Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2021.00032

Research Article

Genocchi polynomial method for the multiterm variable-order fractional differential equations

Sadiye Nergis TURAL POLAT[1,*], Arzu TURAN DINCEL[2]

[1]*Department of Electronics and Communications Engineering, Yıldız Technical University, Istanbul, Turkey*
[2]*Department of , Mathematical Engineering, Yıldız Technical University, Istanbul, Turkey*

ARTICLE INFO

Article history
Received: 25 November 2020
Accepted: 24 March 2021

Key words:
Genocchi polynomials; Collocation method; Variable-order fractional differential equations; Numerical FDE solutions

ABSTRACT

In this paper a numerical solution for multiterm varable-order fractional differential equations (VO-FDEs) using Genocchi polynomials is proffered. By making use of the Genocchi polynomials, a multiterm VO-FDE can be approximated by a corresponding system of algebraic equations. To be able to do that, operational matrices for variable order fractional differentials are obtained using Genocchi polynomials. Then the algebraic equation system is solved for the coefficient values, thus the approximate solution is obtained by using the linear combination of those coefficients. Numerical examples are provided.

Cite this article as: Tural Polat S N, Turan Dincel A. Genocchi polynomial method for the multiterm variable-order fractional differentıal equations. Sigma J Eng Nat Sci 2022;40(1):79–85.

INTRODUCTION

Fractional differential equations (FDEs) are defined as containing differentiator operators of non-integer orders. The FDEs get an increasing amount of attention in recent years owing to their ability to model the various real-life phenomena more precisely. The research areas are very diverse including but not limited to signal processing [1], control engineering [2], optimal control [3], bioengineering [4], solid mechanics [5] etc.

Those fractional orders of derivatives themselves are functions of time in the variable-order fractional differential equations (VO-FDEs). By exploting fractional orders this way, it is possible to accurately model several phenomena with transient regimes such as wave propagation, diffusion, viscoelasticity and damping [6-7].

On the other hand, with the increasing difficulty added by those time-dependent fractional orders, most VO-FDEs do not have analytical solutions. Therefore numerical approximation methods for the solution of VO-FDEs is an active research area. As for the numerical methods proposed up to date, there are various methods such as Haar wavelet collocation method [8], polynomial methods [9-10], collocation method [11], the Galerkin finite element method [12], the modified Adomian decomposition method [13], wavelet methods [14-16].

*Corresponding author.
*E-mail address: nergis@yildiz.edu.tr

This paper was recommended for publication in revised form by Regional Editor Ravindar Kumar

In this paper, we consider the multi-term variable order FDEs of the following form:

$$D^{\alpha(t)}y(t)=F\begin{pmatrix} t, y(t), D^{\beta_1(t)}y(t), \\ D^{\beta_2(t)}y(t),\ldots,D^{\beta_k(t)}y(t)\end{pmatrix}, \quad (1)$$

where $\alpha(t)$ and $\beta_i(t)$ $(i = 1,2,\ldots,k)$ are the variable-order fractional derivative parameters, $D^{\alpha(t)}y(t)$ and $D^{\beta_i(t)}y(t)$ $(i = 1,2,\ldots,k)$ are the fractional derivatives of the variable orders defined in the Caputo sense.

In this study, we approximate the solution of (1) using various orders of Genocchi polynomials. The paper is organized as follows: In Section II, the fundamental properties of the Genocchi polynomials are presented and the Genocchi polynomial method is derived. In Section III, we apply the method to obtain the numerical solutions of multiterm VO-FDEs for various orders of Genocchi polynomials and the conclusion is presented in Section IV.

GENOCCHI POLYNOMIAL METHOD

Fundamental Properties of Fractional Derivatives and Genocchi Polynomials

The Caputo definition of the fractional-order derivative [17] is

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(\tau)}{(t-\tau)^{1-n+\alpha}}d\tau \qquad (2)$$
$$0\le n-1<\alpha\le n, n\in\mathbb{N}$$

where α is the fractional order and n is the smallest integer greater than α. For the Caputo derivative, we have

$$D^{\alpha}t^{q}=\begin{cases}0 & , for\, q\in\mathbb{N}_0\, and\, q<\lceil\alpha\rceil \\ \dfrac{\Gamma(q+1)}{\Gamma(q+1-\alpha)}t^{q-\alpha} & , for\, q\in\mathbb{N}_0\, and\, q\ge\lceil\alpha\rceil\end{cases} \qquad (3)$$

where $\lceil\cdot\rceil$ denotes the ceiling function [17].

