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Research Article

On the rough hausdorff convergence

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ABSTRACT

Apreutesei [1] developed the concept of norm given with the help of the Hausdorff distance from a set A to $\{0\}$ in the almost linear space. This space consists of sets that do not hold the inverse element property with respect to the Minkowski sum. In this paper, we first prove that the rough Hausdorff convergence of a sequence $\{A_n\}$ of sets to the set A requires the rough convergence of the sequence of norms. Then we give the necessary and sufficient conditions for the rough convergence of the sequence $\|A_n - A\|$ to 0.

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INTRODUCTION

Hausdorff [7] introduced the concept of the Hausdorff convergence of sequences of sets in 1927 by defining a function that calculates the distance between two sets. Wijsman [17,18] defined the notion of Wijsman convergence, which corresponds to the pointwise convergence of sequences made up of distance functions. Apreutesei [1] gave some algebraic properties of the Hausdorff convergence. He [2] also showed that Wijsman convergence and Hausdorff convergence are equivalent to each other for monotone sequences of compact sets. Nuray and Rhoades [9] generalized the concepts of Wijsman convergence and Hausdorff convergence using the theory of statistical convergence. Talo and Sever [16] extended the notion of Hausdorff convergence to ideal convergence. Hazarika and Esi [8] defined the idea of asymptotically equivalent sequences of sets in the sense of ideal Wijsman convergence. Nuray et al. [10] extended the definition of Wijsman statistical convergence to double sequences. Moreover, Dündar and Talo [6] introduced the concept of the Wijsman regularly invariant convergence of double sequences of sets.

In 2001, the idea of rough convergence of a sequence was first given by Phu [14] in normed linear spaces. Aytar [3] gave a new definition by combining rough convergence theory with the statistical convergence theory. Debnath and Rakshit [5] obtained some results related to the rough limit set in metric spaces.

Recently, the rough convergence theory has started to be applied in set theory as well. Ölmez and Aytar [12] applied the rough convergence theory to the theory of Wijsman convergence. Ölmez et al. [13] gave the equivalent definitions of rough Wijsman convergence and rough Hausdorff convergence. Subramanian and Esi [15] extended the definition of rough Wijsman convergence to triple sequences.

In this paper, we give the necessary and sufficient conditions for the rough convergence of the sequence $\|A_n - A\|$

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to 0. The proof techniques are similar to that of the classical case (see [1]).

PRELIMINARIES

Throughout this paper, let $(X,\|.\|)$ be a normed linear space. Let $P(X)$, $Pb(X)$ and $K(X)$ be all nonempty subsets, nonempty bounded subsets and compact subsets of X, respectively.

The purpose of the present section is to recall some basic concepts.

Let $A \subset X$. For $x \in X$, the *distance* from x to the set A is defined by

$$d(x,A) = \inf_{a\in A}\|x-a\| \text{ [7]}.$$

The *open ball* with centre $a \in X$ and radius $r > 0$ is the set

$$S(a,r) = \{x \in X : \|a - x\| < r\}.$$

Let r be a nonnegative real number. The sequence $\{x_n\}$ is said to be *rough convergent* to x with the roughness degree r, denoted by $x_n \xrightarrow{r} x$, if for each $\varepsilon > 0$ there exists an $n_0(\varepsilon) \in N$ such that $\|x_n - x\| < r + \varepsilon$ for each $n \geq n_0$ [14].

Throughout this paper, we assume that $A_n \subset X$ for each $n \in N$. The sequence $\{A_n\}$ of sets is said to be *r – Hausdorff convergent* (or *rough Hausdorff convergent* with the roughness degree r) to the set A if for every $\varepsilon > 0$ there exists an $n_0(\varepsilon) \in \mathrm{N}$ such that

$$H(A_n,A) = \max\{h(A_n,A),h(A,A_n)\} < r + \varepsilon \text{ for all } n \geq n_0,$$

where $h(A_n,A) = \sup_{a\in A_n} d(a,A)$ and $h(A,A_n) = \sup_{a\in A} d(a,A_n)$.

In this case, we write $A_n \xrightarrow{r-H} A$ [13].

