Sigma J Eng Nat Sci, Vol. 39, No. 5, Supp 22, ICOMAA 2021, pp. 23–32, December, 2021

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2021.00038

Research Article

Local existence and nonexistence of global solutions for a plate equation with time delay

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ARTICLE INFO

Article history
Received: 17 July 2021
Accepted: 09 November 2021

Key words:
Local existence; Nonexistence; Plate equation; Time delay

ABSTRACT

In this article, we study a plate equation with frictional damping, nonlinear source and time delay. Firstly, we establish the local existence by using the semigroup theory. Then, under suitable conditions, we prove the nonexistence of global solutions for positive initial energy. Time delays often appear in many practical problems such as thermal, economic phenomena, biological, chemical, physical, electrical engineering systems, mechanical applications and medicine.

Cite this article as: Yüksekkaya H, Pişki S. Local existence and nonexistence of global solutions for a plate equation with time delay. Sigma J Eng Nat Sci 2021;39(Supp 22):23–32.

INTRODUCTION

In this work, we deal with the following plate equation with time delay

$$\begin{cases} u_{tt}+\Delta^2 u+\mu_1 u_t(x,t)+\mu_2 u_t(x,t-\tau)=|u|^{p-2}u, & x\in\Omega \quad t>0, \\ u(x,t)=\dfrac{\partial u(x,t)}{\partial v}=0, & x\in\partial\Omega, \\ u_t(x,t-\tau)=f_0(x,t-\tau), & in\ (0,\tau), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, \end{cases} \tag{1}$$

where $p > 2$, μ_1 is a positive constant, μ_2 is a real number, $\tau > 0$ represents the time delay and the functions u_0, u_1, f_0 are the initial data to be specified later. v is the unit outward normal vector.

Problems about the mathematical behavior of solutions for PDEs with time delay effects have become interesting for many authors mainly because time delays often appear in many practical problems such as thermal, economic phenomena, biological, chemical, physical, electrical engineering systems, mechanical applications and medicine. Moreover, it is well known that delay effects may destroy the stabilizing properties of a well-behaved system. In the literature, there are several examples that illustrate how time delays destabilize some internal or boundary control system [6,7].

In 1986, Datko et al. [5] indicated that delay is a source of instability. In [12], Nicaise and Pignotti considered the following wave equation with a linear damping and delay term.

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0. \tag{2}$$

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This paper was recommended for publication in revised form by Regional Editor Yusuf Zeren

Published by Yıldız Technical University Press, İstanbul, Turkey

They obtained some stability results in the case $0 < \mu_2 < \mu_1$. In the absence of delay, Zuazua [27] looked into exponentially for the equation (2).

In [8], Kafini and Messaoudi considered the wave equation with nonlinear source term and constant time delay as follows

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = |u|^{p-2}u. \tag{3}$$

They established the local existence and blow up of solution for positive initial energy of the equation (3).

In [11], Messaoudi studied the equation as follows

$$u_{tt} + \Delta^2 u + |u_t|^{m-2}u_t = |u|^{p-2}u, \tag{4}$$

and obtained the existence results and obtained that, if $m \geq p$, the solution is globally and blows up in finite time if $m < p$. Later, Chen and Zhou [4] extended this result. In the presence of the strong damping term $(-\Delta u_t)$, Polat and Pişkin [22] proved the global existence and decay of solutions for the equation (4).

Xu et al. [23], studied the plate equation with nonlinear damping and source term as follows

$$u_{tt} + \Delta^2 u + \mu|u_t|^{q-2}u_t + au = |u|^{p-2}u, \tag{5}$$

with $2 < q < p, \mu > 0,$ and they proved the well-posedness, decay estimates and blow-up of solution at both subcritical ($E(0) < d$) and critical ($E(0) = d$) initial energy levels. Furthermore, when $p > 2$, $q = 2$ and $\mu > 0$, they established that the solution blows up in finite time at the supercritical initial energy level ($E(0) > d$).

In [2], Al-Gharabli and Messaoudi concerned with the plate equation with logarithmic term as follows

$$u_{tt} + \Delta^2 u + u + h(u_t) = ku \ln|u|. \tag{6}$$

They established the existence results by the Galerkin method and obtained the explicit and decay of solutions utilizing the multiplier method for the equation (6). In recent years, some other authors investigate hyperbolic type equations (see [3, 14–20, 24–26]).

To our best knowledge, there is no research on the plate equation with frictional damping, nonlinear source and time delay. The aim of the present paper is to establish the sufficient conditions for the local existence and nonexistence of global solutions to the plate equation with time delay.

The paper is organized as follows: In section 2, we give some materials that will be used later. In section 3, we establish the local existence by using the semigroup theory similar to the work of Kafini and Messaoudi [8]. In section 4, we prove the nonexistence of global solutions for positive initial energy.

