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Technical Note

Blow up and global existence of solution for a riser problem with logarithmic nonlinearity

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ABSTRACT

In this work, we analyze the influence of the logarithmic source term on solutions to quasilinear riser equation. Firstly, we prove blow up results. Later, we obtain that solutions are global with negative initial energy.

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INTRODUCTION

In this work, we consider the following riser equation with logarithmic nonlinearity

$$\begin{cases} u_{tt}+\alpha u_t+2\beta u_{xxxx}-2\left[(ax+b)u_x\right]_x+\dfrac{\beta}{3}\left(u_x^3\right)_{xxx}-\left[(ax+b)u_x^3\right]_x \\ -\left(\beta u_{xx}^2u_x\right)_x=|u|^{p-2}u\ln|u|, & (x,t)\in[0,1]\times(0,T) \\ u(0,t)=u(1,t),\ \ u_{xx}(0,t)=u_{xx}(1,t)=0, & t\in(0,T) \\ u(x,0)=u_0(x)=0,\ \ u_t(x,0)=u_1(x), & x\in[0,1], \end{cases} \quad (1)$$

where a, b, α, β are nonnegative constants and $p\geq 4$.

In the absence of the logarithmic nonlinearity that is, if source term is taken as $f(u)$, the problem (1) becomes

$$\begin{aligned} &u_{tt}+\alpha u_t+2\beta u_{xxxx}-2\left[(ax+b)u_x\right]_x+\frac{\beta}{3}\left(u_x^3\right)_{xxx} \\ &-\left[(ax+b)u_x^3\right]_x-\left(\beta u_{xx}^2u_x\right)_x=f(u), \end{aligned} \quad (2)$$

where $f(u)$ is a $C(R)$ function satisfying some conditions.

The problem (2) leads to the dynamics of a riser vibrating due to effects of current and waves [17, 33]. Results concerning the global existence and blow-up of solutions for the problem (2) are discussed by many mathematicians, see [1, 5, 7-9, 12, 14, 21, 35] and the references therein.

Wave equations with logarithmic nonlinearity have many applications in many branches of physics, such as

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optics, geophysics and nuclear physics [3, 13]. In recent years, hyperbolic wave equations with logarithmic source term have attracted much attention. We refer the concerned reader to for the study of different kinds of this type problem (see [2, 4, 6, 10, 15, 18-20, 22-25, 29, 31, 32, 36]).

To the best of our knowledge, quasilinear riser problem involving logarithmic nonlinearity has not been studied yet. Our aim in the present paper is to extend the existing blow-up results to our logarithmic nonlinearity quasilinear riser problem.

This paper consists of three sections in addition to the introduction: In Section 2, we prepare some lemmas and notations. Section 3 is concerned with blow up result with negative initial energy. Our approach is similar to the in [14, 18]. In section 4, we establish global existence of the solution.

PRELIMINARIES

In this part we give some some material which will be used in the proof of our results. We denote $(.,.)_2$ the inner product in $L^2 = L^2\,[0,1]$ and $||u||_p$ is the norm in $L^p = L^p\,[0,1]$.

We define the following total energy functional

$$E(t)=\frac{1}{2}\|u_t\|^2+\beta\|u_{xx}\|^2+\int_0^1(ax+b)u_x^2dx+\frac{\beta}{2}\|u_xu_{xx}\|^2 +\frac{1}{4}\int_0^1(ax+b)u_x^4dx-\frac{1}{p}\int_0^1 ln|u|u^pdx+\frac{1}{p^2}\|u\|_p^p, \tag{3}$$

and the initial total energy functional

$$E(0)=\frac{1}{2}\|u_1\|^2+\beta\|u_{0xx}\|^2+\int_0^1(ax+b)u_{0x}^2dx+\frac{\beta}{2}\|u_{0x}u_{0xx}\|^2 +\frac{1}{4}\int_0^1(ax+b)u_{0x}^4dx-\frac{1}{p}\int_0^1 u_0^p ln|u_0|dx+\frac{1}{p^2}\|u_0\|_p^p, \tag{4}$$