An alternative definition of the rough Hausdorff convergence can be given by the following:

$$A_n \xrightarrow{r-H} A \Leftrightarrow \text{for every } \varepsilon>0 \text{ there exists an } n_0(\varepsilon)\in \mathrm{N} \text{ such that } H(A_n,A)=\sup_{x\in X}\left|d(x,A_n)-d(x,A)\right|<r+\varepsilon \text{ for all } n\geq n_0 \text{ [13]}.$$

If a sequence is Hausdorff convergent, then this sequence r – Hausdorff converges to the same set for each r. However, there are some sequences of sets which are r – Hausdorff convergent, but not Hausdorff convergent as can be seen in the following

Example 2.1. Define

$$A_n := \left[\frac{1}{2}+\frac{1}{n},\frac{5}{2}\right]\times\left[-1,1\right]$$

and $A = \{(0,0)\} \times [-1,1]$ in the space $\mathbb{R}^2$ equipped with the Euclid metric. The sequence $\{A_n\}$ is not Hausdorff convergent to the set A. However, this sequence is rough Hausdorff convergent to the set A for all $r \geq \frac{5}{2}$.

Let $A,B \in Pb(X)$ and $\lambda \in \mathbb{R}$. The Minkowski addition and scalar multiplication are defined by

$$A + B = \{a + b : a \in A, b \in B\}$$

and

$$\lambda A = \{\lambda a : a \in A\}$$

respectively.

Let L be a nonempty set on which two operations (vector addition and scalar multiplication) are defined. If the listed axioms are satisfied, then L is called an almost linear space:

1. $(x + y) + z = x + (y + z)$, $\forall x, y, z \in L$ (associative property)
2. L has a zero vector 0 such that $x + 0 = 0+ x = x$, $\forall x \in L$ (additive identity)
3. $x + y = y + x$, $\forall x, y \in L$ (commutative property)
4. $\lambda(\mu x) = (\lambda\mu)x$, $\forall \lambda, \mu \in K$, $\forall x \in L$, where K is a scalar field (associative property)
5. $1 \cdot x = x$, $\forall x \in L$ (scalar identity)
6. $\lambda(x + y) = \lambda x + \lambda y$, $\forall \lambda \in K$, $\forall x, y \in L$ (distributive property)
7. $0 \cdot x = 0$, $\forall x \in L$ (zero property) [1].

We note that this space consists of sets that do not hold the inverse element property with respect to the Minkowski sum and the distributivity with respect to the sum of scalars.

Now we will recall the definition of the norm in almost linear spaces expressed using the Hausdorff distance.

If $A \in Pb(X)$ then we put

$$\|A\| = H(A,\{0\}) = \max\left\{\sup_{a\in A} d(a,\{0\}), d(0,A)\right\} = \sup_{a\in A}\|a\| \text{ [4]}.$$

MAIN RESULTS

In this study, inspired by the work of Apreutesei [1], we obtained some results for rough Hausdorff convergence.

Proposition 3.1. If $A_n \xrightarrow{r-H} A \in P(X)$ then $\|A_n\| \xrightarrow{r} \|A\|$.

Proof. Denote $\|A_n\| = \sup_{u\in A_n}\|u\|$ and $\|A\| = \sup_{v\in A}\|v\|$. Let $\varepsilon > 0$. Since $A_n \xrightarrow{r-H} A$, there exists an $n_0(\varepsilon) \in \mathrm{N}$ such that

$$h(A_n,A) < r + \varepsilon \text{ and } h(A,A_n) < r + \varepsilon \text{ for all } n \geq n_0. \quad (3.1)$$

If A is an unbounded set and $A_n \xrightarrow{r-H} A$ then we also obtain that A_n are unbounded sets for all $n \geq n_0$. This implies that $||A_n|| \to ||A||$. Thus, we have $\|A_n\| \xrightarrow{r} \|A\|$ for each $r \geq 0$. Now we assume that $||A|| < \infty$. If we get the infimum for all $v \in A$ and then the supremum for all $u \in A_n$ on both sides of the inequality

$$||u|| - ||v|| \leq ||u - v||,$$

we obtain

$$\sup_{u\in A_n}\|u\| - \inf_{v\in A}\|v\| = \sup_{u\in A_n}\inf_{v\in A}\left(\|u\| - \|v\|\right) \leq \sup_{u\in A_n}\inf_{v\in A}\|u - v\|.$$

