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Technical Note

Growth of solutions for fourth order viscoelastic system

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ABSTRACT

We consider a system of viscoelastic equations with degenerate damping and source terms under Dirichlet boundary condition. We prove the exponential growth of solutions under some restrictions on the initial data, relaxation functions and degenerate damping terms.

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INTRODUCTION

Let is a bounded domain with a sufficiently smooth boundary in $R^n (n \geq 1)$. We investigate the exponential growth of solutions for the following initial boundary value problem:

$$\begin{cases} u_{tt} + \Delta^2 u - \int_0^t \rho_1(t-s)\Delta^2 u(s)ds + \left(|u|^f + |v|^g\right)|u_t|^{\mu-1} u_t \\ \qquad = f_1(u,v),\ (x,t) \in \Omega \times (0,T), \\ v_{tt} + \Delta^2 v - \int_0^t \rho_2(t-s)\Delta^2 v(s)ds + \left(|v|^h + |u|^j\right)|v_t|^{\eta-1} v_t \\ \qquad = f_2(u,v),\ (x,t) \in \Omega \times (0,T) \\ u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where $\mu,\eta \geq 1, f,g,h,j \geq 0$; $\rho_i(.) : R^+ \to R^+$ $(i = 1,2)$ are positive relaxation functions.

The source terms are defined as follows:

$$f_1(u,v) = a|u + v|^{2(s+1)}(u + v) + b|u|^s u|v|^{s+2},$$
$$f_2(u,v) = a|u + v|^{2(s+1)}(u + v) + b|v|^s v|u|^{s+2},$$

where $a > 0$, $b > 0$, and

$$\begin{cases} -1 < s & if\ n = 1,2, \\ -1 < s \leq \dfrac{3-n}{n-2} & if\ n \geq 3\ . \end{cases} \tag{2}$$

It is easy to show that

$$uf_1(u,v) + vf_2(u,v) = 2(s + 2)F(u,v),\ \forall (u,v) \in R^2, \tag{3}$$

where

$$F(u,v) = \frac{1}{2(s+2)}\left[a|u+v|^{2(s+2)} + 2b|uv|^{s+2}\right]. \tag{4}$$

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Equation (1) can be viewed as a generalization of a plate model. The following plate equation

$$u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u - \int_0^t h(t-s)\Delta^2 u(s)ds = 0, \qquad (5)$$

have been studied by Rivera et al. [1]. The authors proved the asymptotic behaviour of solution with the initial and dynamical boundary conditions. The problem (5) with source term have been investigated by Alabau-Boussouira et al. [2] and the authors discussed exponential and polynomial decay results of solutions when the memory h decay exponentially and polynomially in case $\gamma = 0$, respectively.

On the other hand, Messaoudi [3] discussed the existence result and show that the solution is global in case $m \geq p$ and also blow-up of solutions with negative initial energy in case $m < p$ for the following problem

$$u_{tt} + \Delta^2 u + |u_t|^{m-1} u_t = |u|^{p-2} u.$$

Also, Pişkin and Polat [4] investigated same problem with strong dampin (Δu_t) and obtained decay estimates of solution.

The importance of the viscoelastic properties of materials has been realized by virtue of the quick developments in rubber industry and plastic. During the last few decades, many researchers have been interested in the viscoelastic equation. For example, Cavalcanti et al. [5] discussed the following equation

$$\begin{aligned} |u_t|^{\eta} u_{tt} - \Delta u - \Delta u_{tt} - \int_0^t h(t-s)\Delta u(s)ds \\ -\gamma\Delta u_t = 0,\ x \in \Omega,\ t \geq 0, \end{aligned}$$

and studied a global existence result for $\gamma \geq 0$ and uniform exponential decay result for $\gamma \geq 0$ Then, Messaoudi and Tatar [6] investigated same problem with a nonlinear source term and proved global existence and an exponential decay result by using the potential well method. Also, the authors investigated exponential growth of solutions.