PRELIMINARIES

In this part, we introduce some needed materials for the proof of our result. As usual, the notation $\|.\|_p$ denotes L^p norm, and (.,.) is the L^2 inner product. In particular, we write $\|.\|$ instead of $\|.\|_2$.

Let $B_p > 0$ be the constant satisfying [1, 21]

$$\|\nabla v\|_p \leq B_p\|\Delta v\|_p, \text{ for } v \in H_0^2(\Omega.) \tag{7}$$

Similar to the [12], we introduce the new function

$$z(x,\rho,t) = u_t(x,t-\tau\rho),\ x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Hence, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0,\ x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Thus, the problem (1) transforms into:

$$\begin{cases} u_{tt} + \Delta^2 u + \mu_1 u_t(x,t) + \mu_2 (z,1,t) = |u|^{p-2}u, & in\Omega\times(0,\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & in\Omega\times(0,1)\times(0,\infty), \\ z(x,\rho,0) = f_0(x,-\rho\tau), & in\Omega\times(0,1), \\ u(x,t) = \dfrac{\partial u(x,t)}{\partial v} = 0, & x\in\partial\Omega\times[0,1), t\in[0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & in\Omega. \end{cases} \tag{8}$$

The energy functional related to the problem (8) is

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{\xi}{2}\int_\Omega\int_0^1 |z(x,\rho,t)|^2 d\rho dx - \frac{1}{p}\|u\|_p^p,\ \forall t \geq 0, \tag{9}$$

where

$$\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|),\ |\mu_2| < \mu_1. \tag{10}$$

Now, we give the technical lemmas as follows:

Lemma 1. *The solution of (8) satisfies*

$$E'(t) \leq -C_0 \int_\Omega \left(|u_t|^2 + |z(x,1,t)|^2\right) dx \leq 0, \tag{11}$$

for some $C_0 > 0$.

Proof. We multiply the first equation in (8) by u_t and integrating over Ω, and use integration by parts, to get

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 - \frac{1}{p}\|u\|_p^p\right) \\ &= -\mu_1 \int_\Omega |u_t(t)|^2 dx - \mu_2 \int_\Omega u_t z(x,1,t) dx. \end{aligned} \tag{12}$$

Also, we multiply the second equation in (8) by $(\xi/\tau)z$ and integrate over $\Omega \times (0,1)$, $\xi > 0$, to have

$$\frac{\xi}{2}\frac{d}{dt}\int_\Omega\int_0^1 |z(x,\rho,t)|^2\,d\rho dx + \frac{\xi}{\tau}\int_\Omega\int_0^1 z(x,\rho,t)z_\rho(x,\rho,t)\,d\rho dx = 0. \tag{13}$$

We note that

$$\begin{aligned}-\frac{\xi}{\tau}\int_\Omega\int_0^1 z(x,\rho,t)z_\rho(x,\rho,t)\,d\rho dx &= -\frac{\xi}{2\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}z^2(x,\rho,t)\,d\rho dx\\ &= \frac{\xi}{2\tau}\int_\Omega (z^2(x,0,t)-z^2(x,1,t))dx\\ &= \frac{\xi}{2\tau}\left(\int_\Omega u_t^2 dx - \int_\Omega z^2(x,1,t)dx\right).\end{aligned} \tag{14}$$

By combining (12) and (13) and taking into consideration (14), we obtain

$$\begin{aligned}E'(t) = &-\left(\mu_1 - \frac{\xi}{2\tau}\right)\int_\Omega |u_t(x,t)|^2 dx - \frac{\xi}{2\tau}\int_\Omega |z(x,1,t)|^2 dx\\ &-\mu_2\int_\Omega u_t z(x,1,t)dx,\end{aligned} \tag{15}$$

for $t \in (0,T)$.

Utilizing Young's inequality, we estimate

$$-\mu_2\int_\Omega u_t z(x,1,t)dx \le \frac{|\mu_2|}{2}\int_\Omega \left(|u_t|^2 + |z(x,1,t)|^2\right)dx. \tag{16}$$

Therefore, by (15), we get

$$\begin{aligned}E'(t) \le &-\left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_\Omega |u_t|^2 dx\\ &-\left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_\Omega |z(x,1,t)|^2 dx.\end{aligned} \tag{17}$$

From (10), for some $C_0 > 0$, we have

$$E'(t) \le -C_0\int_\Omega \left(|u_t(x,t)|^2 + |z(x,1,t)|^2\right)dx \le 0. \tag{18}$$

Now, we introduce quantities as follows:

$$\alpha = B^{\frac{-p}{p-2}} \text{ and } E_1 = \left(\frac{1}{2}-\frac{1}{p}\right)\alpha^2, \tag{19}$$

where B is the constant of the Sobolev embedding $H_0^2(\Omega) \hookrightarrow L^p(\Omega)$.