Genocchi numbers G_n and Genocchi polynomials $G_n(t)$ of order n are generally defined using exponential generating functions as given below [18]:

$$\frac{2x}{e^x+1}=\sum_{n=0}^{\infty}G_n\frac{x^n}{n!},\ |x|<\pi \qquad (4)$$

$$\frac{2xe^{xt}}{e^x+1}=\sum_{n=0}^{\infty}G_n(t)\frac{x^n}{n!} \qquad (5)$$

Genocchi polynomials can be defined using Genocchi numbers as:

$$G_n(t)=\sum_{i=0}^{n}\binom{n}{i}G_{n-i}t^i \qquad (6)$$

First few Genocchi polynomials are:

$$\begin{aligned}&G_0(t)=0,\ G_1(t)=1,\ G_2(t)=2t-1,\ G_3(t)=3t^2-3t,\\ &G_4(t)=4t^3-6t^2+1,\ G_5(t)=5t^4-10t^3+5t\end{aligned} \qquad (7)$$

Genocchi polynomials have the following properties:

$$\begin{aligned}&G_n(0)+G_n(1)=0,\quad n>1\\ &\frac{dG_n}{dt}(t)=nG_{n-1}(t),\quad n>1\\ &\int_0^1 G_n(t)G_m(t)dt=\frac{2(-1)^n n!m!}{(m+n)!}G_{m+n},\quad i\end{aligned} \qquad (8)$$

Function Approximation using Genocchi Polynomials

We can easily construct a Genocchi vector $G(t)$ of size $1\times n$ containing Genocchi polynomials as

$$G(t)=\left[G_1(t)\ \ G_2(t)\ \ \cdots\ \ G_n(t)\right]^T \qquad (9)$$

The Genocchi vector can further be divided into a coefficient matrix A of size $n\times n$ and a polynomial vector $T_n(t)$ of the powers of t of size $1\times n$ such as

$$G(t)=A.T_n(t) \qquad (10)$$

where $T_n(t) = [1\ \ t\ \ t^2\ \ \ldots\ \ t^{n-1}]^T$. It is clear that

$$T_n(t)=A^{-1}G(t) \qquad (11)$$

The first-order derivative of the Genocchi vector can be constructed using (8) as

$$\underbrace{\begin{bmatrix}G_1'(t)\\ G_2'(t)\\ G_3'(t)\\ \vdots\\ G_{n-1}'(t)\\ G_n'(t)\end{bmatrix}}_{G'(t)}=\underbrace{\begin{bmatrix}0&0&0&\cdots&0&0&0\\ 2&0&0&\cdots&0&0&0\\ 3&3&0&\cdots&0&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\ 0&0&0&\cdots&n-1&0&0\\ 0&0&0&\cdots&0&n&0\end{bmatrix}}_{D^{(1)}}\underbrace{\begin{bmatrix}G_1(t)\\ G_2(t)\\ G_3(t)\\ \vdots\\ G_{n-1}(t)\\ G_n(t)\end{bmatrix}}_{G(t)} \qquad (12)$$

where $D^{(1)}$ of size $n\times n$ is called the operational matrix of the first-order derivative. We also have

$$\begin{aligned}D^{(1)}G(t)&=\frac{d}{dt}\left[AT_n(t)\right]\\ &=A\frac{d}{dt}\left\{\left[1\ \ t\ \ t^2\ \ \cdots\ \ t^{n-1}\right]^T\right\}\\ &=AD^{(1)}A^{-1}G(t)\end{aligned} \qquad (13)$$