In addition, Tahamtani et al. [7] studied uniform decay results and exponential growth of solutions with positive initial energy for the following viscoelastic system with $\eta > 0$, j, $s \geq 1$

$$\begin{cases} |u_t|^{\eta} u_{tt} + \Delta^2 u - \int_0^t h_1(t-s)\Delta^2 u(s)ds - \Delta u_{tt} + |u_t|^{j-1} u_t = f_1(u,v), \\ |v_t|^{\eta} v_{tt} + \Delta^2 v - \int_0^t h_2(t-s)\Delta^2 v(s)ds - \Delta v_{tt} + |v_t|^{s-1} v_t = f_2(u,v). \end{cases} \qquad (6)$$

During the last few decades, many researchers have been interested in the nonlinear wave equations with degenerate damping terms. Now, we state some present results in the literature: Rammaha and Sakuntasathien [8] firstly discussed coupled equations with degenerate damping such as

$$\begin{cases} u_{tt} - \Delta u + \left(|u|^f + |v|^g\right)|u_t|^{\mu-1} u_t = f_1(u,v), \\ v_{tt} - \Delta v + \left(|v|^h + |u|^j\right)|v_t|^{\eta-1} v_t = f_2(u,v). \end{cases} \qquad (7)$$

The authors considered the global well posedness of the solution under some restriction on the parameters. Then, in [9, 10], the authors studied the same problem treated in [8], and discussed the exponential growth and blow up of solutions. In recent years, some other authors investigate the hyperbolic type system with degenerate damping terms see [11–22].

The present work is organized as follows: In Section 2, we present some assumptions, lemmas needed for our work. In Section 3, the exponential growth of solutions with negative initial energy is proved.

PRELIMINARIES

In this section, we will present some assumptions, notations, and lemmas that will be used later for our main result. Throughout this paper, we denote the standart $L^2(\Omega)$ norm by $\|.\| = \|.\|_{L^2(\Omega)}$ and $L^p(\Omega)$ norm by $\|.\|_p = \|.\|_{Lp(\Omega)}$.

To state and prove our result, we need some assumptions:

(A1) Regarding $\rho_i(.)$: $R^+ \to R^+$, $(i = 1,2)$ are C^1 -nonincreasing functions satisfying

$$\rho_i(\alpha) > 0,\ \ \rho_i'(\alpha) < 0,\ \ 1 - \int_0^\infty \rho_i(\alpha)d\alpha = l_i > 0,\ \alpha \geq 0.$$

(A2) For the nonlinearity, we assume that

$$\begin{cases} 1 \leq \mu, \eta & if\ n = 1,2, \\ 1 \leq \mu, \eta \leq \dfrac{n+2}{n-2} & if\ n \geq 3. \end{cases}$$

In addition, we present the following notation:

$$(\rho_i \Diamond \Delta w)(t) = \int_0^t \rho_i(t-s)\|\Delta w(t) - \Delta w(s)\|^2 ds.$$

Lemma 1 *(Sobolev-Poincare inequality)* [23]. Let q be a number with $2 \leq q < \infty$ $(n = 1,2,3,4)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 5)$, then there is a constant $C_* = C_*(\Omega,q)$ such that

$$\|u\|_q \leq C_* \|\Delta u\| \text{ for } u \in H_0^2(\Omega).$$

Lemma 2 [24]. Assume that

$$s \leq 2\frac{n-1}{n-2},\ \ n \geq 3$$

holds. Then, there exists a positive constant $C > 1$ depending on Ω only such that

$$\|u\|_s^{\alpha} \leq C\left(\|\nabla u\|^2 + \|u\|_s^s\right)$$

for any u ∈ $H_0^1(\Omega)$, $2 \le \alpha \le s$.