Lemma 2. *Let u be a solution of (8), with initial data, such that*

$$E(0) < E_1 \text{ and } \left(\|\Delta u_0\|^2 + \xi\int_\Omega\int_0^1 |f_0(x,-\rho\tau)|^2\,d\rho dx\right)^{\frac{1}{2}} > \alpha, \tag{20}$$

Then, there exists a constant $\beta > \alpha$, so that

$$\left[\|\Delta u\|^2 + \xi\int_\Omega\int_0^1 |z(x,\rho,t)|^2\,d\rho dx\right]^{\frac{1}{2}} \ge \beta, \forall t \ge 0, \tag{21}$$

and

$$\|u\|_p \ge \beta B. \tag{22}$$

Proof. By (9), we have

$$\begin{aligned}E(t) &\ge \frac{1}{2}\|\Delta u\|^2 + \frac{\xi}{2}\int_\Omega\int_0^1 |z(x,\rho,t)|^2\,d\rho dx - \frac{1}{p}\|u\|_p^p\\ &\ge \frac{1}{2}\|\Delta u\|^2 + \frac{\xi}{2}\int_\Omega\int_0^1 |z(x,\rho,t)|^2\,d\rho dx - \frac{1}{p}B^p\|\Delta u\|_2^p\\ &\ge \left(\frac{1}{2}\|\Delta u\|^2 + \frac{\xi}{2}\int_\Omega\int_0^1 |z(x,\rho,t)|^2\,d\rho dx\right)\\ &\quad -\frac{B^p}{p}\left(\|\Delta u\|^2 + \xi\int_\Omega\int_0^1 |z(x,\rho,t)|^2\,d\rho dx\right)^{\frac{p}{2}}\\ &= \frac{1}{2}\varsigma^2 - \frac{B^p}{p}\varsigma^p = h(\varsigma),\end{aligned} \tag{23}$$

where

$$\varsigma = \left[\|\Delta u\|^2 + \xi\int_\Omega\int_0^1 |z(x,\rho,t)|^2\,d\rho dx\right]^{\frac{1}{2}}. \tag{24}$$

We see that, h is increasing for $0 < \varsigma < \alpha$, decreasing for $\varsigma > \alpha$,

$$h(\varsigma) \to -\infty \text{ as } \varsigma \to +\infty$$

and

$$h(\alpha) = \left(\frac{1}{2}-\frac{1}{p}\right)\alpha^2. \tag{25}$$

Thus, since $E(0) < E_1$, there exists $\beta > \alpha$ so that $h(\beta) = E(0)$. Setting

$$\alpha_0 = \left[\|\Delta u_0\|^2 + \xi \int_\Omega \int_0^1 \left| f_0(x, -\rho\tau) \right|^2 d\rho dx \right]^{\frac{1}{2}}$$

Then, from (23), we obtain

$$h(\alpha_0) < E(0) = h(\beta).$$

As a result, $\alpha_0 > \beta$.

Establishing (21), we assume, by contradiction, there exists $t_0 > 0$ such that

$$\left[\|\Delta u(t_0)\|^2 + \xi \int_\Omega \int_0^1 \left| z(x, \rho, t_0) \right|^2 d\rho dx \right]^{\frac{1}{2}} < \beta.$$

From continuity of,

$$\left\|\Delta u^2\right\| + \xi \int_\Omega \int_0^1 \left| z(x, \rho, t) \right|^2 d\rho dx,$$

choosing t_0 such that, we have

$$\left[\|u(t_0)\|^2 + \xi \int_\Omega \int_0^1 |z(x, \rho, t_0)|^2 d\rho dx \right]^{\frac{1}{2}} > \alpha.$$

By using (23), we conclude that

$$E(t_0) \geq h\left(\left[\|\Delta u(t_0)\|^2 + \xi \int_\Omega \int_0^1 |z(x, \rho, t_0)|^2 d\rho dx \right]^{\frac{1}{2}} \right)$$
$$> h(\beta) = E(0).$$

Since $E(t) \leq E(0)$, this is impossible, for all $t \in [0,T)$. Therefore, (21) is established.

To prove (22), we exploit (18) and (21) to get

$$\frac{1}{2}\|\Delta u\|^2 + \frac{\xi}{2} \int_\Omega \int_0^1 |z(x, \rho, t)|^2 d\rho dx \leq E(0) + \frac{1}{p}\|u\|_p^p.$$

Hence,

$$\frac{1}{p}\|u\|_p^p \geq \frac{1}{2}\|\Delta u\|^2 + \frac{\xi}{2} \int_\Omega \int_0^1 |z(x, \rho, t)|^2 d\rho dx - E(0)$$
$$\geq \frac{1}{2}\beta^2 - E(0) \geq \frac{1}{2}\beta^2 - h(\beta) = \frac{(\beta B)^p}{p},$$

from which (22) follows.