For the mth integer-order derivative, we obtain

$$\frac{d^m}{dt^m}G(t)=\left(D^{(1)}\right)^m G(t) \tag{14}$$

Now, let us define the function approximation. Any function $y(t)$ can be approximated using the finite length Genocchi matrix such that

$$y(t)\cong\sum_{k=1}^{n}c_k G_k(t)=c^T G(t) \tag{15}$$

By combining (12) and (14), the first-order derivative of the $y(t)$ can be approximated by

$$\begin{aligned}\frac{d}{dt}y(t)&=\frac{d}{dt}\left[c^T G(t)\right]=c^T\frac{d}{dt}G(t)\\&=c^T A D^{(1)}A^{-1}G(t)\end{aligned} \tag{16}$$

where $AD^{(1)}A^{-1}$ is the operational matrix for the first-order derivative. For the time-dependent fractional order derivatives $D^{\alpha(1)}$, employing Caputo definition for fractional order derivatives, we can obtain

$$\begin{aligned}D^{\alpha(t)}G(t)&=D^{\alpha(t)}\left[AT_n(t)\right]\\&=AD^{\alpha(t)}\left\{\left[1\ \ t\ \ t^2\ \ \cdots\ \ t^{n-1}\right]^T\right\}\\&=AH^{\alpha(t)}(t)A^{-1}G(t)\end{aligned} \tag{17}$$

where

$$H^{\alpha(t)}(t)=\begin{bmatrix}0 & 0 & 0 & \cdots & 0\\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{-\alpha(t)} & 0 & \cdots & 0\\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha(t))}t^{-\alpha(t)} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{\Gamma(n)}{\Gamma(n-\alpha(t))}t^{-\alpha(t)}\end{bmatrix} \tag{18}$$

where $AH^{\alpha(1)}(t)A^{-1}$ is called the operational matrix for the variable-order fractional derivative $D^{\alpha(t)}$. Combining (15) and (17) yields following equation for the approximation for the fractional derivatives

$$D^{\alpha(t)}y(t)=D^{\alpha(t)}\left[c^T G(t)\right]=c^T A.H^{\alpha(t)}(t).A^{-1}G(t) \tag{19}$$

For the solution of VO-FDEs, firstly, the operational matrices for all the variable and integer-order fractional derivatives must be obtained. Then replacing the differentials with their corresponding operational matrices, an algebraic equation is obtained in vector-matrix form. Calculating that equation for a few collocation points t_i provides a system of algebraic equations where the only unknown is the coefficient vector c. Solving for c provides the approximate solution given in (15).

Error bound

In this section we provide the error bound for the approximated function $y(t)$. Assume that the function $y(t)\in C^{n+1}[0,1]$ and $c^T G(t)$ is the best approximation of $y(t)$. Then we have [19],

$$\left\|y(t)-c^T G(t)\right\|\le\frac{(\ell)^{(2n+3)/2}S}{(n+1)!\sqrt{2n+3}},\ t\in \tag{20}$$

where $S=\max t\in\left[t_i,t_{i+1}\right]\left|y^{(n+1)}(t)\right|$ and $\ell=t_{i+1}-t_i$.

NUMERICAL EXAMPLES

In this section the Genocchi vector and its operational matrices for integer and fractional derivatives are used to solve several FDEs.

Example 1

As the first example, consider the linear FDE:

$$D^{1.5}y(t)-t^{1.5}y(t)=4\sqrt{\frac{t}{\pi}}-t^{3.5},0<t\le 1 \tag{21}$$

with the initial conditions $y(0) = y'(0) = 0$. The exact solution to this FDE is known and given as $y(t) = t^2$. Applying the Genocchi polynomial method to (21), we obtain

$$c^T A.H^{1.5}(t).A^{-1}G(t)+t^{1.5}c^T G(t)=f(t) \tag{22}$$

where $f(t)=4\sqrt{\frac{t}{\pi}}-t^{3.5}$ and for the initial conditions, the following equations must be met:

$$\begin{aligned}y(0)&=c^T G(0)=0\\y'(0)&=c^T D^{(1)}.G(0)=0\end{aligned} \tag{23}$$

By calculating (21) for several collocation points of $t=t_i=\frac{2i+1}{2(n+1)},i=0,1,\ldots,n$ in the interval [0,1] and incorporating the initial conditions (23) we obtain a system of algebraic equations, the solution of which gives us the c coefficients, and c in turn provides us with the approximate solution for $y(t)$. The absolute errors for several n values are given in Table 1 and approximate and exact solution values are plotted in Fig.1. A comparison for the error values of the Bernstein polynomial method in [20] is also presented. As can be seen from the Table 1 and Figure 1, the absolute errors are on the order of E-13 in the worst case and therefore the method is a very accurate approximation even for the smaller values of n.

