h(\alpha) = 1$, then the h-convexity reduces to convexity and definition of P-function, respectively.

Readers can look at [1, 13] for studies on h-convexity.

In [9], H. Hudzik and L. Maligranda considered among others the class of functions which are s-convex in the second sense.

Definition 3. A function $f:[0,\infty)\to\mathbb{R}$ is said to be s–convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y),$$

for all $x,y \in 0,\infty)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0,1]$. This class of s-convex functions in the second sense is usually denoted by K_s^2.

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [6], S. S. Dragomir and S. Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense.

Theorem 2. Suppose that $f:[0,\infty) \to 0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1)$, and let $a,b \in 0, \infty)$, $a < b$. If $f \in L[a,b]$ then the following inequalities hold

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \quad (1.3)$$

Both inequalities hold in the reversed direction if f is s-concave. The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3).

The main purpose of this paper is to introduce the concept of (s,P)-function which is connected with the concepts of P-function and s-convex function and establish some new Hermite-Hadamard type inequality for these classes of functions. In recent years many authors have studied error estimations of Hermite-Hadamard type inequalities; for refinements, counterparts, generalizations, for some related papers see [2, 3, 4, 5, 7, 10, 11, 12].

THE DEFINITION OF (s,P)-FUNCTION

In this section, we introduce a new concept, which is called (s,P)-function and we give by setting some algebraic properties for (s,P)-function, as follows:

Definition 4. Let $s \in (0,1]$. A function $f: I \subset \mathbb{R} \to \mathbb{R}$ is called (s,P)-function if

$$f(tx + (1-t)y) \leq (t^s + (1-t)^s)[f(x) + f(y)], \quad (2.1)$$

for every $x,y \in I$ and $t \in [0,1]$.

We will denote by $P_s(I)$ the class of all (s,P)-functions on interval I. Clearly, the definition of $(1,P)$-function is coincide with the definition of P-function.

We note that, every (s,P)-function is an h-convex function with the function $h(t) = t^s + (1-t)^s$. Therefore, if $P_s(I)$, then

i) $f+g \in P_s(I)$ and for $c \in \mathbb{R}$ $(c \geq 0)$ $cf \in P_s(I)$ (see [13], Proposition 9).

ii) if f and g be a similarly ordered functions on I, then $fg \in P_s(I)$.(see [13], Proposition 10).

Also, if $f: I \to J$ is a convex and $g \in P_s(J)$ and nondecreasing, then $g \circ f \in P_s(I)$ (see [13], Theorem 15).

Remark 1. We note that if f is satisfy (2.1), then f is a nonnegative function. Indeed, if we rewrite the inequality (2.1) for $t = 0$ then

$$f(y) \leq f(x) + f(y),$$

for every $x,y \in I$. Thus we have $f(x) \geq 0$ for all $x \in I$.

Proposition 1. Every nonnegative s-convex function is also an (s,P)-function.

Proof. Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be an arbitrary nonnegative s-convex function. Then

$$\begin{aligned} f(tx + (1-t)y) &\leq t^s f(x) + (1-t)^s f(y) \\ &\leq (t^s + (1-t)^s)[f(x) + f(y)], \end{aligned}$$

for every $x,y \in I$ and $t \in [0,1]$.

Proposition 2. Every P-function is also an (s,P)-function.

Proof. Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be an arbitrary P-function. The proof is clear from the following inequalities

$$t \leq t^s \text{ and } 1 - t \leq (1-t)^s,$$

for all $t \in [0,1]$. In this case, we can write

$$1 \leq t^s + (1-t)^s.$$

Therefore,

$$\begin{aligned} f(tx + (1-t)y) &\leq f(x) + f(y) \\ &\leq (t^s + (1-t)^s)[f(x) + f(y)], \end{aligned}$$

for every $x,y \in I$, $t \in [0,1]$ and $s \in (0,1]$ Thus desired result is obtained.

We can give the following corollary for every nonnegative convex function is also an P-function.

Corollary 1. Every nonnegative convex function is also an (s,P)-function.

Theorem 3. If $f:[a,b] \subset \mathbb{R} \to \mathbb{R}$ is an (s,P)-function, then f is bounded on $[a,b]$.