Similar to the work of [10] and by using Sobolev Embedding theorem, we have the following lemma:

Lemma 3. *Assume that* $2 < p \leq \frac{2(n-2)}{n-4}$, $n \geq 5$ *and* $|\mu_2| \leq \mu_1$ *hold. Then, there exists a positive constant* $C > 1$ *such that*

$$\|u\|_p^s \leq C\left(\|\Delta u\|^2 + \|u\|_p^p \right) \tag{26}$$

for any $u \in H_0^2$ *and* $2 \leq s \leq p$.

Lemma 4. *Let u be the solution of (8). Suppose that* $2 < p \leq \frac{2(n-2)}{n-4}$, $n \geq 5$ *and* $|\mu_2| \leq \mu_1$ *hold.*

Then, for any $2 \leq s \leq p$, *we obtain*

$$\|u\|_p^s \leq C\left(-H(t) - \|u_t\|^2 - \frac{\xi}{2} \int_\Omega \int_0^1 |z(x, \rho, t)|^2 d\rho dx + \|u\|_p^p \right), \tag{27}$$
$$\forall t \in [0, T),$$

where

$$H(t) = E_1 - E(t), \qquad t \geq 0. \tag{28}$$

Proof. From (9), noting that

$$\frac{1}{2}\|\Delta u\|^2 = E(t) - \frac{1}{2}\|u_t\|^2 - \frac{\xi}{2} \int_\Omega \int_0^1 |z(x, \rho, t)|^2 d\rho dx + \frac{1}{p}\|u\|_p^p$$
$$= E_1 - H(t) - \frac{1}{2}\|u_t\|^2 - \frac{\xi}{2} \int_\Omega \int_0^1 |z(x, \rho, t)|^2 d\rho dx + \frac{1}{p}\|u\|_p^p.$$

By exploiting (19), (22) and reminding that $\beta > \alpha$, we get

$$\frac{p-2}{2p}\|u\|_p^p - E_1 = \frac{p-2}{2p}\|u\|_p^p \left[1 - \alpha^2 u_p^{-p} \right]$$
$$> \frac{p-2}{2p}\|u\|_p^p \left[1 - \beta^2 (\beta B)^{-p} \right]$$
$$= \frac{p-2}{2p}\|u\|_p^p \left[1 - \beta^{2-p} B^{-p} \right] > 0.$$

Therefore,

$$E_1 < \frac{p-2}{2p}\|u\| \tag{29}$$

and the result is obtained.

LOCAL EXISTENCE

In this part, we establish the local existence result by using the semigroup theory. Assume that $v = u_t$ and denote by

$$\Phi = (u,v,z)^T, \Phi(0) = \Phi_0 = (u_0,u_1,f_0(.,-\rho\tau))^{\mathrm{T}}$$
$$\text{and } J(\Phi) = \left(0,|u|^{p-2}u,0\right)^T.$$

Hence, (8) can be written as an initial-value problem:

$$\begin{cases} \partial_t\Phi + \breve{A}\Phi = J(\Phi) \\ \Phi(0) = \Phi_0, \end{cases} \tag{30}$$

where the linear operator $\breve{A}: D(\breve{A}) \to \mathcal{H}$ is defined by

$$\tilde{A}\Phi = \begin{pmatrix} -v \\ \Delta^2 u + \mu_1 v + \mu_2 z(1,.) \\ \frac{1}{\tau} z_\rho \end{pmatrix}.$$

The state space of Φ is the Hilbert space

$$\mathcal{H} = H_0^2(\Omega)\times L^2(\Omega)\times L^2(\Omega\times(0,1)),$$

equipped with the inner product

$$\left\langle \Phi,\tilde{\Phi}_{\mathcal{H}} \right\rangle = \int_\Omega (\Delta u\Delta\tilde{u} + v\tilde{v})dx + \tau|\mu_2|\int_0^1\int_\Omega z\tilde{z}dxd\rho,$$

for all $\Phi = (u,v,z)^T$ and $\tilde{\Phi} = (\tilde{u},\tilde{v},\tilde{z})^T$ in $\mathcal{H}$. The domain of $\breve{A}$ is

$$D(\breve{A}) = \left\{ \begin{matrix} \Phi\in\mathcal{H}: u\in H^4(\Omega), & v\in H_0^1(\Omega), & z(1,.)\in L^2(\Omega), \\ z,z_\rho \in L^2(\Omega\times(0,1)), & & z(0,.) = v \end{matrix} \right\}.$$

Now, we give the local existence theorem as follows:

Theorem 5. *Suppose that* $\mu_1 \geq |\mu_2|$ *and*

$$2 < p \leq \frac{2(n-2)}{n-4}, n \geq 5. \tag{31}$$

Then, for any $\Phi_0 \in \mathcal{H}$*, the problem (30) has a unique weak solution* $\Phi \in C(R^+;\mathcal{H})$.