Table 1. Absolute errors of Genocchi Polynomial method for several *n* values

t	**Genocchi Polynomial Method for several *n* values**				**Bernstein Polynomial Method [20]**
	n=2	***n=3***	***n=4***	***n=5***	
0.1	2.07473E-15	1.98522E-14	3.32998E-14	3.19529E-13	3.62030E-11
0.3	1.79023E-15	2.04281E-14	3.55271E-14	3.18467E-13	3.40588E-09
0.5	1.60982E-15	2.12053E-14	3.77476E-14	3.24046E-13	8.04365E-09
0.7	1.60982E-15	2.23155E-14	4.02456E-14	3.38229E-13	1.32341E-08
0.9	1.66533E-15	2.40918E-14	4.29656E-14	3.65041E-13	1.61286E-08

Table 2. *c* coefficients and maximum absolute errors of Genocchi polynomial method for several n values for example 2

n	c^T	$E_{\max}$
2	[0.5, 0.5, 0.333333333333]	3.573530E-16
3	[0.5, 0.5, 0.333333333333, 0]	2.220446E-16
4	[0.5, 0.5, 0.333333333333325, 6.6613E-16, -3.7747E-15]	3.99680E-16
5	[0.5, 0.5, 0.333333333333336, 5.5511E-16, 6.6613E-16, 6.9389E-17]	4.44089E-15

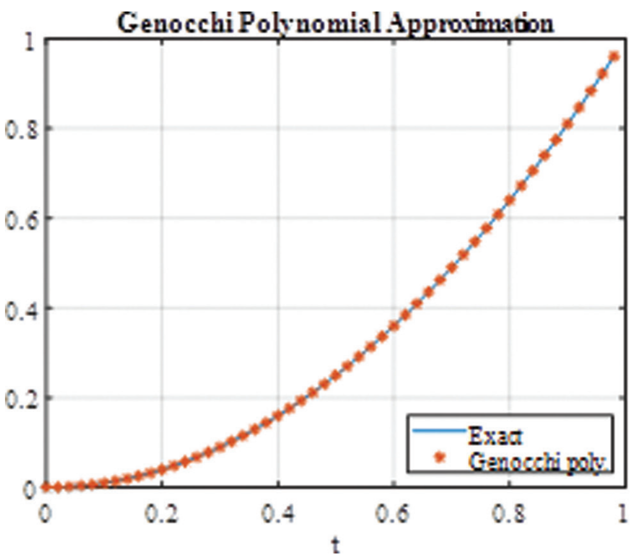

Figure 1. Genocchi Polynomial Approximation solution for *n=2.*

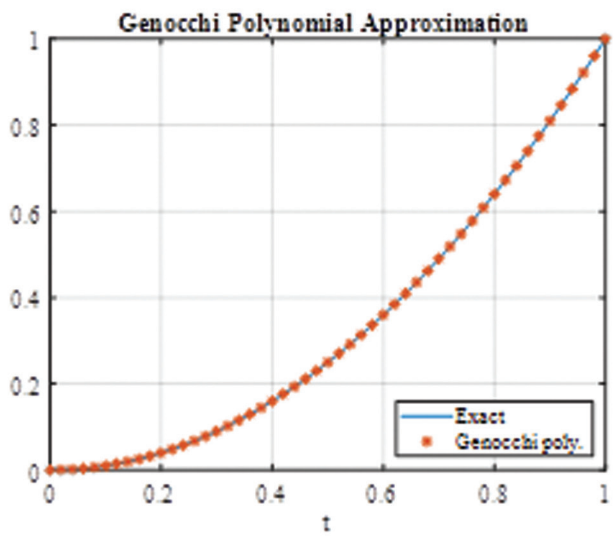

Figure 2. Exact and approximate solutions for *n=2* for example 2.

Example 2

Consider the multi-term VO-FDE:

$$D^{\alpha(t)}y(t)+2D^{\beta(t)}y(t)+4y(t)=g(t) \tag{24}$$

with $y(0)=0, y'(0)=0, \alpha(t)=2t, \beta(t)=(1+t)/2$ and $g(t)=\frac{\Gamma(3)}{\Gamma(3-\alpha(t))}t^{2-\alpha(t)}+\frac{2\Gamma(3)}{\Gamma(3-\beta(t))}t^{2-\beta(t)}+4t^2.$