Lemma 1. *Let u be the solution of the problem (1). Then the $E(t)$ is decreasing with respect to t and*

$$E'(t) = -\alpha||u_t||^2 \le 0. \tag{5}$$

Proof. We multiply both sides of the equation (1) by u_t and then integrating from 0 to 1, we obtain

$$\begin{aligned} &\int_0^1 u_{tt}u_tdx+\alpha\int_0^1 u_tu_tdx+2\beta\int_0^1 u_{xxxx}u_tdx \\ &-2\int_0^1\left[(ax+b)u_x\right]_x u_tdx+\frac{\beta}{3}\int_0^1\left(u_x^3\right)_{xxx}u_tdx \\ &-\int_0^1\left[(ax+b)u_x^3\right]_x u_tdx-\beta\int_0^1\left(u_{xx}^2u_x\right)_x u_tdx \\ &=\int_0^1|u|^{p-2}\,u ln|u|u_tdx \end{aligned} \tag{6}$$

By direct calculation, we obtain

$$\int_0^1 u_{tt}u_tdx=\frac{1}{2}\left(\int_0^1\frac{d}{dt}\left(|u_t|^2\right)dx\right)=\frac{1}{2}\|u_t\|^2, \tag{7}$$

and

$$\alpha\int_0^1 u_tu_tdx=\alpha\int_0^1|u_t|^2\,dx=\alpha\|u_t\|^2. \tag{8}$$

By using integration by parts and boundary value of the problem (1), we get:

$$\begin{aligned} &2\beta\int_0^1 u_{xxxx}u_tdx=2\beta\int_0^1\left(u_{xxx}u_t\right)_x dx-2\beta\int_0^1 u_{xxx}u_{tx}dx \\ &=2\beta[u_{xxx}(1)-u_{xxx}(0)u_t(0)]-2\beta\int_0^1\left(u_{xx}u_{tx}\right)_x dx+2\beta\int_0^1 u_{xx}u_{txx}dx \\ &=-2\beta[u_{xx}(1)u_{tx}(1)-u_{xx}(0)u_{tx}(0)]+2\beta\int_0^1 u_{xx}u_{txx}dx \\ &=2\beta\int_0^1 u_{xx}u_{txxx}dx=\beta\int_0^1\frac{d}{dt}\left(u_{xx}^2\right)dx=\beta\frac{d}{dt}\|u_{xx}\|^2, \end{aligned} \tag{9}$$

and

$$\begin{aligned} &-2\int_0^1\left[(ax+b)u_x\right]_x u_tdx=-2\int_0^1\left[(ax+b)u_xu_t\right]_x dx \\ &+2\int_0^1(ax+b)u_xu_{tx}dx=\frac{d}{dt}\int_0^1(ax+b)u_x^2dx, \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\frac{\beta}{3}\int_0^1\left(u_x^3\right)_{xxx}u_tdx=\frac{\beta}{3}\int_0^1\left(\left(u_x^3\right)_{xx}u_t\right)_x dx-\frac{\beta}{3}\int_0^1\left(u_x^3\right)_{xx}u_{tx}dx \\ &=-\frac{\beta}{3}\int_0^1\left(\left(u_x^3\right)_x u_{tx}\right)_x dx+\frac{\beta}{3}\int_0^1\left(u_x^3\right)_x u_{txx}dx \\ &=\frac{\beta}{3}\int_0^1\left(u_x^3\right)_x u_{txx}dx=\frac{\beta}{3}\int_0^1 3u_x^2u_{xx}u_{txx}dx \\ &=\frac{\beta}{2}\int_0^1 u_x^2\frac{d}{dt}\left(u_{xx}^2\right)dx-\frac{\beta}{2}\int_0^1 u_{xx}^2\frac{d}{dt}\left(u_x^2\right)dx \\ &=\frac{\beta}{2}\frac{d}{dt}\|u_xu_{xx}\|^2-\beta\int_0^1 u_{xx}^2u_xu_{tx}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \beta\int_0^1\left(u_{xx}^2u_x\right)_x u_tdx=&-\beta\int_0^1\left(u_{xx}^2u_xu_t\right)_x dx \\ &+\beta\int_0^1 u_{xx}^2u_xu_{tx}dx, \end{aligned} \tag{12}$$