Proof. Let $M = \max\{f(a), f(b)\}$. For any $x \in [a,b]$, there exists a $t_0 \in [0,1]$ such that $x = t_0 a + (1 - t_0)b$. Since f is an (s,P)-function on $[a,b]$, and $t^s + (1 - t)^s \leq 2^{1-s}$ for all $t \in [0,1]$ and $s \in (0,1]$, we have

$$f(x) \leq \left(t^s + (1-t)^s\right)\left[f(a) + f(b)\right] \leq 2^{2-s} M \leq 4M.$$

This shows that f is bounded from above. For any $x \in [a,b]$, there exists a $t_0 \in [0,1]$ such that either $x = \frac{a+b}{2} + t_0$ or $x = \frac{a+b}{2} - t_0$. Since it will lose nothing generality we can assume $x = \frac{a+b}{2} + t_0$. Thus we can write

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}\left[\frac{a+b}{2} + t_0\right] + \frac{1}{2}\left[\frac{a+b}{2} - t_0\right]\right) \\ &\leq 2^{1-s}\left[f(x) + f\left(\frac{a+b}{2} - t_0\right)\right], \end{aligned}$$

and from here we have

$$\begin{aligned} f(x) &\geq 2^{s-1} f\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2} - t_0\right) \\ &\geq \frac{1}{2} f\left(\frac{a+b}{2}\right) - 4M = m. \end{aligned}$$

This completes the proof.

Theorem 4. Let $b > a$ and $f_\alpha:[a,b] \to \mathbb{R}$ be an arbitrary family of (s,P)-function and let $f(x) = sup_\alpha f_\alpha(x)$. If $J = \{u \in [a,b]: f(u) < \infty\}$ is nonempty, then J is an interval and f is an (s,P)-function on J.

Proof. Let $t \in [0,1]$ and $x,y \in J$ be arbitrary. Then

$$\begin{aligned} f\left(tx + (1-t)y\right) &= \sup_\alpha f_\alpha\left(tx + (1-t)y\right) \\ &\leq \sup_\alpha\left\{\left(t^s + (1-t)^s\right)\left[f_\alpha(x) + f_\alpha(y)\right]\right\} \\ &\leq \left(t^s + (1-t)^s\right)\left[\sup_\alpha f_\alpha(x) + \sup_\alpha f_\alpha(y)\right] \\ &= \left(t^s + (1-t)^s\right)\left[f(x) + f(y)\right] < \infty. \end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is an exponential type P-function on J. This completes the proof of theorem.

HERMITE-HADAMARD'S INEQUALITY FOR (*s*,*P*)-FUNCTIONS

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for (s,*P*)-functions. In this section, we will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on $[a,b]$.

Theorem 5. Let $s \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be a (s,P)-function. If $a < b$ and $f \in L[a,b]$, then the following Hermite-Hadamard type inequalities hold:

$$\begin{aligned} 2^{s-2} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{2}{s+1}\left[f(a) + f(b)\right]. \end{aligned} \tag{3.1}$$

Proof. Since f is a (s,P)-function, we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}\left[ta + (1-t)b\right] + \frac{1}{2}\left[(1-t)a + tb\right]\right) \\ &\leq 2^{1-s}\left[f\left(ta + (1-t)b\right) + f\left((1-t)a + tb\right)\right]. \end{aligned}$$

By taking integral in the last inequality with respect to $t \in [0,1]$, we deduce that

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{2-s}}{b-a}\int_a^b f(x)dx.$$

By using the property of (s,P)-function of f, if the variable is changed as $x = ta + (1 - t)b$, then

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)du &= \int_0^1 f\left(ta + (1-t)b\right)dt \\ &\leq \left[f(a) + f(b)\right]\int_0^1 t^s + (1-t)^s dt \\ &= \frac{2}{s+1}\left[f(a) + f(b)\right]. \end{aligned}$$

This completes the proof of theorem.

Remark 2. In Theorem 5, if we choose $s = 1$, then inequality (3.1) reduce to inequality (1.2).

