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Research Article

A sine-cosine wavelet method for the approximation solutions of the fractional Bagley-Torvik equation

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ABSTRACT

Fractional Bagley-Torvik differential equations can be used to model a variety of natural phenomena in many branches of applied mathematics and engineering in general. The focus of this study is on solving the fractional Bagley-Torvik equations by using sine-cosine wavelet. To this end, the operational matrix of fractional integration is obtained for sine-cosine wavelets. By utilizing this matrix, fractional Bagley-Torvik differential equation is transformed to a system of linear algebraic equations with unknown coefficients, which in turn can readily be solved using numerical solution methods. Test examples are provided to demonstrate the validity and efficiency of this approach.

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INTRODUCTION

Many physical and engineering problems are modeled accurately via fractional order differential equations (FDEs). [1–11] Therefore, applied mathematicians and scientists focus on analyzing FDEs. The paper is focused on the numerical solution of fractional Bagley–Torvik (B-T) equation because it plays an important role in modelling dynamic physical phenomena. One such field is fluid mechanics, in particular, the analysis of the motion of a rigid plate soaked in a Newtonian fluid.

Analytical, semi-analytical and numerical algorithms have been established for the solution of B-T equations. To name a few, a fractional linear multistep method was considered in [12]. The Hybrid functions method was proposed in [13]. Legendre artificial neural network method [14] was studied for solving B-T equations. Bagley-Torvik equations was examined with Fermat Operational Tau Method [15]. Sumudu transformation method [16] was applied to solve B-T equations. Legendre collocation method was used for solving the B-T equations [17]. The operational formulation of collocation methods [18], wavelet methods [19–20] and fractional Taylor methods [21] are implemented for the solution of this fractional equation.

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The basis functions used to obtain sine-cosine wavelets have compact support and they are orthogonal. Therefore the resulting operational matrices for fractional integration using sine-cosine wavelets end up being sparser. Which in turn is important for less computational load and speed considerations which is the reason why we intend to explore the sine-cosine wavelet in this study.

In this paper, the following form of Bagley-Torvik (B-T) equation is analyzed:

$$mD^2 y(t)+cD^{\frac{3}{2}} y(t)+ky(t)=g(t), \tag{1}$$

with initial conditions

$$y(0)=0,\ y'(0)=0, \tag{2}$$

where m, c, k *and* $g(t)$ represent the mass, damping, stiffness coefficients and external force, respectively. *y(t)* is the displacement function and D^{α} is Caputo fractional differential operator of order 3/2. We utilize a new operational matrix method based on the sine-cosine wavelet (SCW) to solve initial value problem of B-T equation. By means of operational matrix of fractional integration, we obtain a system of linear algebraic equations.

The paper is organized as follows. The fundamental concepts and definitions are provided in section II. The definition and the properties of sine-cosine wavelets are covered in section III and also the operational matrices for fractional integration for sine-cosine wavelets are obtained. In section IV the method is applied to the fractional B-T equation. The numerical results are presented in section V.

PRELIMINIARIES

In this section we provide the definition and the basic properties of the Caputo fractional derivative which are used in this paper as follows:

The Caputo definition of fractional derivative operator is expressed in the form:

$$\left(D^{\alpha} f\right)(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{1-n+\alpha}}d\tau, 0\le n-1<\alpha\le n. \tag{3}$$

For the Caputo derivative, we have

$$\left(D^{\alpha} I^{\alpha} f\right)(t)=f(t) \tag{4}$$

and

$$\left(I^{\alpha} D^{\alpha} f\right)(t)=f(t)-\sum_{k=0}^{n-1} f^{(k)}\left(0^+\right)\frac{t^k}{k!} \tag{5}$$

SINE-COSINE WAVELET APPROXIMATION

Sine-Cosine Wavelet

Wavelets involve a mother wavelet and dilated and translated versions of the mother wavelet. If the dilation parameter a and the translation parameter b vary continuously, family of continuous wavelets is written as follows [22]:

$$\psi_{a,b}(t)=|a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), a,b\in R, a\neq 0 \tag{6}$$

When the dilation parameter and the translation parameter are chosen 2 and 1, the family of discrete wavelets are obtained:

$$\psi_{kn}(t)=2^{\frac{k}{2}}\psi\left(2^k t-n\right),\quad k,n\in Z \tag{7}$$

where ψ_{kn} (t) forms an orthonormal basis [23].