and

$$\begin{aligned} &-\int_0^1\left[(ax+b)u_x^3\right]_x u_tdx=-\int_0^1\left[(ax+b)u_x^3u_t\right]_x dx \\ &+-\int_0^1(ax+b)u_x^3u_{tx}dx=\frac{1}{4}\int_0^1(ax+b)\frac{d}{dt}u_x^4dx \\ &=\frac{1}{4}\frac{d}{dt}\int_0^1(ax+b)u_x^4dx. \end{aligned} \tag{13}$$

By using

$$|u|^{p-2} uu_t = \frac{1}{p}\frac{d}{dt}\left(|u_t|^p\right)$$

we find

$$\begin{aligned}
&\int_0^1 |u|^{p-2} uu_t lnudx = \int_0^1 \frac{1}{p}\frac{d}{dt}\left(|u_t|^p\right)lnudx \\
&= \int_0^1 \frac{1}{p}\frac{d}{dt}\left(|u|^p\right)lnudx = \frac{1}{p}\int_0^1 \frac{d}{dt}\left(|u|^p\, lnu\right)dx \\
&-\frac{1}{p}\int_0^1 |u|^p \frac{d}{dt}(lnu)dx = \frac{1}{p}\int_0^1 \frac{d}{dt}\left(|u|^p\, lnu\right)dx \\
&-\frac{1}{p}\int_0^1 |u|^{p-2} uu_t dx = \frac{1}{p}\int_0^1 \frac{d}{dt}\left(|u|^p\, lnu\right)dx \\
&-\frac{1}{p^2}\int_0^1 \frac{d}{dt}|u|^p dx = \frac{d}{dt}\left[\frac{1}{p}\int_0^1\left(|u|^p\, lnu\right)dx - \frac{1}{p^2}\|u\|_p^p\right].
\end{aligned} \tag{14}$$

By replacing (7)-(14) in (6), we have

$$\begin{aligned}
&\frac{d}{dt}\left[\begin{array}{l}\frac{1}{2}\|u_t\|^2 + \beta\|u_{xx}\|^2 + \int_0^1 (ax+b)u_x^2 dx + \frac{\beta}{2}\|u_x u_{xx}\|^2 \\ +\frac{1}{4}\int_0^1 (ax+b)u_x^4 dx - \frac{1}{p}\int_0^1 ln|u| u^p dx + \frac{1}{p^2}\|u\|_p^p\end{array}\right]. \\
&= -\alpha\|u_t\|^2
\end{aligned} \tag{15}$$

By inserting (3) in (15), we have

$$E'(t) = -\alpha||u_t||^2 \le 0 \tag{16}$$

By integration (16) from 0 *to t*, we obtain

$$E(t) = -\alpha\int_0^t\int_0^1 |u_t|^2 dx + E(0). \tag{17}$$

We give some lemmas which be used in our proof. For proof of Lemmas 2-5 and Corollary 4, we refer the readers to Kafini and Messaoudi [18].

Lemma 2. *There exists a positive constant C > 0 depending on [0,1] only such that*

$$\left(\int_0^1\left(|u|^p\, lnu\right)dx\right)^{\frac{s}{p}} \le C\left[\int_0^1\left(|u|^p\, lnu\right)dx + \|u_x\|_4^4\right], \tag{18}$$

for any $u \in L^p\,[0,1]$ and $4 \le s \le p$, provided that $\int_0^1(|u|^p\, lnu)\,dx \ge 0$.