Proof. For all $\Phi \in D(\breve{A})$, we obtain

$$\begin{aligned} \left\langle \breve{A}\Phi,\Phi \right\rangle_{\mathcal{H}} &= -\int_\Omega \Delta v\Delta u dx + \int_\Omega v[\Delta^2 u + \mu_1 v + \mu_2 z(1,.)]dx \\ &+ |\mu_2|\int_0^1\int_\Omega zz_\rho dxd\rho \\ &= \mu_1\int_\Omega |v|^2 dx + \mu_2\int_\Omega vz(1,.)dx \\ &+ \frac{|\mu_2|}{2}\int_\Omega |z(1,.)|^2 dx - \frac{|\mu_2|}{2}\int_\Omega |v|^2 dx. \end{aligned} \tag{32}$$

Utilizing Young's inequality, estimate (32) becomes

$$\begin{aligned} \left\langle \tilde{A}\Phi,\Phi_{\mathcal{H}} \right\rangle &\geq \mu_1\int_\Omega |v|^2 dx - \frac{|\mu_2|}{2}\int_\Omega |v|^2 dx - \frac{|\mu_2|}{2}\int_\Omega |v|^2 dx \\ &\geq (\mu_1 - |\mu_2|)\int |v|^2 dx \geq 0. \end{aligned}$$

Hence, $\breve{A}$ is a monotone operator.
To show that $\breve{A}$ is maximal, we prove that for each

$$F = (f,g,h)^T \in \mathcal{H},$$

there exists $V = (u,v,z)^T \in D(\breve{A})$ so that $(I + \breve{A})V = F$. Thus,

$$\begin{cases} u - v = f \\ v + \Delta^2 u + \mu_1 v + \mu_2 z(1,.) = g \\ \tau z + z_\rho = \tau h. \end{cases} \tag{33}$$

We note that $v = u - f$, from the third equation of (33), we conclude that

$$z(\rho,.) = (u-f)e^{-\rho\tau} + \tau e^{-\rho\tau}\int_0^\rho h(\gamma,.)e^{\gamma\tau}d\gamma. \tag{34}$$

By substituting (34) in the second equation of (33), we have

$$ku + \Delta^2 u = G,$$

Where

$$k = 1 + \mu_1 + \mu_2 e^{-\tau}, G = g + kf - \tau\mu_2 e^{-\tau}\int_0^1 h(\gamma,.)e^{\gamma\tau}d\gamma \in L^2(\Omega). \tag{35}$$

Now, we define, over $H_0^2(\Omega)$, the bilinear and linear forms:

$$B(u,w)=k\int_\Omega uw+\int_\Omega \Delta u\Delta w,\ \ L(w)=\int_\Omega Gw.$$

We see that B is coercive and continuous, and L is continuous on $H_0^2(\Omega)$. Then, Lax-Milgram theorem specifies that the equation

$$B(u,w) = \mathrm{L}(w), \forall w \in H_0^2(\Omega), \tag{36}$$

has a unique solution $u \in H_0^2(\Omega)$. Therefore, $v = u - f \in H_0^2(\Omega)$. As a result, by (34), we have $z, z_\rho \in L^2(\Omega \times (0,1))$. Hence, $V \in \mathcal{H}$.

By using (36), we obtain

$$k\int_\Omega uw+\int_\Omega \Delta u\Delta w=\int_\Omega G(w),\ \forall w\in H_0^2(\Omega).$$

From the elliptic regularity theory, we have $u \in H^4(\Omega)$ and utilizing Green's formula and the second equation of (33) give

$$\int_\Omega [(1+\mu_1)u+\Delta^2 u+\mu_2 z(1,.)-g]w=0,\ \ \forall w\in H_0^2(\Omega).$$

Thus,

$$(1 + \mu_1)u + \Delta^2 u + \mu_2\, z(1,.) = \mathrm{g} \in L^2(\Omega).$$

Hence,

$$V = (u,v,z)^{\mathrm{T}} \in D(\breve{A}).$$

As a result, $I + \breve{A}$ is surjective and then $\breve{A}$ is maximal.

Consequently, we denote that $J:\mathcal{H}\to\mathcal{H}$ is locally Lipschitz. Hence,

$$\begin{aligned}\left\|J(\Phi)-J(\tilde{\Phi})\right\|_{\mathcal{H}}^2&=\left\|\left(0,|u|^{p-2}u-|\tilde{u}|^{p-2}\tilde{u},0\right)\right\|_{\mathcal{H}}^2\\&=\left\||u|^{p-2}u-|\tilde{u}|^{p-2}\tilde{u}\right\|_{L^2}^2\\&=\int_\Omega\left||u|^{p-2}u-|\tilde{u}|^{p-2}\tilde{u}\right|^2dx.\end{aligned}$$

From the mean value theorem, we obtain

$$\left\|J(\Phi)-J(\tilde{\Phi})\right\|_{\mathcal{H}}^2=(p-1)^2\int_\Omega|\theta u+(1-\theta)\tilde{u}|^{2(p-2)}|u-\tilde{u}|^2\,dx,$$