The exact solution of example 2 is $y(t)$ = *t*2. Applying Genocchi polynomials method to the example yields the following algebraic equation

$$\begin{aligned}&c^T AH^{\alpha(t)}(t)A^{-1}G(t)\\&+2c^T AH^{\beta(t)}A^{-1}G(t)+4c^T G(t)=g(t).\end{aligned} \tag{25}$$

Table 3. *c* coefficients and maximum absolute errors of Genocchi polynomial method for several *n* values for example 2

n	*cT*	*E***max**
2	[1.75, -0.25, 166666666666667]	4.440892E-16
3	[1.75, -0.25, -0.166666666666667, -3.46944695E-17]	1.110223E-15
4	[1.75, -0.25, -0.166666666666659, -5.32907E-15, 5.273559E-15]	3.019806E-14
5	[1.75, -0.25, -0.16666666666667, 1.4016565E-15, 5.68989E-16, 6.730727E-16]	5.329070E-15

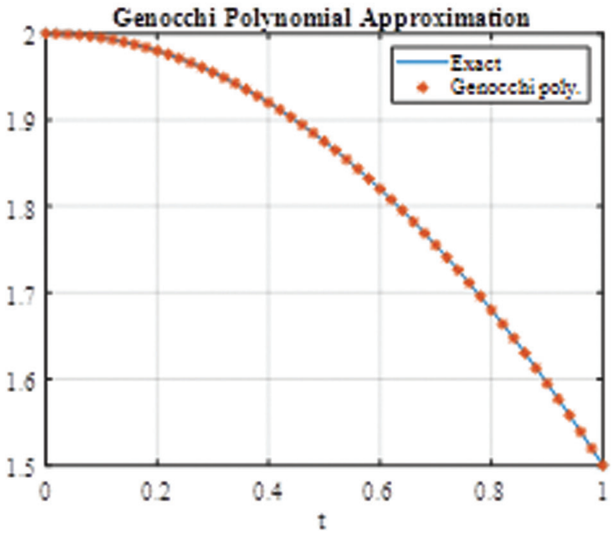

Figure 3. Exact and approximate solutions for *n=2* for example 3.

Following the same steps as the previous example, we solve for the *c* coefficients for several *n*. Table 2 presents *c* and maximum absolute error (E_{max}) values for various degrees of *n* and Figure 2 presents the exact and approximate solutions. Again, the numerical results follow the exact results closely.

Example 3

Consider the multi-term VO-FDE:

$$\begin{aligned} &pD^{\alpha(t)}y(t)+q_1(t)D^{\beta_1(t)}y(t)+q_2(t)D^{\beta_2(t)}y(t)\\ &+q_3(t)D^{\beta_3(t)}y(t)+r(t)y(t)=g(t) \end{aligned} \tag{26}$$

with the initial conditions $y(0)=0, y'(0)=0$ and

$$p=1, q_1(t)=t^{1/2}, q_2(t)=t^{1/3}, q_3(t)=t^{1/4}, r(t)=t^{1/5}$$

$$\alpha(t)=2t,\ \beta_1(t)=\frac{t}{3},\ \beta_2(t)=\frac{t}{4},\ \beta_3(t)=\frac{t}{5} \tag{27}$$

$$\begin{aligned} g(t)=&-p\frac{t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))}-q_1(t)\frac{t^{2-\beta_1(t)}}{\Gamma(3-\beta_1(t))}\\ &-q_2(t)\frac{t^{2-\beta_2(t)}}{\Gamma(3-\beta_2(t))}-q_3(t)\frac{t^{2-\beta_3(t)}}{\Gamma(3-\beta_3(t))}\\ &-r(t)\left(2-\frac{t^2}{2}\right) \end{aligned}$$

The exact solution of the problem is given as $y(t)=2-\frac{t^2}{2}$. We obtain following algebraic equation by applying Genocchi polynomials method

$$\begin{aligned} &c^T AH^{\alpha(t)}(t)A^{-1}G(t)+q_1(t)c^T AH^{\beta_1(t)}A^{-1}G(t)\\ &+q_2(t)c^T AH^{\beta_2(t)}A^{-1}G(t)+q_3(t)c^T AH^{\beta_3(t)}A^{-1}G(t)\\ &+r(t)c^T G(t)=g(t) \end{aligned} \tag{28}$$

As stated above, we can solve for the *c* coefficients for several degrees *n* of Genocchi polynomials. Table 3 gives *c* and maximum absolute error (E_{max}) values for various degrees of *n* and Figure 3 presents the exact and approximate solutions. Again, the numerical results follow the exact results closely and the maximum error is on the range of E-15.