We define the energy function as follows

$$E(t)=\frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{1}{2}\left[\left(\rho_1\lozenge\Delta u\right)(t)+\left(\rho_2\lozenge\Delta v\right)(t)\right]$$
$$+\frac{1}{2}\begin{bmatrix}\left(1-\int_0^t\rho_1(s)ds\right)\|\Delta u(t)\|^2\\ +\left(1-\int_0^t\rho_2(s)ds\right)\|\Delta v(t)\|^2\end{bmatrix}$$
$$-\int_\Omega F(u,v)dx.$$

By computation, we get

$$\frac{d}{dt}E(t)\le\frac{1}{2}\left[\left(\rho_1'\lozenge\Delta u\right)(t)+\left(\rho_2'\lozenge\Delta v\right)(t)\right]$$
$$-\frac{1}{2}\left(\rho_1(t)\|\Delta u\|^2+\rho_2(t)\|\Delta v\|^2\right) \quad (9)$$
$$-\int_\Omega\left(|u|^f+|v|^g\right)|u_t|^{\mu+1}dx$$
$$-\int_\Omega\left(|v|^h+|u|^j\right)|v_t|^{\eta+1}dx\le 0.$$

GROWTH

In this section, we aim to prove that the energy grows up as an exponential function as time as goes to infinity.

Theorem Assume that $2(s + 2) > max\{f + \mu + 1, g + \mu + 1, h + \eta + 1, j + \eta + 1\}$, and the initial energy $E(0) < 0$. Then, the solution of the system (1) grows exponentially.

Proof. Let us define the functional

$$L(t)=H(t)+\varepsilon\left(\int_\Omega u_t u dx+\int_\Omega v_t v dx\right) \quad (10)$$

where $H(t) = -E(t)$ and $0 < \varepsilon \le 1$.

By differentiating (10) and using Eq.(1), we get

$$L'(t)=H'(t)+\varepsilon\left(\int_\Omega|u_t|^2dx+\int_\Omega|v_t|^2dx\right)$$
$$+\varepsilon\left(\int_\Omega u_{tt}udx+\int_\Omega v_{tt}vdx\right)$$
$$=H'(t)+\varepsilon\left(\|u_t\|^2+\|v_t\|^2\right)-\varepsilon\left(\|\Delta u\|^2+\|\Delta v\|^2\right)$$
$$+2\varepsilon(s+2)\int_\Omega F(u,v)dx \quad (11)$$
$$+\varepsilon\int_\Omega\int_0^t\rho_1(t-s)\Delta u(s)\Delta u(t)dsdx$$
$$+\varepsilon\int_\Omega\int_0^t\rho_2(t-s)\Delta v(s)\Delta v(t)dsdx$$
$$-\varepsilon\begin{pmatrix}\int_\Omega u\left(|u|^f+|v|^g\right)u_t|u_t|^{\mu-1}dx\\ -\int_\Omega v\left(|v|^h+|u|^j\right)v_t|v_t|^{\eta-1}dx\end{pmatrix}.$$

In order to estimate the last terms in (11), we use the following Young's inequality for $X, Y > 0$, $\delta > 0$, $k, l \in R^+$

$$XY\le\frac{\delta^k X^k}{k}+\frac{\delta^{-l}Y^l}{l},$$

where $\frac{1}{k}+\frac{1}{l}=1$ So we have for all $\delta_1 > 0$

$$\left|uu_t|u_t|^{\mu-1}\right|\le\frac{\delta_1^{\mu+1}}{\mu+1}|u|^{\mu-1}+\frac{\mu\delta_1^{\frac{\mu+1}{\mu}}}{\mu+1}|u|^{\mu-1},$$

and therefore

$$\int_\Omega\left(|u|^f+|v|^g\right)\left|uu_t|u_t|^{\mu-1}\right|dx\le\frac{\delta_1^{\mu+1}}{\mu+1}\int_\Omega\left(|u|^f+|v|^g\right)|u|^{\mu+1}dx$$
$$+\frac{\mu\delta_1^{\frac{\mu+1}{\mu}}}{\mu+1}\int_\Omega\left(|u|^f+|v|^g\right)|u_t|^{\mu+1}dx. \quad (12)$$