Theorem 6. Let $a < b$, $s \in (0,1]$ and $f:[a,b] \to \mathbb{R}$ be an (s,P)-function. If f is symmetric with respect to $(a + b)/2$(i.e. $f(x) = f(a + b - x)$ for all $x \in [a,b]$), then the following inequalities hold:

$$2^{s-2} f\left(\frac{a+b}{2}\right) \leq f(x) \leq 2^{1-s}\left[f(a) + f(b)\right],$$

for all $x \in I$.

Proof. Let $x \in [a,b]$ be arbitrary point. Since $t^s + (1 - t)^s \leq 2^{1-s}$ for all $t \in [0,1]$, we get

$$f(x) = f\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a\right)$$

$$\leq\left(\left(\frac{x-a}{b-a}\right)^{s}+\left(\frac{b-x}{b-a}\right)^{s}\right)[f(a)+f(b)]$$
$$\leq 2^{1-s}[f(a)+f(b)],$$

and

$$f\left(\frac{a+b}{2}\right)=f\left(\frac{1}{2}x+\frac{1}{2}[a+b-x]\right)$$
$$\leq 2^{1-s}[f(x)+f(a+b-x)]$$
$$=2^{2-s}f(x).$$

This completes the proof.

SOME NEW INEQUALITIES APPLICATIONS FOR (*s*,*P*)-FUNCTIONS

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value is exponential type *P*-function. S. S. Dragomir and R. P. Agarwal [4] used the following lemma:

Lemma 1. Let $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If $f'\in L[a,b]$, then

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1(1-2t)$$
$$f'(ta+(1-t)b)dt.$$

Notation 1. We'll use the following notation for brevity throughout this section

$$T_f(a,b)=\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx.$$

Theorem 7. Let $f:I\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$ and assume that $f'\in L[a,b]$ and $s\in(0,1]$. If $|f'|$ is (*s*,*P*)-function on interval $[a,b]$, then the following inequality holds

$$\left|T_f(a,b)\right|\leq\frac{b-a}{(s+1)(s+2)}\left(\frac{1+s2^{s}}{2^{s-1}}\right)$$
$$A\left(|f'(a)|,|f'(b)|\right). \tag{4.1}$$

Proof. Using Lemma 1 and the inequality

$$\left|f'(ta+(1-t)b)\right|\leq\left(t^{s}+(1-t)^{s}\right)\left[|f'(a)|+|f'(b)|\right],$$

we get

$$\left|T_f(a,b)\right|\leq\frac{b-a}{2}\left[|f'(a)|+|f'(b)|\right]\int_0^1|1-2t|$$
$$\left(t^{s}+(1-t)^{s}\right)dt$$
$$=\frac{b-a}{(s+1)(s+2)}\left(\frac{1+s2^{s}}{2^{s-1}}\right)A\left(|f'(a)|,|f'(b)|\right),$$

where

$$\int_0^1|1-2t|\left(t^{s}+(1-t)^{s}\right)dt=\frac{1}{(s+1)(s+2)}\left(\frac{1+s2^{s}}{2^{s-1}}\right),$$

and *A* is the arithmetic mean. This completes the proof of theorem.

Theorem 8. Let $f:I\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$ and assume that $f'\in L[a,b]$ and $s\in(0,1]$. If $|f'|^{q}$, $q>1$, is a (*s*,*P*)-function on interval $[a,b]$, then the following inequality holds

$$\left|T_f(a,b)\right|\leq\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{4}{s+1}\right)^{\frac{1}{q}}$$
$$A^{\frac{1}{q}}\left(|f'(a)|^{q},|f'(b)|^{q}\right),$$

where $\frac{1}{p}+\frac{1}{q}=1$ and *A* is the arithmetic mean.

Proof. Using Lemma 1, Hölder's integral inequality and the following inequality

$$\left|f'(ta+(1-t)b)\right|^{q}\leq\left(t^{s}+(1-t)^{s}\right)\left[|f'(a)|^{q}+|f'(b)|^{q}\right]$$

which is (*s*,*P*)-function of $|f'|^{q}$, we get

$$\left|T_f(a,b)\right|\leq\frac{b-a}{2}\left(\int_0^1|1-2t|^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}$$
$$\leq\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left[|f'(a)|^{q}+|f'(b)|^{q}\right]\int_0^1 t^{s}+(1-t)^{s}dt\right)^{\frac{1}{q}}$$
$$=\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{4}{s+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|f'(a)|^{q},|f'(b)|^{q}\right).$$

This completes the proof of theorem.