CONCLUSIONS

In this paper the numerical solutions for the multi-term VO FDEs are obtained using the Genocchi Polynomial method. The method converts the FDE to a system of algebraic equations by approximating integer and fractional-order derivatives using the Genocchi Polynomials and the definition of fractional derivatives in the Caputo sense. The system of algebraic equations is solved together with another set of equations obtained from the initial conditions to obtain the approximation coefficients. Once those coefficients are calculated, the approximate solution has also been obtained using (13). The absolute errors for various *n* values are presented. The results demonstrate that even for the smaller values of *n*, the method is very accurate. One other advantage of the method is that the absolute error does not increase with increasing *t*.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

[1] Podlubny I. Fractional Differential Equations, New York: Academic Press, 1999.

[2] Duarte FBM, Machado JAT. Chaotic phenomena and fractional order dynamics in the trajectory control of redundant manipulators, Nonlinear Dyn 2002;29:342–362.

[3] Bahaa M. Fractional optimal control problem for differential system with delay argument, Adv Differ Equat 2017;69:1–19. [CrossRef]

[4] Magin RL. Fractional calculus models of complex dynamics in biological tissues. Comput Math Appl 2010;59:1586–1593. [CrossRef]

[5] Rossikhin YA, Shitikova MV. Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids. Appl Mech Rev 1997;50:15–67. [CrossRef]

[6] Coimbra CFM. Mechanics with variable-order differential operators, Ann. Phys. 2003;12692–703.

[7] Soon CM, Coimbra CFM, Kobayashi MH. The variable viscoelasticity oscillator. Ann Phys 2005;14:378–389.

[8] Shiralashetti SC, Deshi AB. An efficient Haar wavelet collocation method for the numerical solution of multi-term fractional differential equations, Nonlinear Dyn 2016;83:293–303. [CrossRef]

[9] Liu J, Li X, Wu L. An operational matrix of fractional differentiation of the second kind of Chebyshev polynomial for solving multi-term variable order fractional differential equation. Math Probl Eng 2016:7126080. [CrossRef]

[10] Ganji RM, Jafari H. A numerical approach for multi-variable orders differential equations using Jacobi polynomials. Int J Appl Comput Math 2019;5:34. [CrossRef]

[11] Ghoreishi F, Mokhtary P. Spectral collocation method for multi-order fractional differential equations. Int J Comput Methods 2014;11:1350072. [CrossRef]

[12] Jin B, Lazarov R, Liu Y, Zhou Z. The Galerkin finite element method for a multi-term time-fractional diffusion equation. J Comput Phys 2015;281:825–843. [CrossRef]

[13] Srivastava V, Rai KN. A multi-term fractional diffusion equation for oxygen delivery through a capillary to tissues. Math Comput Model 2010;51:616–624. [CrossRef]

[14] Heydari MH, Avazzadeh Z, Haromi MF. A wavelet approach for solving multi-term variable-order time fractional diffusion-wave equation. Appl Math Comput 2019;341:215–228. [CrossRef]

[15] Heydari MH, Hooshmandasl MR, Ghaini FMM, Mohammadi F. Wavelet collocation method for solving multi order fractional differential equations. J Appl Math 2012;19:542401. [CrossRef]

[16] Heydari MH, Avazzadeh Z. A new wavelet method for variable-order fractional optimal control problems. Asian J Control 2018;20:1–14. [CrossRef]

[17] Liu J, Li X, Wu L. An operational matrix of fractional differentiation of the second kind of Chebyshev polynomial for solving multi-term variable order fractional differential equation. Math Probl Eng 2016;7126080. [CrossRef]

[18] Araci S. Novel identities involving Genocchi numbers and polynomials arising from applications of umbral calculus. Appl Math Comput 2014;233:599–607. [CrossRef]

[19] Isah A, Phang C. New operational matrix of derivative for solving non-linear fractional differential equations via Genocchi polynomials, J. King Saud Univ. Sci., 2019;31:1–7. [CrossRef]

[20] Alipour M, Rostamy D. Solving nonlinear fractional differential equations by bernstein polynomials operational matrices. Int J Math Comput Sci 2012;5:185–196. [CrossRef]