Lemma 3. *There exists a positive constant C > 0 depending on [0,1] only such that*

$$\|u\|_p^p \le C\left[\int_0^1\left(|u|^p\, lnu\right)dx + \|u\|_4^4\right], \tag{19}$$

for any $u \in L^p\,[0,1]$, provided that $\int_0^1(|u|^p\, lnu)dx \ge 0$.

Corollary 4. *There exists a positive constant C > 0 depending on [0,1] only such that*

$$\|u\|_4^4 \le C\left[\left(\int_0^1\left(|u|^p\, lnu\right)dx\right)^{\frac{4}{p}} + \|u_x\|_4^{\frac{16}{p}}\right]. \tag{20}$$

Lemma 5. *There exists a positive constant C>0 depending on [0,1] only such that*

$$\|u\|_p^s \le C\left[\|u\|_p^p + \|u_x\|_4^4\right], \tag{21}$$

for any $u \in L^p\,[0,1]$, and $4 \le s \le p$.

BLOW UP RESULT

In this part, we state and prove a blow up result for problem (1) in finite time with $E(0) < 0$.

Theorem 6. *Let $u_0 \in H_0^2(0,1)$, $u_1 \in L^2\,(0,1)$. Suppose that $E(0) < 0$ and $\int_0^1 u_0 u_1\, dx > 0$ are satisfied. Then the solution u of the problem (1) blows up in a finite time T^*. Moreover, T^* can be estimated by*

$$T^* = \frac{1-\sigma}{\xi\sigma L^{\frac{\sigma}{1-\sigma}}(0)},$$

where L is defined by (25) below, ξ and σ are positive contants with σ < 1.

Proof. We define

$$H(t) = -E(t), \tag{22}$$

Multiplying (1) by u_t and integrating over [0,1], we get

$$H'(t) = -E'(t) = \alpha||u_t||^2. \tag{23}$$

By virtue of (4) and (22) we note that

$$0 < H(0) \le H(t) \le \frac{1}{p}\int_0^1\left(|u|^p\, lnu\right)dx. \tag{24}$$

We define the following function

$$L(t) = H^{1-\sigma}(t) + \varepsilon\int_0^1 uu_t dx, \tag{25}$$

where ε small to be chosen later and σ is a positive constant and

$$\frac{4(p-2}{p^2} < \sigma < \frac{(p-2)}{4p}. \tag{26}$$

Now, taking derivative of $L(t)$ with respect to t and using equation (1) and (23), we get

$$L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_0^1 |u_t|^2 dx + \varepsilon\int_0^1 uu_{tt} dx$$

$$= \alpha(1-\sigma)H^{-\sigma}(t)\|u_t\|^2 + \varepsilon\|u_t\|^2 - 2\varepsilon\beta\|u_x u_{xx}\|^2$$
$$-2\varepsilon\beta\|u_{xx}\|^2 - \varepsilon\alpha\int_0^1 uu_t dx - 2\varepsilon\int_0^1 (ax+b)u_x^2 dx, \quad (27)$$
$$-\varepsilon\int_0^1 (ax+b)u_x^4 dx + \varepsilon\int_0^1 \left(|u|^p \ln u\right)dx.$$

For estimating the fifth term of the (27), by using Young.s inequality, for any $\delta > 0$ we have

$$\int_0^1 uu_t dx \le \frac{\delta^2}{2}\|u\|^2 + \frac{1}{2\delta^2}\|u_t\|^2. \quad (28)$$

Therefore, by inserting (28) into (27), we have

$$L'(t) \ge \alpha(1-\sigma)H^{-\sigma}(t)\|u_t\|^2 + \varepsilon\left(1 - \frac{\alpha}{2\delta^2}\right)\|u_t\|^2$$
$$-\frac{\varepsilon\alpha\delta^2}{2}\|u\|^2 - 2\varepsilon\beta\|u_x u_{xx}\|^2 - 2\varepsilon\beta\|u_{xx}\|^2 \quad (29)$$
$$-2\varepsilon\int_0^1 (ax+b)u_x^2 dx - \varepsilon\int_0^1 (ax+b)u_x^4 dx$$
$$+\varepsilon\int_0^1 \left(|u|^p \ln u\right)dx.$$