Sine-cosine wavelets $\psi_{nm} = \psi(k,n,m,t)$ consist of four arguments: $k = 1,2,3,\ldots$, $n = 1,2,3,\ldots,2^{k-1}$, the values m are given in (9) and t represents normalized time. Sine-cosine wavelets defined as follows:

$$\psi_{nm}(t)=\begin{cases} 2^{\frac{k+1}{2}} f_m\left(2^k t-n\right), & \frac{n}{2^k}\le t<\frac{n+1}{2^k} \\ 0, & otherwise \end{cases} \tag{8}$$

and

$$f_m(t)=\begin{cases} \frac{1}{\sqrt{2}}, & m=0, \\ \cos(2m\pi t), & m=1,2,\ldots,L, \\ sin\left(2(m-L)\pi t\right) & m=L+1,L+2,\ldots,2L, \end{cases} \tag{9}$$

where $\in [0,1)$, $L\in Z^+$. It is obvious that the set of sine-cosine wavelet forms orthonormal set.

Approximating a function

A function f defined over [0,1) is given by the following expanding

$$f(t)=\sum_{m=0}^{2L}\sum_{n=0}^{2^{k-1}} c_{nm}\Psi_{nm}(t)=C^T\psi(t), \tag{10}$$

where C and $\psi(t)$ are $m'\times 1$ ($m' = 2^k(2L + 1)$) vectors expressed as:

$$C=\begin{bmatrix} c_{0,0}, c_{0,1},\ldots,c_{0,2L}, c_{1,0},\ldots, \\ c_{1,2L},\ldots,c_{2^{k-1},0},\ldots,c_{2^{k-1},2L} \end{bmatrix}^T \tag{11}$$

and

$$\Psi=\begin{bmatrix} \Psi_{0,0}, \Psi_{0,1},\ldots,\Psi_{0,2L}, \Psi_{1,0},\ldots, \\ \Psi_{1,2L},\ldots,\Psi_{2^{k-1},0},\ldots,\Psi_{2^{k-1},2L} \end{bmatrix}^T \tag{12}$$

We give sine-cosine wavelet matrix as follows:

$$\phi_{m'xm'} = \left[\Psi(t_1)\ \ \Psi(t_2)\ \ \Psi(t_3)\ \cdots\ \ \Psi(t_{m'})\right], \quad (13)$$

where $t_i = \frac{i-0.5}{m'}$ are collocation points. $(i = 1,2,3,...,m')$

Operational matrix of the fractional integration

We determine the operational matrix of fractional order integration by means of block pulse functions (BPFs). The set of BPFs is defined as

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{m}, \le t < \frac{i}{m}, \\ 0, & otherwise \end{cases} \quad (i = 1,2,3,...,m') \quad (14)$$

For $t \in [0,1)$, $b_i(t)$ has following properties:

$$b_i(t)b_j(t) = \begin{cases} 0, & i \ne j, \\ b_i(t), & i = j, \end{cases} \quad (15)$$

$$\int_0^1 b_i(\tau)b_j(\tau)d\tau = \begin{cases} 0, & i \ne j \\ \frac{1}{m}, & i = j. \end{cases} \quad (16)$$