$0\le\theta\le1.$

As $u,\ \tilde{u} \in H_0^2(\Omega)$ utilizing Sobolev embedding $H_0^2(\Omega)\hookrightarrow L^{2^*}, 2^*=\dfrac{2n}{n-4}$, Hölder's inequality and (31), we get

$$\begin{aligned}\left\|J(\Phi)-J(\tilde{\Phi})\right\|_{\mathcal{H}}^2&\le(p-1)^2\left(\int_\Omega|u-\tilde{u}|^{\frac{2n}{n-4}}dx\right)^{\frac{n-4}{n}}\\&\quad\times\left(\int_\Omega\left|\theta u+\left(1-\theta\right)\tilde{u}\right|^{\frac{n}{2}(p-2)}dx\right)^{\frac{4}{n}}\\&\le\left(p-1\right)^2\|u-\tilde{u}\|_{L^{2^*}}^2\left\|\theta u+\left(1-\theta\right)\tilde{u}\right\|_{L^{\frac{n}{2}(p-2)}}^{2(p-2)}\\&\le\left(p-1\right)^2\|u-\tilde{u}\|_{L^{2^*}}^2\left\|\theta u+\left(1-\theta\right)\tilde{u}\right\|_{L^{n(p-2)}}^{2(p-2)}\\&\le C\|u-\tilde{u}\|_{H_0^2(\Omega)}^2\left(\|u\|_{L^{n(p-2)}}+\|\tilde{u}\|_{L^{n(p-2)}}\right)^{2(p-2)},\end{aligned}$$

where, we used $L^{n(p-2)}(\Omega)\hookrightarrow L^{\frac{n}{2}(p-2)}(\Omega)$.

Therefore,

$$\left\|J(\Phi)-J(\Phi)\right\|_{\mathcal{H}}^2\le C\left(\|u\|_{H_0^2(\Omega)}+\|\tilde{u}\|_{H_0^2(\Omega)}\right)^{2(p-2)}\left\|\Phi-\tilde{\Phi}\right\|_{\mathcal{H}}^2.$$

Hence, J is locally Lipschitz. Similar to the theorems in Komornik [9] (See also Pazy [13], we completed the proof.)

NONEXISTENCE OF SOLUTIONS

In this part, we prove the nonexistence of global solutions for positive initial energy of the problem (8).

Theorem 6. *Suppose that* $2<p\le\dfrac{2(n-2)}{n-4}, n\ge5$ *and* $|\mu_2| \le |\mu_1|$ *hold. Then, the solution of (8), with initial data satisfying (20), blows up in finite time.*

Proof. From (9), (18) and (28), we get

$$\begin{aligned}0&<H(0)\le H(t)\\&\le E_1-\frac{1}{2}\left[\|u_t\|^2+\Delta^2u+\frac{\xi}{2}\int_\Omega\int_0^1|z(x,\rho,t)|^2d\rho dx\right]+\frac{1}{p}\|u\|_p^p.\end{aligned}\tag{37}$$

and, by (19) and (21), we deduce

$$\begin{aligned}&E_1-\frac{1}{2}\left[\|u\|_t^{\,2}+\Delta^2u+\frac{\xi}{2}\int_\Omega\int_0^1\left|z\left(x,\rho,t\right)\right|^2d\rho dx\right]\\&\le E_1-\frac{1}{2}\beta^2\le\left(\frac{1}{2}-\frac{1}{p}\right)\alpha^2-\frac{1}{2}\beta^2\le\left(\frac{1}{2}-\frac{1}{p}\right)\beta^2-\frac{1}{2}\beta^2\\&=-\frac{1}{p}\beta^2<0,\forall t\ge0.\end{aligned}$$

Therefore,

$$0 < H(0) \le H(t) \le \frac{1}{p}\|u\|_p^p, \forall t \ge 0. \tag{38}$$

We define

$$L(t) = H^{1-\sigma}(t) + \varepsilon \int_\Omega u u_t dx,$$

for small ε to be chosen later and for

$$0 < \sigma \le \frac{p-2}{2p}. \tag{39}$$

By taking a derivative of $L(t)$ and utilizing (8) and (18), we obtain

$$\begin{aligned} L'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon \int_\Omega |u_t|^2 + \varepsilon \int_\Omega u u_{tt} dx \\ &\ge C_0(1-\sigma)H^{-\sigma}(t)\left[\|u_t\|^2 + \|z(x,1,t)\|^2\right] \\ &+\varepsilon\left[\|u_t\|^2 - \|\Delta u\|^2 + \varepsilon\mu_1 \int_\Omega u u_t dx\right] \\ &+\varepsilon\mu_2 \int_\Omega u z(x,1,t) dx + \varepsilon\|u\|_p^p. \end{aligned} \tag{40}$$