We use the definition of $H(t)$ to substitute for $\int_0^1(|u|^p \ln u)dx \ge 0$ for $0 < m < \frac{p-4}{p}$ Therefore (29) takes form

$$L'(t) \ge \alpha\left[(1-\sigma)H^{-\sigma}(t) - \frac{\varepsilon}{2\delta^2}\right]\|u_t\|^2 + \varepsilon\left(1 + \frac{p(1-m)}{2}\right)\|u_t\|^2$$
$$+\varepsilon\beta\left(p(1-m)-2\right)\|u_{xx}\|^2 + \varepsilon\beta\left(\frac{p(1-m)}{2} - 2\right)\|u_x u_{xx}\|^2$$
$$+\varepsilon p(1-m)H(t) + \varepsilon\left(p(1-m)-2\right)\int_0^1 (ax+b)u_x^2 dx \quad (30)$$
$$+\varepsilon\left(\frac{p(1-m)}{4}\right) - 1\Bigg)\int_0^1 (ax+b)u_x^4 dx - \frac{\varepsilon\alpha\delta^2}{2}\|u\|^2$$
$$+\varepsilon\frac{(1-m)}{p}\|u\|_p^p + \varepsilon m\int_0^1 \left(|u|^p \ln u\right)dx.$$

Of course (30) holds valid even if δ is time dependent since the integral is taken over the x variable. Morever, taking δ so that $\delta^2 = \frac{1}{2h}H^{\sigma}(t)$ for sufficiently large h to be specified later, and substituting in (30), we have

$$L'(t) \ge \alpha\left[(1-\sigma) - h\varepsilon\right]H^{-\sigma}(t)\|u_t\|^2 + \varepsilon\left(1 + \frac{p(1-m)}{2}\right)\|u_t\|^2$$
$$+\varepsilon\beta\left(p(1-m)-2\right)\|u_{xx}\|^2 + \varepsilon\beta\left(\frac{p(1-m)}{2} - 2\right)\|u_x u_{xx}\|^2$$
$$+\varepsilon p(1-m)H(t) + \varepsilon\left(p(1-m)-2\right)\int_0^1 (ax+b)u_x^2 dx \quad (31)$$
$$+\varepsilon\left(\frac{p(1-m)}{4}\right) - 1\Bigg)\int_0^1 (ax+b)u_x^4 dx - \frac{\varepsilon\alpha H^{\sigma}(t)}{4h}\|u\|^2$$
$$+\varepsilon\frac{(1-m)}{2}\|u\|_p^p + \varepsilon m\int_0^1 \left(|u|^p \ln u\right)dx.$$

Making using of (31), Young's inequality and Corollary 4, we encounter

$$H^{\sigma}(t)\|u\|^2 \le \frac{1}{p}\left(\int_0^1 \left(|u|^p \ln u\right)dx\right)^{\sigma}\|u\|^2$$
$$\le \frac{1}{p}\left(\int_0^1 \left(|u|^p \ln u\right)dx\right)^{\sigma}\left(\|u\|_4^4\right)^{\frac{1}{2}}$$
$$\le C\left(\int_0^1 \left(|u|^p \ln u\right)dx\right)^{\sigma}\left[\left(\int_0^1 \left(|u|^p \ln u\right)dx\right)^{\frac{4}{p}} + \|u_x\|_4^{\frac{16}{p}}\right]^{\frac{1}{2}}$$
$$\le C\left(\int_0^1 \left(|u|^p \ln u\right)dx\right)^{\sigma+\frac{4}{p}} + \left(\int_0^1 \left(|u|^p \ln u\right)dx\right)^{\sigma}\|u_x\|_4^{\frac{8}{p}} \quad (32)$$
$$\le C\left(\int_0^1 \left(|u|^p \ln u\right)dx\right)^{\sigma+\frac{4}{p}} + \frac{(p-2)}{p}\left(\int_0^1 \left(|u|^p \ln u\right)dx\right)^{\frac{\sigma p}{p-2}}$$
$$+\frac{2}{p}\|u_x\|_4^4 \le C_1\left[\begin{array}{l}\left(\int_0^1 \left(|u|^p \ln u\right)dx\right)^{\sigma+\frac{4}{p}} \\ +\left(\int_0^1 \left(|u|^p \ln u\right)dx\right)^{\frac{\sigma p}{p-2}} + \|u_x\|_4^4\end{array}\right],$$