Similarly, for all $\delta_2 > 0$

$$\left|vv_t|v_t|^{\eta-1}\right|\le\frac{\delta_2^{\eta+1}}{\eta+1}|v|^{\eta+1}+\frac{\eta\delta_2^{\frac{\eta+1}{\eta}}}{\eta+1}|v_t|^{\eta+1},$$

which gives

$$\int_\Omega\left(|v|^h+|u|^j\right)\left|vv_t|v_t|^{\eta-1}\right|dx\le\frac{\delta_2^{\eta+1}}{\eta+1}\int_\Omega\left(|v|^h+|u|^j\right)|v|^{\eta+1}dx$$
$$+\frac{\eta\delta_2^{\frac{\eta+1}{\eta}}}{\eta+1}\int_\Omega\left(|v|^h+|u|^j\right)|v_t|^{\eta+1}dx. \quad (13)$$

Inserting the estimates (12), (13) into (11), we have

$$L'(t)\ge H'(t)+\varepsilon\left(\|u_t\|^2+\|v_t\|^2\right)-\varepsilon\left(\|\Delta u\|^2+\|\Delta v\|^2\right)$$
$$+2\varepsilon(s+2)\int_\Omega F(u,v)dx+\varepsilon\int_\Omega\int_0^t\rho_1(t-s)\Delta u(s)\Delta u(t)dsdx$$
$$+\varepsilon\int_\Omega\int_0^t\rho_2(t-s)\Delta v(s)\Delta v(t)dsdx-\varepsilon\frac{\delta_1^{\mu+1}}{\mu+1}\int_\Omega\left(|u|^f+|v|^g\right)|u|^{\mu+1}dx \quad (14)$$
$$-\varepsilon\frac{\mu\delta_1^{\frac{\mu+1}{\mu}}}{\mu+1}\int_\Omega\left(|u|^f+|v|^g\right)|u_t|^{\mu+1}dx-\varepsilon\frac{\delta_2^{\eta+1}}{\eta+1}\int_\Omega\left(|v|^h+|u|^j\right)|v|^{\eta+1}dx$$
$$-\varepsilon\frac{\eta\delta_2^{\frac{\eta+1}{\eta}}}{\eta+1}\int_\Omega\left(|v|^h+|u|^j\right)|v_t|^{\eta+1}dx.$$

Now, the seventh term in the right hand side of (14) can be estimated, as follows (see [25]):

$$\int_\Omega\Delta u(t)\int_0^t\rho_1(t-s)\Delta u(s)dsdx\le\frac{1}{2}\|\Delta u\|^2$$
$$+\frac{1}{2}\int_\Omega\left(\int_0^t\rho_1(t-s)\left(|\Delta u(s)-\Delta u(t)|+|\Delta u(t)|\right)ds\right)^2dx. \quad (15)$$

Thanks to Young's inequality and in view of the fact that

$$\int_0^t\rho_1(s)ds\le\int_0^\infty\rho_1(s)ds\le 1-l_i$$

we have, for any $\eta_1 > 0$,

$$\begin{aligned}
&\int_\Omega \Delta u(t)\int_0^t \rho_1(t-s)\Delta u(s)dsdx \le \frac{1}{2}\|\Delta u\|^2 \\
&+\frac{1}{2}(1+\eta_1)\int_\Omega\left(\int_0^t \rho_1(t-s)\Delta u(s)ds\right)^2 dx \\
&+\frac{1}{2}\left(1+\frac{1}{\eta_1}\right)\int_\Omega\left(\int_0^t \rho_1(t-s)|\Delta u(s)-\Delta u(t)|ds\right)^2 dx \\
&\le \frac{1+(1+\eta_1)(1-l_1)^2}{2}\|\Delta u\|^2 \\
&+\frac{\left(1+\frac{1}{\eta_1}\right)(1-l_1)}{2}(\rho_1\lozenge\Delta u)(t).
\end{aligned} \tag{16}$$

Similar calculations also yield, for any $\eta_2 > 0$

$$\begin{aligned}
&\int_\Omega \Delta v(t)\int_0^t \rho_2(t-s)\Delta v(s)dsdx \le \frac{1+(1+\eta_2)(1-l_2)^2}{2}\|\Delta v\|^2 \\
&+\frac{\left(1+\frac{1}{\eta_2}\right)(1-l_2)}{2}(\rho_2\lozenge\Delta v)(t).
\end{aligned} \tag{17}$$