By exploiting (37), we have

$$\begin{aligned} L'(t) &\ge C_0(1-\sigma)H^{-\sigma}(t)\left[\|u_t\|^2 + \|z(x,1,t)\|^2\right] + \varepsilon\left[\|u_t\|^2 - \|\Delta u\|^2\right] \\ &+\varepsilon\left(2H(t) - 2E_1 + \|u_t\|^2 + \|\Delta u\|^2 + \xi \int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx\right) \\ &+\varepsilon\left(1 - \frac{2}{p}\right)\|u\|_p^p + \varepsilon\mu_1 \int_\Omega u u_t dx + \varepsilon\mu_2 \int_\Omega u z(x,1,t) dx \end{aligned}$$

and (22), to get

$$\begin{aligned} L'(t) &\ge C_0(1-\sigma)H^{-\sigma}(t)\left[\|u_t\|^2 + \|z(x,1,t)\|^2\right] + 2\varepsilon\|u_t\|^2 \\ &+2\varepsilon H(t) + \tilde{c}\varepsilon\|u\|_p^p + \varepsilon\xi \int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx \\ &+\varepsilon\mu_1 \int_\Omega u u_t dx + \varepsilon\mu_2 \int_\Omega u z(x,1,t) dx, \end{aligned} \tag{41}$$

where, thanks to (29), $\tilde{c} > 0$.
Reminding Young's inequality

$$XY \le \frac{\delta^2}{2}X^2 + \frac{\delta^{-2}}{2}Y^2, \; X,Y \ge 0, \; \forall \delta > 0,$$

we estimate the last two terms as follows

$$\int_\Omega u u_t dx \le \frac{\delta^2}{2}\|u\|^2 + \frac{\delta^{-2}}{2}\|u_t\|^2 \tag{42}$$

and

$$\int_\Omega u z(x,1,t) dx \le \frac{\delta^2}{2}\|u\|^2 + \frac{\delta^{-2}}{2}\|z(x,1,t)\|^2. \tag{43}$$

Similar to [10], the estimates (42) and (43) remain valid even if δ is time dependent. Hence, taking δ such that

$$\delta^{-2} = kH^{-\sigma}(t), \tag{44}$$

for large k to be specified later, we obtain

$$\begin{aligned} \frac{\delta^{-2}}{2}\|z(x,1,t)\|^2 &= \frac{k}{2}H^{-\sigma}(t)\|z(x,1,t)\|^2, \\ \frac{\delta^{-2}}{2}\|u_t\|^2 &= \frac{k}{2}H^{-\sigma}(t)\|u_t\|^2. \end{aligned}$$

And

$$\frac{\delta^2}{2}\|u\|^2 \le \frac{k^{-1}}{2}H^\sigma(t)\|u\|^2.$$

By using the inequality $||u||^2 \le C||u||_p^2$ and exploiting (38), we reach

$$H^\sigma(t)\|u\|^2 \le \left(\frac{1}{p}\right)^\sigma C\|u\|_p^{2+\sigma p}. \tag{45}$$

Hence, (41) becomes

$$\begin{aligned} L'(t) &\ge \left[C_0(1-\sigma) - \varepsilon\frac{\mu_1 k}{2}\right]H^{-\sigma}(t)\|u_t\|^2 \\ &+\left[C_0(1-\sigma) - \varepsilon\frac{\mu_2 k}{2}\right]H^{-\sigma}(t)\|z(x,1,t)\|^2 \\ &+\varepsilon\left[pH(t) - (\mu_1+\mu_2)\frac{k^{-1}}{2}\left(\frac{1}{p}\right)^\sigma C\|u\|_p^{2+\sigma p}\right] \\ &+\widetilde{c\varepsilon}\|u\|_p^p + \varepsilon\left(1+\frac{p}{2}\right)u_t^2 \\ &+\varepsilon\frac{p\xi}{2}\int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx. \end{aligned} \tag{46}$$

Since, from (39), $s = 2 + \sigma p \le p$, then Lemma 4 satisfies

$$\begin{aligned} L'(t) \ge & \left[C_0\left(1-\sigma\right) - \varepsilon \frac{\mu_1 k}{2} \right] H^{-\sigma}(t) \left\| u_t \right\|^2 \\ & + \left[C_0(1-\sigma) - \varepsilon \frac{\mu_2 k}{2} \right] H^{-\sigma}(t) \left\| z(x,1,t) \right\|^2 \\ & + \varepsilon(\tilde{c} - C_1 k^{-1}) \left\| u \right\|_p^p + \varepsilon(p + C_1 k^{-1}) H(t) \\ & + \varepsilon\left(1 + \frac{p}{2} + C_1 k^{-1} \right) \left\| u_t \right\|^2 \\ & + \varepsilon\left(C_1 k^{-1} + \frac{p\xi}{2} \right) \int_\Omega \int_0^1 \left| z(x,\rho,t) \right|^2 d\rho dx, \end{aligned} \tag{47}$$

where $C_1 = \frac{C}{2}\left(\mu_1 + \mu_2\right)\left(\frac{1}{p}\right)^\sigma$.