where C and C_1 are positive contants.

Utilizing (26), we obtain

$$4 < \sigma p + 2 \le p \text{ and } 4 < \frac{\sigma p^2}{(p-2)} \le p.$$

Thus, by Lemma 2 yields

$$H^{\sigma}(t)\|u\|^2 \le C\left(\int_0^1 \left(|u|^p \ln u\right)dx + \|u_x\|_4^4\right)$$
$$\le C\left(\int_0^1 \left(|u|^p \ln u\right)dx + \int_0^1 (ax+b)u_x^4 dx\right). \quad (33)$$

Combining (33) and (31) we arrive at

$$L'(t) \ge \alpha\left[(1-\sigma) - h\varepsilon\right]H^{-\sigma}(t)\|u_t\|^2 + \varepsilon\left(1 + \frac{p(1-m)}{2}\right)\|u_t\|^2$$
$$+\varepsilon\beta\left(p(1-m)-2\right)\|u_{xx}\|^2 + \varepsilon\beta\left(\frac{p(1-m)}{2} - 2\right)\|u_x u_{xx}\|^2$$
$$+\varepsilon p(1-m)H(t) + \varepsilon\left(p(1-m)-2\right)\int_0^1 (ax+b)u_x^2 dx \quad (34)$$
$$+\varepsilon\left(\frac{p(1-m)}{4}\right) - 1 - \frac{\varepsilon\alpha}{4h}\Bigg)\int_0^1 (ax+b)u_x^4 dx$$
$$+\varepsilon\frac{(1-m)}{p}\|u\|_p^p + \varepsilon\left(m - \frac{\alpha}{4h}\right)\int_0^1 \left(|u|^p \ln u\right)dx.$$

At this point we choose $0 < m < \frac{p-4}{p}$ so small that

$$p(1 - \mathrm{m}) - 2 > 0,$$

and h so large that

$$\frac{p(1-m)}{4} - 1 - \frac{\varepsilon\alpha}{4h} > 0 \text{ and } m - \frac{\alpha}{4h} > 0.$$

Since *h and m* are fixed, taking ε small enough yields

$$(1-\sigma)-h\varepsilon \geq 0,$$

and

$$L(0)=H^{1-\sigma}(0)+\varepsilon\int_0^1 u_0u_1dx. \tag{35}$$

Therefore, (34) takes the form

$$L'(t)\geq \lambda\varepsilon\begin{bmatrix} H(t)+\|u_t\|^2+\|u_{xx}\|^2+\|u_xu_{xx}\|^2 \\ +\int_0^1(ax+b)u_x^2dx+\int_0^1(ax+b)u_x^4dx \\ +\|u\|_p^p+\int_0^1\left(|u|^p\, lnu\right)dx\end{bmatrix}, \tag{36}$$

in which $\lambda > 0$ is the minimum of the coefficients of $H(t)$, $||u_t||^2$, $||u_{xx}||^2$, $||u_xu_{xx}||^2$, $\int_0^1(ax+b)u_x^2dx$, $\int_0^1(ax+b)u_x^4dx$, $||u||_p^p$, $\int_0^1(|u|^p\, lnu)dx$.