Theorem 9. Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$ and assume that $f\in L[a,b]$ and $s\in(0,1]$ If $|f'|^{q}$, $q\geq 1$, is a (*s*,*P*)-function on the interval [a,b], then the following inequality holds

$$\left|T_f(a,b)\right| \le \frac{b-a}{2^{2-\frac{2}{q}}}\left(\frac{1}{(s+1)(s+2)}\left(\frac{1+s2^s}{2^{s-1}}\right)\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right). \quad (4.3)$$

Proof. From Lemma 1, well known power-mean integral inequality and the property of (s,P)-function of $|f'|^q$, we obtain

$$\left|T_f(a,b)\right| \le \frac{b-a}{2}\left(\int_0^1|1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2t|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$

$$\le \frac{b-a}{2^{2-\frac{1}{q}}}\left(\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]\int_0^1|1-2t|\left(t^s+(1-t)^s\right)dt\right)^{\frac{1}{q}}$$

$$= \frac{b-a}{2^{2-\frac{2}{q}}}\left(\frac{1}{(s+1)(s+2)}\left(\frac{1+s2^s}{2^{s-1}}\right)\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right).$$

This completes the proof of theorem.

Corollary 2. Under the assumption of Theorem 9, If we take $q = 1$ in the inequality (4.3), then we get the following inequality:

$$\left|T_f(a,b)\right| \le \frac{b-a}{(s+1)(s+2)}\left(\frac{1+s2^s}{2^{s-1}}\right) A\left(\left|f'(a)\right|,\left|f'(b)\right|\right).$$

This inequality coincides with the inequality (4.1).

AN EXTENTION OF THEOREM 7

In this section we will denote by K an open and convex set of $\mathbb{R}^n$ $(n \ge 1)$.

Let $s \in (0,1]$. We say that a function $f:K \to \mathbb{R}$ is an (s,P)-function on K if

$$f\left(tx+(1-t)y\right) \le \left(t^s+(1-t)^s\right)\left[f(x)+f(y)\right],$$

for all $x,y \in K$ and $t \in [0,1]$.

Lemma 2. Let $f:K \to \mathbb{R}$ be a function. Then f is (s, P)-function on K if and only if for all $x,y \in K$ the function $\Phi:[0,1] \to \mathbb{R}$, $\Phi(t) = f\left(tx+(1-t)y\right)$ is (s, P)-function on $[0, 1]$.

Proof. $(\Leftarrow)$ Let $x,y \in K$ be fixed. Assume that $\Phi:[0,1] \to \mathbb{R}$, $\Phi(t) = f\left(tx+(1-t)y\right)$ is (s, P)-function on $[0, 1]$.

Let $t \in [0,1]$ be arbitrary, but fixed. Clearly, $t = (1-t).0 + t.1$ and thus,

$$f\left(tx+(1-t)y\right) = \Phi(t) = \Phi\left(t.1+(1-t).0\right)$$

$$\le \left(t^s+(1-t)^s\right)\left[\Phi(0)+\Phi(1)\right]$$

$$= \left(t^s+(1-t)^s\right)\left[f(x)+f(y)\right].$$

It follows that f is (s, P)-function on K

$(\Leftarrow)$ Assume that f is (s, P)-function on K Let $x,y \in K$ be fixed and define $\Phi:[0,1] \to \mathbb{R}$, $\Phi(t) = f\left(tx+(1-t)y\right)$ We must show that Φ is (s, P)-function on $[0, 1]$.

Let $u_1,u_2 \in [0,1]$ and $t \in [0,1]$. Then

$$\Phi\left(tu_1+(1-t)u_2\right) = f\left(\left(tu_1+(1-t)u_2\right)x+\left(1-tu_1-(1-t)u_2\right)y\right)$$

$$= f\left(t(u_1x+\left(1-u_1\right)y+(1-t)(u_2x+\left(1-u_2\right)y\right)$$

$$\le \left(t^s+(1-t)^s\right)\left[f\left(u_1x+\left(1-u_1\right)y\right)+f\left(u_2x+\left(1-u_2\right)y\right)\right]$$

$$= \left(t^s+(1-t)^s\right)\left[\Phi\left(u_1\right)+\Phi\left(u_2\right)\right].$$

We deduce that Φ is (s, P)-function on $[0, 1]$.