Then, by (15) - (17) and add $2H(t)$ to both side of (14), we deduce to

$$\begin{aligned}
&L'(t) \ge H'(t)+2\varepsilon\left(\|u_t\|^2+\|v_t\|^2\right) \\
&+\varepsilon\left((1-l_1)+\frac{1+(1+\eta_1)(1-l_1)^2-1}{2}\right)\|\Delta u\|^2 \\
&+\varepsilon\left((1-l_2)+\frac{1+(1+\eta_2)(1-l_2)^2-1}{2}\right)\|\Delta v\|^2 \\
&+2\varepsilon(s+2)\int_\Omega F(u,v)dx+2\varepsilon H(t) \\
&+\varepsilon\left(1+\frac{\left(1+\frac{1}{\eta_1}\right)(1-l_1)}{2}\right)(\rho_1\lozenge\Delta u)(t) \\
&+\varepsilon\left(1+\frac{\left(1+\frac{1}{\eta_2}\right)(1-l_2)}{2}\right)(\rho_2\lozenge\Delta v)(t) \\
&-\varepsilon\frac{\delta_1^{\mu+1}}{\mu+1}\int_\Omega\left(|u|^f+|v|^g\right)|u|^{\mu+1}dx-\varepsilon\frac{\mu\delta_1^{\frac{\mu+1}{\mu}}}{\mu+1}\int_\Omega \left(|u|^f+|v|^g\right)|u_t|^{\mu+1}dx \\
&-\varepsilon\frac{\delta_2^{\eta+1}}{\eta+1}\int_\Omega\left(|v|^h+|u|^j\right)|v|^{\eta+1}dx-\varepsilon\frac{\eta\delta_2^{\frac{\eta+1}{\eta}}}{\eta+1}\int_\Omega \left(|v|^h+|u|^j\right)|v_t|^{\eta+1}dx.
\end{aligned} \tag{18}$$

Then, by Young's inequality, we have

$$\begin{aligned}
&\int_\Omega\left(|u|^f+|v|^g\right)|u|^{\mu+1}dx \le \int_\Omega |u|^{f+\mu+1}dx+\int_\Omega |v|^g|u|^{\mu+1}dx \\
&\le \int_\Omega |u|^{f+\mu+1}dx+\frac{g}{g+\mu+1}\gamma_1^{\frac{g+\mu+1}{g}}\int_\Omega |v|^{g+\mu+1}dx \\
&+\frac{\mu+1}{g+\mu+1}\gamma_1^{\frac{g+\mu+1}{\mu+1}}\int_\Omega |u|^{g+\mu+1}dx \\
&\|u\|_{f+\mu+1}^{f+\mu+1}+\frac{g}{g+\mu+1}\gamma_1^{\frac{g+\mu+1}{g}}\|v\|_{g+\mu+1}^{g+\mu+1} \\
&+\frac{\mu+1}{g+\mu+1}\gamma_1^{\frac{g+\mu+1}{\mu+1}}\|u\|_{g+\mu+1}^{g+\mu+1}
\end{aligned}$$

And

$$\begin{aligned}
&\int_\Omega\left(|v|^h+|u|^j\right)|v|^{\eta+1}dx \le \int_\Omega |v|^{h+\eta+1}dx+\int_\Omega |u|^j|v|^{\eta+1}dx \\
&\le \int_\Omega |v|^{h+\eta+1}dx+\frac{j}{j+\eta+1}\gamma_2^{\frac{j+\eta+1}{j}}\int_\Omega |u|^{j+\eta+1}dx \\
&+\frac{\eta+1}{j+\eta+1}\gamma_2^{\frac{j+\eta+1}{\eta+1}}\int_\Omega v^{j+\eta+1}dx \\
&=\|v\|_{h+\eta+1}^{h+\eta+1}+\frac{j}{j+\eta+1}\gamma_2^{\frac{j+\eta+1}{j}}\|u\|_{j+\eta+1}^{j+\eta+1} \\
&+\frac{\eta+1}{j+\eta+1}\gamma_2^{\frac{j+\eta+1}{\eta+1}}\|v\|_{j+\eta+1}^{j+\eta+1}.
\end{aligned}$$