Consequently we obtain

$$L(t) > L(0),\ t \geq 0. \tag{37}$$

Next, we esimate $L^{\frac{1}{1-\sigma}}(t)$ Using Hölder inequality, we obtain

$$\left|\int_0^1 uu_tdx\right|\leq \|u\|\|u_t\|\leq C\left(\|u\|_p\|u_t\|\right),$$

which implies

$$\left|\int_0^1 uu_tdx\right|^{\frac{1}{1-\sigma}}\leq C\left(\|u\|_p^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}}\right).$$

Applying Young's inequality we get for $\frac{1}{\mu}+\frac{1}{\kappa}=1$

$$\left|\int_0^1 uu_tdx\right|^{\frac{1}{1-\sigma}}\leq C\left(\|u\|_p^{\frac{\mu}{1-\sigma}}+\|u_t\|^{\frac{\kappa}{1-\sigma}}\right). \tag{38}$$

To be able to use Lemma 5, we take = $2/(1-\sigma)$, to get $\mu = 2(1-\sigma)/(1-2\sigma)$. Therefore (38) has the form

$$\left|\int_0^1 uu_tdx\right|^{\frac{1}{1-\sigma}}\leq C\left(\|u\|_p^s+\|u_t\|^2\right), \tag{39}$$

where $s = 2/(1-2\sigma) \leq p$. By using Lemma 5 we get

$$\begin{aligned}\left|\int_0^1 uu_tdx\right|^{\frac{1}{1-\sigma}}&\leq C\left(\|u_t\|^2+\|u\|_p^p+\|u\|_4^4\right)\\ &\leq C\left(\|u_t\|^2+\|u\|_p^p+\int_0^1(ax+b)u_x^4dx\right).\end{aligned} \tag{40}$$

Here we establish $L^{\frac{1}{1-\sigma}}(t)$ from (40). On the other hand by $(\theta+\varsigma)^k \leq 2^{k-1}(\theta^k+\varsigma^k)$ for $\theta, \varsigma > 0$ and $k > 1$, we have

$$\begin{aligned}L^{\frac{1}{1-\sigma}}(t)&=\left[H^{1-\sigma}(t)+\varepsilon\int_0^1 uu_tdx\right]^{\frac{1}{1-\sigma}}\\ &\leq 2^{\frac{1}{1-\sigma}}\left[H(t)+\varepsilon\left|\int_0^1 uu_tdx\right|^{\frac{1}{1-\sigma}}\right]\\ &\leq 2^{\frac{1}{1-\sigma}}\left[H(t)+\varepsilon\left|\int_0^1 uu_tdx\right|^{\frac{1}{1-\sigma}}\right]\\ &\leq C\left[H(t)\|u_t\|^2+\|u\|_p^p+\int_0^1(ax+b)u_x^4dx\right]\\ &\leq C\begin{bmatrix} H(t)+\|u_t\|^2+\|u_{xx}\|^2+\|u_xu_{xx}\|^2 \\ +\int_0^1(ax+b)u_x^2dx \\ +\int_0^1(ax+b)u_x^4dx+\|u\|_p^p+\int_0^1\left(|u|^p\, lnu\right)dx\end{bmatrix},\end{aligned} \tag{41}$$

where $p=\frac{1}{1-\sigma}>1$.

By associatining (36) and (41) we arrive at

$$L'(t)\geq \xi L^{\frac{1}{1-\sigma}}(t), \tag{42}$$

where ξ is a positive constant depending only on λ, ε and C. Integration of (42) over $(0,t)$, we arrive at

$$L^{\frac{1}{1-\sigma}}(t)\geq \frac{1}{L^{\frac{1}{1-\sigma}}(0)-\frac{\xi\sigma t}{(1-\sigma)}}. \tag{43}$$

Moreover (43) shows that $L(t)$ blows up in a finite time

$$T\leq T^*\leq \frac{1-\sigma}{\xi\sigma L^{\frac{\sigma}{1-\sigma}}(0)}.$$

GLOBAL EXISTENCE

In this part, we state the global existence of the problem (1) without damping terms (for $\alpha = 0$).