The proof of Lemma 2 is complete.

Using the above lemma we will prove an extension of Theorem 7 to functions of several variables.

Proposition 3. Assume $f:K \subseteq \mathbb{R}^n \to \mathbb{R}^+$ is a (s, P)-function on K. Then for any $x,y \in K$ and any $u,v \in (0,1)$ with $u < v$ the following inequality holds

$$\left|\frac{1}{2}\int_0^u f\left(sx+(1-s)y\right)ds+\frac{1}{2}\int_0^v f\left(sx+(1-s)y\right)ds - \frac{1}{v-u}\int_u^v\left(\int_0^\theta f\left(sx+(1-s)y\right)ds\right)d\theta\right|$$

$$\le \frac{v-u}{(s+1)(s+2)}\left(\frac{1+s2^s}{2^{s-1}}\right) A\left(f\left(ux+(1-u)y\right),f\left(vx+(1-v)y\right)\right).$$

Proof. We fix $x, y \in K$ and $u, v \in (0,1)$ with $u < v$. Since f is (s, P)-function, by Lemma 2 it follows that the function

$$\Phi : [0,1] \to \mathbb{R}, \Phi(t) = f(tx + (1-t)y),$$

is (s, P)-function on [0, 1].

Define $\Psi : [0,1] \to \mathbb{R}$,

$$\Psi(t) = \int_0^t \Phi(s)ds = \int_0^t f(sx + (1-s)y)ds.$$

Obviously, $\Psi'(t) = \Phi(t)$ for all $t \in (0,1)$.

Since $f(K) \subseteq \mathbb{R}^+$ it results that $\Phi \geq 0$ on [0, 1] and thus, $\Psi' \geq 0$ on (0, 1).

Applying Theorem 7 to the function ψ we obtain

$$\left| \frac{\Psi(u)+\Psi(v)}{2} - \frac{1}{v-u}\int_u^v \Psi(\theta)d\theta \right| \leq \frac{v-u}{(s+1)(s+2)}\left(\frac{1+s2^s}{2^{s-1}}\right) A\left(\left|\Psi'(u)\right|, \left|\Psi'(v)\right|\right),$$

and we deduce that relation (5.1) holds true.

Remark 3. We point out that a similar result as those of Proposition 3 can be stated by using Theorem 8 and Theorem 9.

APPLICATIONS TO TRAPEZOIDAL FORMULA

Assume $\wp$ is a division of the interval $[a,b]$ such that

$$\wp : \ a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

For a given function $f : [a,b] \to \mathbb{R}$ we consider the trapezoidal formula

$$T(f, \wp) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i).$$

It is well known that if f is twice differentiable on (a,b) and $M = sup_{x \in (a,b)}|f''(x)| < \infty$ then

$$\int_a^b f(x)dx = T(f, \wp) + E(f, \wp),$$

where $E(f, \wp)$ is the approximation error of the integral $\int_a^b f(x)dx$ by the trapezoidal formula and satisfies,

$$\left|E(f, \wp)\right| \leq \frac{M}{12}\sum_{i=0}^{n-1}(x_{i+1} - x_i)^3. \tag{6.1}$$

Clearly, if the function f is not twice differentiable or the second derivative is not bounded on (a,b), then (6.1) does not hold true. In that context, the following results are important in order to obtain some estimates of $E(f, \wp)$

Proposition 4. Let $s \in (0,1]$, $a,b \in \mathbb{R}$ with $a < b$ and assume that $f : [a,b] \to \mathbb{R}$ is a differentiable function on (a,b). If $|f'|$ is (s,P)-function on $[a,b]$ then for each division $\wp$ of the interval $[a,b]$ we have,

$$\left|E(f, \wp)\right| \leq \frac{1+s2^s}{2^{2s-3}(s+1)(s+2)} A\left(\left|f'(a)\right|, \left|f'(b)\right|\right) \sum_{i=0}^{n-1}(x_{i+1} - x_i)^2. \tag{6.2}$$