Then, (18) becomes

$$\begin{aligned}
&L'(t) \ge H'(t)+2\varepsilon\left(\|u_t\|^2+\|v_t\|^2\right)+2\varepsilon H(t) \\
&+\varepsilon\left((1-l_1)+\frac{(1+\eta_1)(1-l_1)^2-1}{2}\right)\|\Delta u\|^2 \\
&+\varepsilon\left((1-l_2)+\frac{(1+\eta_2)(1-l_2)^2-1}{2}\right)\|\Delta v\|^2 \\
&+2\varepsilon(s+2)\left[\|u\|_{2(s+2)}^{2(s+2)}+\|v\|_{2(s+2)}^{2(s+2)}\right] \\
&+\varepsilon\left(1+\frac{\left(1+\frac{1}{\eta_1}\right)(1-l_1)}{2}\right)(\rho_1\lozenge\Delta u)(t) \\
&+\varepsilon\left(1+\frac{\left(1+\frac{1}{\eta_2}\right)(1-l_2)}{2}\right)(\rho_2\lozenge\Delta v)(t) \\
&-\varepsilon\frac{\delta_1^{\mu+1}}{\mu+1}\left(\|u\|_{f+\mu+1}^{f+\mu+1}+\frac{g}{g+\mu+1}\gamma_1^{\frac{g+\mu+1}{g}}\|v\|_{g+\mu+1}^{g+\mu+1}+\frac{\mu+1}{g+\mu+1}\gamma_1^{\frac{g+\mu+1}{\mu+1}}\|u\|_{g+\mu+1}^{g+\mu+1}\right)
\end{aligned} \tag{19}$$

$$-\varepsilon\frac{\delta_2^{\eta+1}}{\eta+1}\left[\begin{array}{l}\|v\|_{h+\eta+1}^{h+\eta+1}+\frac{j}{j+\eta+1}\gamma_2^{\frac{j+\eta+1}{j}}\|u\|_{j+\eta+1}^{j+\eta+1}\\+\frac{\eta+1}{j+\eta+1}\gamma_2^{\frac{j+\eta+1}{\eta+1}}\|v\|_{j+\eta+1}^{j+\eta+1}\end{array}\right]$$
$$-\varepsilon\frac{\mu\delta_1^{\frac{\mu+1}{\mu}}}{\mu+1}\int_\Omega\left(|u|^f+|v|^g\right)|u_t|^{\mu+1}dx$$
$$-\varepsilon\frac{\eta\delta_2^{\frac{\eta+1}{\eta}}}{\eta+1}\int_\Omega\left(|v|^h+|u|^j\right)|v_t|^{\eta+1}dx.$$

Since $2(s + 2) > \max\{f + \mu + 1, g + \mu + 1, h + \eta + 1, j + \eta + 1\}$ and using the following algebraic inequality

$$x^v \le x+1\le\left(1+\frac{1}{a}\right)(x+a),\quad \forall x\ge 0, 0<v\le 1, a\ge 0,$$

we have for all $t \ge 0$

$$\|v\|_{h+\eta+1}^{h+\eta+1}\le c_1\|v\|_{2(s+2)}^{h+\eta+1}\le d\left(\|v\|_{2(s+2)}^{2(s+2)}+H(t)\right),$$

where $d=1+\frac{1}{H(0)}$. In the same way, we obtain

$$\|u\|_{j+\eta+1}^{j+\eta+1}\le c_2\|u\|_{2(s+2)}^{j+\eta+1}\le d\left(\|u\|_{2(s+2)}^{2(s+2)}+H(t)\right),$$
$$\|v\|_{g+\mu+1}^{g+\mu+1}\le c_3\|v\|_{2(s+2)}^{g+\mu+1}\le d\left(\|v\|_{2(s+2)}^{2(s+2)}+H(t)\right),$$
$$\|u\|_{g+\mu+1}^{g+\mu+1}\le c_4\|u\|_{2(s+2)}^{g+\mu+1}\le d\left(\|u\|_{2(s+2)}^{2(s+2)}+H(t)\right),$$
$$\|u\|_{f+\mu+1}^{f+\mu+1}\le c_5\|u\|_{2(s+2)}^{f+\mu+1}\le d\left(\|u\|_{2(s+2)}^{2(s+2)}+H(t)\right),$$