Theorem 7. *Assume that* $u_0 \in H_0^2(0,1)$ *and* $u_1 \in L^2\,[0,1]$ *Assume further that*

$$\int_0^1\left(|u|^p\, lnu\right)dx\leq 0$$

and the initial value satisfy $E(0) < 0$. *Then, the problem (1) admits a global weak solution.*

Proof. Set

$$B(t)=E(t)+\frac{1}{p}\int_0^1\left(|u|^p\, lnu\right)dx-\frac{1}{p^2}\int_0^1|u|^p\,dx. \tag{44}$$

Then by definition of $E(t)$ and (44) we reach

$$B(t)=\frac{1}{2}\|u_t\|^2+\beta\|u_{xx}\|^2+\int_0^1(ax+b)u_x^2dx+\frac{\beta}{2}\|u_xu_{xx}\|^2 + \frac{1}{4}\int_0^1(ax+b)u_x^4dx\geq 0. \tag{45}$$

By using $E(0) < 0$ and (23) with $\alpha = 0$, we obtain

$$\frac{dE(t)}{dt}=0, \tag{46}$$

$$E(t) = E(0) = 0. \tag{47}$$

By taking a derivation of $B(t)$ and using of the (46), we have

$$B'(t)=E'(t)+\int_0^1|u|^{p-1}u_t lnudx+\frac{1}{p}\int_0^1|u|^{p-1}u_tdx - \frac{1}{p}\int_0^1|u|^{p-1}u_tdx=\int_0^1|u|^{p-1}u_t lnudx. \tag{48}$$

Since $x < ln|x|$ and in wiev of using Hölder and Young's inequality, definition of the $B(t)$, (48) replaces

$$|B'(t)|<\int_0^1|u|^p u_tdx\leq\|u\|_p^{\frac{p}{2}}\|u_t\|^2\leq\frac{1}{4\rho^2}\|u\|_p^p+\rho^2\|u_t\|^2 \leq\frac{1}{4\rho^2}\left[E(t)-B(t)+\frac{1}{p}\int_0^1|u|^p lnudx\right]+\rho^2\|u_t\|^2, \tag{49}$$

where ρ is a positive constant.

Because of (45), (47) and $\int_0^1|u|^p lnudx \leq 0$, we attain

$$E(t)-B(t)+\frac{1}{p}\int_0^1|u|^p lnudx\leq 0, \tag{50}$$

which means

$$|B'(t)| \leq \rho^2 ||u_t||^2. \tag{51}$$

On the other hand, by using (45), we deduce

$$2B(t)=\|u_t\|^2+2\beta\|u_{xx}\|^2+2\int_0^1(ax+b)u_x^2dx + \beta\|u_xu_{xx}\|^2+\frac{1}{2}\int_0^1(ax+b)u_x^4dx.$$

It follows that

$$||u_t||^2 \leq 2B(t). \tag{52}$$

From (51) and (52), we have

$$|B'(t)| \leq 2\rho^2 B(t).$$

Therefore

$$B(t)exp(-2\rho^2 t) < B(t) < B(t)exp(2\rho^2 t).$$

With the last estimate, the definition of B(t) and continuation principle, the proof was completed.

CONCLUSION

As mentioned earlier in the introduction nonlinear equations with logarithmic source term and their analytical solutions have received much attention from physicists and mathematicians. However, in the last time, many authors have made great progress and adopted various techniques to study the analytical side of these problems(see, for example;[16, 22, 26, 27, 28, 30, 34, 36]).

The result of Theorem 6 tells us the blow up property at in finite time of the solution, but we don't know whether the solution blows up infinitely time. We conclude that the normal Georgive-Todorova method (e.g.[11]) no longer emploies in this specific situation when the logarithmic source term appears, for blow-up in finite time of the solutions. Theorem 6 shows us that there is quite a difference in cases of the equations with polynomial nonlinear term or logarithmic nonlinear term.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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