The sine–cosine wavelet may be expanded in terms of BPFs as:

$$\psi(t) = \phi_{m'xm'} B_{m'}(t), \quad (17)$$

where $B_{m'}(t) = [b_1(t), b_2(t),..., b_m(t)]^T$. The operational matrix of the fractional integral F^α related with the BPFs vector can be represented as:

$$\left(I^\alpha B_{m'}\right)(t) \approx F^\alpha B_{m'}(t), \quad (18)$$

where

$$F^\alpha = \frac{1}{m^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 \cdots & \xi_{m'-1} \\ 0 & 1 & \xi_1 & \xi_2 \cdots & \xi_{m'-2} \\ 0 & 0 & 1 & \xi_1 \cdots & \xi_{m'-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 \cdots & 0 & 1 & \xi_1 \\ 0 & 0 \cdots & 0 & 0 & 1 \end{bmatrix}, \quad (19)$$

with $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$. [24]

By using F^α, sine-cosine wavelet operational matrix of the fractional integration $P^\alpha_{m'xm'}$ can be expressed

$$P^\alpha_{m'xm'} \approx \phi_{m'xm'} F^\alpha \phi^{-1}_{m'xm'}. \quad (20)$$

The error estimate of the SCW basis is provided by [25].

DESCRIPTION OF THE METHOD

We solve the Bagley–Torvik equation (1) and (2) via sine-cosine wavelet method and we approximate $D^2y(t)$ and $g(t)$ as follows:

$$D^2 y(t) \approx C^T \psi(t) \quad (21)$$

and

$$g(t) \approx G^T \psi(t), \quad (22)$$

where $G = [g_0, g_1, \dots g_{m'-1}]$. By equations (4) and (20), we have

$$D^{3/2} y(t) = (I^{0.5} D^2 y)t \approx C^T P^{1/2}_{m'xm'} \psi(t), \quad (23)$$

$$Dy(t) \approx C^T P^1_{m'xm'} \psi(t) \quad (24)$$

and

$$y(t) \approx C^T P^2_{m' \times m'} \psi(t) \quad (25)$$

by substituting Eqs. (21), (22), (23) and (25) in Eq. (1) we obtain a system of algebraic equations as:

$$mC^T \psi(t) + cC^T P^{1/2}_{m'xm'} \psi(t) + kP^2_{m'xm'} \psi(t) = G^T \psi(t). \quad (26)$$

The vector of unknown coefficients C is computed. By using Eq. (25), we can obtain y(t).

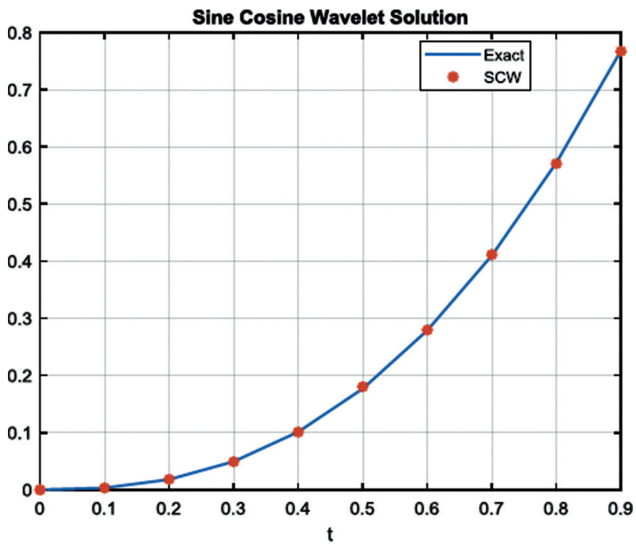

Figure 1. Sine-cosine wavelet and the exact solution for Example 1.