For large enough and for any $\varepsilon > 0$, (47) becomes

$$\begin{aligned} L'(t) \ge & \left[C_0\left(1-\sigma\right) - \varepsilon \frac{\mu_1 k}{2} \right] H^{-\sigma}(t) \left\| u_t \right\|^2 \\ & + \left[C_0(1-\sigma) - \varepsilon \frac{\mu_2 k}{2} \right] H^{-\sigma}(t) \left\| z(x,1,t) \right\|^2 \\ & + \gamma \left(H\left(t\right) + \left\| u \right\|_t^2 + \int_\Omega \int_0^1 \left| z\left(x,\rho,t\right) \right|^2 d\rho dx + \left\| u \right\|_p^p \right), \end{aligned} \tag{48}$$

where $\gamma > 0$. Then, choosing ε small enough, such that

$$C_0(1-\sigma) - \varepsilon \frac{\mu_1 k}{2} > 0 \text{ and } C_0(1-\sigma) - \varepsilon \frac{\mu_2 k}{2} > 0.$$

And

$$L(0) = H^{1-\sigma}(0) + \varepsilon \int_\Omega u_0 u_1 dx,$$

will make

$$L'(t) \ge \gamma \left(H(t) + \left\| u_t \right\|^2 + \int_\Omega \int_0^1 \left| z(x,\rho,t) \right|^2 d\rho dx + \left\| u \right\|_p^p \right). \tag{49}$$

As a result, we conclude that

$$L(\mathrm{t}) \ge L(0) > 0, \; t \ge 0.$$

Now, we get, from Hölder's inequality

$$\int_\Omega u u_t dx \le C \left\| u \right\|_2 \left\| u_t \right\|_2 \le C \left\| u \right\|_p \left\| u_t \right\|_2,$$

and from Young's inequality

$$\left| \int_\Omega u u_t dx \right|^{1/(1-\sigma)} \le C\left(\left\| u \right\|_p^{\frac{\mu}{(1-\sigma)}} + \left\| u_t \right\|_2^{\frac{\theta}{(1-\sigma)}} \right), for \frac{1}{\mu} + \frac{1}{\theta} = 1. \tag{50}$$

By Lemma 4, taking $\theta = 2(1 - \sigma)$ which satisfies $\frac{\mu}{(1-\sigma)} = \frac{2}{(1-2\sigma)} \le p$. Hence, (50) becomes

$$\left| \int_\Omega u u_t dx \right|^{1/(1-\sigma)} \le C\left(\left\| u \right\|_p^s + \left\| u_t \right\|_2^2 \right),$$

for $s = \frac{2}{\left(1-2\sigma\right)}$. From Lemma 4, we get

$$\begin{aligned} & \left| \int_\Omega u u_t dx \right|^{1/(1-\sigma)} \\ & \le C\left[H(t) + \left\| u_t \right\|^2 + \frac{\xi}{2} \int_\Omega \int_0^1 \left| z(x,\rho,t) \right|^2 d\rho dx + \left\| u \right\|_p^p \right]. \end{aligned} \tag{51}$$

As a result, we obtain

$$\begin{aligned} L^{1/(1-\sigma)}(t) = & \left(H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t dx \right)^{1/(1-\sigma)} \\ & \le 2^{\frac{1}{(1-\sigma)}} \left[H(t) + \left(\int_\Omega u u_t dx \right)^{\frac{1}{(1-\sigma)}} \right] \\ & \le C\left[H(t) + \left\| u_t \right\|^2 + \frac{\xi}{2} \int_\Omega \int_0^1 \left| z(x,\rho,t) \right|^2 d\rho dx + \left\| u \right\|_p^p \right], \\ & t \ge 0. \end{aligned} \tag{52}$$

By combining (49) and (52), we conclude that

$$L'(t) \ge \Lambda L^{\frac{1}{(1-\sigma)}}(t), t \ge 0, \tag{53}$$

where Λ is a positive constant depending only on γ and C. Taking a simple integration of (53) over $(0,t)$ satisfies

$$L^{\sigma/(1-\sigma)}(t) \ge \frac{1}{L^{-\frac{\sigma}{(1-\sigma)}}\left(0\right) - \frac{\Lambda \sigma t}{\left(1-\sigma\right)}}.$$

Thus, $L(t)$ blows up in time

$$T \leq T^{*} = \frac{1-\sigma}{\Lambda \sigma L^{\frac{\sigma}{(1-\sigma)}}(0)}.$$

Hence, the proof is completed.

CONCLUSION

In recent years, there has been published much work concerning the wave equations (Kirchhoff, Petrovsky, Bessel,... etc.) with different state of delay time (constant delay, time-varying delay,... etc.). However, to the best of our knowledge, there were no local existence and nonexistence of global solutions for the plate equation with time delay. Firstly, we have been obtained the local existence by using the semigroup theory. Later, we have been proved the nonexistence of global solutions for positive initial energy.

ACKNOWLEDGEMENT

The authors are grateful to DUBAP (ZGEF.20.009) for research funds.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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