And

$$\|v\|_{j+\eta+1}^{j+\eta+1}\le c_6\|v\|_{2(s+2)}^{j+\eta+1}\le \mathrm{d}\left(\|v\|_{2(s+2)}^{2(s+2)}+H(t)\right).$$

We choose M_1, M_2, M_3, M_4, M_5 such that

$$M_1=\frac{\mu\delta_1^{\frac{\mu+1}{\mu}}}{\mu+1},\quad M_2=\frac{\eta\delta_2^{\frac{\eta+1}{\eta}}}{\eta+1},$$
$$M_3=\frac{\delta_1^{\mu+1}}{\mu+1}\left(1+\frac{\mu+1}{g+\mu+1}\gamma_1^{\frac{g+\mu+1}{\mu+1}}\right)+\frac{\delta_2^{\eta+1}}{\eta+1}\frac{j}{j+\eta+1}\gamma_2^{\frac{j+\eta+1}{j}},$$
$$M_4=\frac{\delta_2^{\eta+1}}{\eta+1}\left(1+\frac{\eta+1}{j+\eta+1}\gamma_2^{\frac{j+\eta+1}{\eta+1}}\right)+\frac{\delta_1^{\mu+1}}{\mu+1}\frac{g}{g+\mu+1}\gamma_1^{\frac{g+\mu+1}{g}},$$

And

$$M_5=\frac{\delta_1^{\mu+1}}{\mu+1}\left(1+\frac{g}{g+\mu+1}\gamma_1^{\frac{g+\mu+1}{g}}+\frac{\mu+1}{g+\mu+1}\gamma_1^{\frac{g+\mu+1}{\mu+1}}\right)$$
$$+\frac{\delta_2^{\eta+1}}{\eta+1}\left(1+\frac{\eta+1}{j+\eta+1}\gamma_2^{\frac{j+\eta+1}{\eta+1}}+\frac{j}{j+\eta+1}\gamma_2^{\frac{j+\eta+1}{j}}\right).$$

Also, we pick δ_1, δ_2, γ_1 and γ_2 to find small enough M_1, M_2, M_3, M_4 and M_5.

This implies

$$\begin{aligned}L'(t)&\ge H'(t)+2\varepsilon\left(\|u_t\|^2+\|v_t\|^2\right)+\varepsilon\left(2-dM_5\right)H(t)\\&+\varepsilon\alpha_1\|\Delta u\|^2+\varepsilon\alpha_2\|\Delta v\|^2\\&+\varepsilon\left(2(s+2)-dM_3\right)\|u\|_{2(s+2)}^{2(s+2)}+\varepsilon\left(2(s+2)-dM_4\right)\|v\|_{2(s+2)}^{2(s+2)}\\&+\varepsilon\beta_1\left(\rho_1\Diamond\Delta u\right)(t)+\varepsilon\beta_2\left(\rho_2\Diamond\Delta v\right)(t)\\&+\left(1-\varepsilon M_1\right)\int_\Omega\left(|u|^f+|v|^g\right)|u_t|^{\mu+1}dx\\&+\left(1-\varepsilon M_2\right)\frac{\eta\delta_2^{\frac{\eta+1}{\eta}}}{\eta+1}\int_\Omega\left(|v|^h+|u|^j\right)|v_t|^{\eta+1}dx\end{aligned}\qquad(21)$$

where $\alpha_i=\left((1-l_i)+\frac{(1+\eta_i)(1-l_i)^2-1}{2}\right)>0$ and $\beta_i=\left(1+\frac{\left(1+\frac{1}{\eta_i}\right)(1-l_i)}{2}\right)>0\,(i=1,2)$ for choosing $\eta_i=\frac{l_i}{1-l_i}$.