Table 1. The absolute errors of sine-cosine wavelet method for different k values for Example 1

t	(k=6, L=1)	(k=7, L=1)	(k=8, L=1)
0	1.6248E-06	2.0329E-07	2.5428E-08
0.2	1.2640E-04	1.1938E-05	3.0823E-05
0.4	9.5463E-04	2.4670E-05	2.3622E-05
0.6	2.1195E-04	5.5888E-04	5.3325E-05
0.8	1.9748E-04	1.8901E-04	4.9591E-05

Table 2. The absolute errors of sine-cosine wavelet method for different k values for Example 2

t	(k=6, L=1)	(k=7, L=1)	(k=8, L=1)
0	1.2881E-05	2.2776E-06	4.0270E-07
0.2	2.3397E-04	2.2190E-05	5.7513E-06
0.4	1.2548E-04	3.2547E-04	3.1143E-05
0.6	2.2836E-04	6.0073E-04	5.7348E-04
0.8	1.8421E-04	1.7620E-04	4.6190E-04

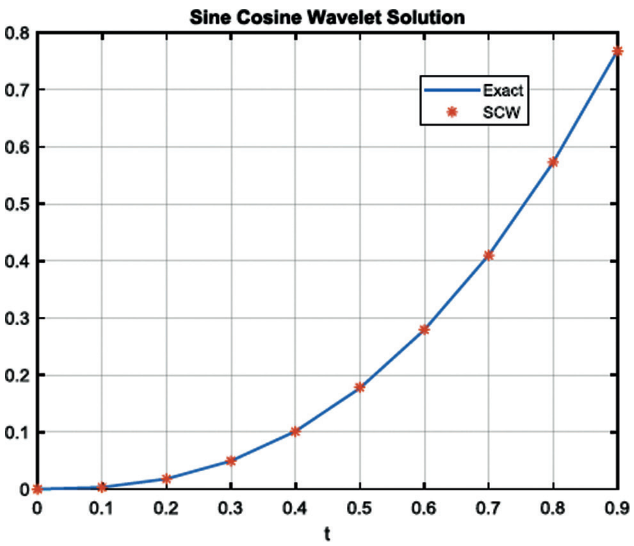

Figure 2. Sine-cosine wavelet and the exact solution for Example 2.

ILLUSTRATIVE EXAMPLES

In this section, we apply the method to two examples and we have carried out all of the numerical calculations using Matlab R2020a.

Example 1: We first consider following B-T equation

$$D^{3/2}y(t)+D^2y(t)+y(t)=t^3+6t+\frac{8}{\Gamma(0.5)t^{0.5}},$$
$$y(0)=y'(0)=0, \qquad t\in[0,1),$$

with exact solution is $y(t) = t^3$.

By applying the method described in section IV, the values of the unknown matrix C^T are obtained. Figure 1 represents the sine -cosine wavelet solution for k = 6, L = 1 and the exact solution. From Figure 1, it is clear that sine-cosine wavelet solution provides good approximation with exact solution.

The absolute errors for different k values of sine-cosine wavelet solution (L = 1) are given in Table 1. As results suggest, when the values of k increase, the absolute error decreases and our solution converges to the exact solution which proves the validity of the method.

Example 2: In this example, we consider the following B-T equation:

$$D^{3/2}y(t)+D^2y(t)+y(t)=\frac{15}{4}\sqrt{t}+\frac{15}{8}\sqrt{\pi t}+t^2\sqrt{t},$$
$$y(0)=y'(0)=0, \qquad t\in[0,1),$$

with exact solution is $y(t) = t^{2.5}$.

By using the proposed technique presented in section IV, we obtain the approximate solution of B-T equation. Table 2 represents absolute errors for k = 6, 7, 8 and L=1. In addition, the comparison of the numerical solutions and the exact solutions for k = 6 and L=1 is given in Figure 2. As can be seen, numerical results show the efficiency of our solution. Numerical results also demonstrate that the method is fast and can be applied to real-time problems.

CONCLUSIONS

In this study, we propose sine-cosine wavelet method with block pulse functions to solve fractional Bagley–Torvik differential equations. This method converts this fractional differential equation to system of linear algebraic equations. We obtain the numerical solutions of the Bagley-Torvik equations by solving this system. Since the basis functions of sine–cosine wavelets are orthogonal, the implementation of present approach is simple and easy. Numerical examples are included to illustrate the validity and the accuracy of the proposed numerical method.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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