Proof. We apply Theorem 7 on the sub-intervals $[x_i, x_{i+1}]$, $i = 0,1,\ldots,n-1$ given by the division $\wp$. Adding from $i = 0$ to $i = n-1$ we deduce

$$\left|T(f, \wp) - \int_a^b f(x)dx\right| \leq \frac{1+s2^s}{2^{s-1}(s+1)(s+2)} \sum_{i=0}^{n-1}(x_{i+1} - x_i)^2 A\left(\left|f'(x_i)\right|, \left|f'(x_{i+1})\right|\right). \tag{6.3}$$

On the other hand, for each $x_i \in [a,b]$ there exists $t_i \in [0,1]$ such that $x_i = t_i a + (1 - t_i)b$. Since $|f'|$ is (s,P)-function and $t^s + (1 - t)^s \leq 2^{1-s}$) for all $t \in [0,1]$, we deduce

$$\begin{aligned} \left|f'(x_i)\right| &\leq \left(t_i^s + (1-t_i)^s\right)\left[f(a) + f(b)\right] \\ &\leq 2^{2-s} A\left(\left|f'(a)\right|, \left|f'(b)\right|\right), \end{aligned} \tag{6.4}$$

for each $i = 0,1,\ldots,n-1$. Relations (6.3) and (6.4) imply that relation (6.2) holds true. Thus, Proposition 4 is completely proved.

A similar method as that used in the proof of Proposition 4 but based on Theorem 8 and Theorem 9 shows that the following results are valid.

Proposition 5. Let $s \in (0,1]$, $a,b \in \mathbb{R}$ with $a < b$ and assume that $f : [a,b] \to \mathbb{R}$ is a differentiable function on (a,b). If $|f'|^q$, $q > 1$, is an (s,P)-function on interval $[a,b]$, then for each division $\wp$ of the interval $[a,b]$ we have,

$$\left|E(f, \wp)\right| \leq \frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2^{4-s}}{s+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)\sum_{i=0}^{n-1}(x_{i+1} - x_i)^2,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 6. Let $s \in (0,1]$, $a,b \in \mathbb{R}$ with $a < b$ and assume that $f:[a,b] \to \mathbb{R}$ is a differentiable function on (a,b). If $|f'|^q$, $q > 1$ is an (s,P)-function on interval $[a,b]$, then for each division $\wp$ of the interval $[a,b]$ we have,

$$\left|E(f,\wp)\right| \le \frac{1}{4}\left(\frac{1+s2^s}{2^{2s-5}(s+1)(s+2)}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)^2.$$

SOME APPLICATIONS FOR SPECIAL MEANS

Let us recall the following special means of two non-negative number a,b with $b > a$:

1. The arithmetic mean:

$$A = A(a,b) := \frac{a+b}{2}, a,b \ge 0.$$

2. The geometric mean:

$$G = G(a,b) := \sqrt{ab}, a,b \ge 0.$$

3. The harmonic mean:

$$H = H(a,b) := \frac{2ab}{a+b}, a,b > 0.$$

4. The Logarithmic mean

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}, a,b > 0.$$

5. The p-Logarithmic mean:

$$L_p = L_p(a,b) := \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, \quad p \in \mathbb{R}\setminus\{-1,0\}, a,b > 0.$$

6. The Identric mean:

$$I = I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, a,b > 0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature: $H \le G \le L \le I \le A$.

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 7. Leta, $b \in [0,\infty)$ with $a < b$, $s \in (0,1]$ and $n \in (-\infty,0) \cup [1,\infty)\setminus\{-1\}$. Then, the following inequalities are obtained:

$$2^{s-2}A^n(a,b) \le L_n^n(a,b) \le \frac{4}{s+1}A\left(a^n,b^n\right).$$

Proof. The assertion follows from the inequalities (3.1) for the function $f(x) = x^n$, $x \in [0,\infty)$.

Proposition 8. Let $a, b \in (0,\infty)$ with $a < b$ and $s \in (0,1]$. Then, the following inequalities are obtained:

$$2^{s-2}A^{-1}(a,b) \le L^{-1}(a,b) \le \frac{4}{s+1}H^{-1}(a,b).$$

Proof. The assertion follows from the inequalities (3.1) for the function $f(x) = x^{-1}$, $x \in (0,\infty)$.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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