We can find positive constants K_1, K_2, K_3 and M_6 such that

$$\begin{aligned}L'(t)&\ge\left(1-\varepsilon M_6\right)H'(t)+2\varepsilon\left(\|u_t\|^2+\|v_t\|^2\right)+\varepsilon K_1H(t)\\&+\varepsilon\alpha_1\|\Delta u\|^2+\varepsilon\alpha_2\|\Delta v\|^2+\varepsilon K_2\|u\|_{2(s+2)}^{2(s+2)}+\varepsilon K_3\|v\|_{2(s+2)}^{2(s+2)}.\end{aligned}\qquad(22)$$

We choose ε small enough such that $(1 - \varepsilon M_6) \ge 0$ and

$$L(0)=H(0)+\varepsilon\left(\int_\Omega u_tu_0dx+\int_\Omega v_tv_0dx\right)>0.$$

Consequently, there exists $\Gamma > 0$ such that (22) becomes

$$\begin{aligned}&L'(t)\ge\varepsilon\Gamma\\&\left(H(t)+\|u_t\|^2+\|v_t\|^2+\|\Delta u\|^2+\|\Delta v\|^2+\|u\|_{2(s+2)}^{2(s+2)}+\|v\|_{2(s+2)}^{2(s+2)}\right).\end{aligned}\qquad(23)$$

Therefore, $L(t)$ is strictly positive and increasing for all $t \ge 0$.

Now, by using Holder's and Young's inequalities, we estimate

$$\begin{aligned}\left|\int_\Omega u_tudx\right|&\le\|u_t\|\|u\|\\&\le C\left(\|u_t\|\|u\|_{2(s+2)}\right)\\&\le\frac{C}{2}\left(\|u_t\|^2+\|u\|_{2(s+2)}^2\right)\\&\le\frac{C}{2}\left(\|u_t\|^2+\left(\|u\|_{2(s+2)}^{2(s+2)}\right)^{\frac{1}{s+2}}\right).\end{aligned}$$

Using (20) for $\left(\|u\|_{2(s+2)}^{2(s+2)}\right)^{\frac{1}{s+2}}$ we obtain

$$\left|\int_\Omega u_tudx\right|\le\frac{C}{2}\left(\|u_t\|^2+\left(1+\frac{1}{H(0)}\right)\left(\|u\|_{2(s+2)}^{2(s+2)}+H(t)\right)\right).$$

Similarly, we obtain

$$\left|\int_{\Omega} v_t v dx\right| \leq \frac{C}{2}\left(\|v_t\|^2 + \left(1+\frac{1}{H(0)}\right)\left(\|v\|_{2(s+2)}^{2(s+2)} + H(t)\right)\right).$$

Also, by noting that

$$L(t) = H(t) + \varepsilon\left(\int_{\Omega} u_t u dx + \int_{\Omega} v_t v dx\right) \leq C\begin{pmatrix} H(t) + \|u_t\|^2 + \|v_t\|^2 + \|\Delta u\|^2 + \|\Delta v\|^2 \\ + \|u\|_{2(s+2)}^{2(s+2)} + \|v\|_{2(s+2)}^{2(s+2)} \end{pmatrix} \quad (24)$$

and combining with (24) and (23), we arrive at

$$\frac{dL(t)}{dt} \geq \xi L(t), \forall t \geq 0 \quad (25)$$

where ξ is a positive constant.

Integration of (25) between 0 and t gives us

$$L(t) \geq L(0) exp(\xi t)$$

and this completes the proof.

CONCLUSION

In this paper, we are interested in the growth of solutions for a viscoelastic system with degenerate damping. This type of problem is frequently found in some mathematical models in applied sciences, especially in the theory of viscoelasticity. What interests us in this current work is the combination of these terms of damping (viscoelastic term, degenerate damping, and source terms), which dictates the emergence of these terms in the system.

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