From $\sup_{v\in A}\|v\| \geq \inf_{v\in A}\|v\|$ for all $v \in A$, we get

$$\sup_{u\in A_n}\|u\| - \sup_{v\in A}\|v\| \leq \sup_{u\in A_n}\|u\| - \inf_{v\in A}\|v\| \leq \sup_{u\in A_n}\inf_{v\in A}\|u - v\|.$$

Thus, we have

$$\|A_n\| - \|A\| \leq h(A_n, A).$$

Similarly,

$$\|A\| - \|A_n\| \leq h(A, A_n)$$

can be easily calculated. From (3.1), we obtain $\left|\|A_n\| - \|A\|\right| < r + \varepsilon$. This proves that $\|A_n\| \xrightarrow{r} \|A\|$.

Let us give an example to explain the Proposition 3.1.

Example 3.1. Define

$$A_n := \begin{cases} [-4,-2]\times[-2,2] & \text{, if } n \text{ is an odd integer} \\ [2,4]\times[-2,2] & \text{, if } n \text{ is an even integer} \end{cases}$$

and

$$A = [-2,2] \times [-2,2]$$

in the space $\mathbb{R}^2$ equipped with the Euclid metric. Then the sequence $\{A_n\}$ is rough Hausdorff convergent to the set A for $r = 4$. Moreover,

$$\|A_n\| \xrightarrow{r} \|A\| \text{ for } r = 4.$$

First we show that the sequence $\{A_n\}$ is rough Hausdorff convergent to the set A. Let $\varepsilon > 0$ and $(x^*,y^*) \in \mathbb{R}^2$. Then we calculate

$$d((x^*, y^*), A) = \begin{cases} \sqrt{(x^*-2)^2 + (y^*-2)^2} & \text{, if } x^* > 2 \text{ and } y^* > 2 \\ \left|x^* - 2\right| & \text{, if } x^* > 2 \text{ and } -2 \leq y^* \leq 2 \\ \sqrt{(x^*-2)^2 + (y^*+2)^2} & \text{, if } x^* > 2 \text{ and } y^* < -2 \\ \sqrt{(x^*+2)^2 + (y^*-2)^2} & \text{, if } x^* < -2 \text{ and } y^* > 2 \\ \left|x^* + 2\right| & \text{, if } x^* < -2 \text{ and } -2 \leq y^* \leq 2 \\ \sqrt{(x^*+2)^2 + (y^*+2)^2} & \text{, if } x^* < -2 \text{ and } y^* < -2 \\ \left|y^* - 2\right| & \text{, if } -2 \leq x^* \leq 2 \text{ and } y^* > 2 \\ \left|y^* + 2\right| & \text{, if } -2 \leq x^* \leq 2 \text{ and } y^* < -2 \\ 0 & \text{, if } -2 \leq x^* \leq 2 \text{ and} -2 \leq y^* \leq 2 \end{cases}.$$

Similarly, $d((x^*,y^*),A_n)$ can be easily calculated. Then there exists an $n_0 = n_0((x^*,y^*),\varepsilon)$ such that it can be easily obtained

$$\sup_{(x^*,y^*)\in\mathbb{R}^2}\left|d((x^*, y^*), A_n) - d((x^*, y^*), A)\right| \leq 4 + \varepsilon$$

for each $n > n_0$ using the inequality

$$\sqrt{(x^* - x)^2 + (y^* - y)^2} \leq \left|x^* - x\right| + \left|y^* - y\right|.$$

Hence, it is proved that $A_n \xrightarrow{r-H} A$ for every $r \geq 4$. On the other hand, since

$$\|A_n\| = \sqrt{20} \text{ and } \|A\| = \sqrt{8}$$
$$\left|\|A_n\| - \|A\|\right| = \left|\sqrt{20} - \sqrt{8}\right| < 4 + \varepsilon$$

we have $\|A_n\| \xrightarrow{r} \|A\|$ for every $r \geq 4$.

We note that the r value at the norm convergence may be smaller than the r value that provides the Hausdorff convergence of the sequence.

Following proposition shows that the sum of rough Hausdorff convergent sequences is also rough Hausdorff convergent.

Proposition 3.2. [11] Let A, A_n, B, B_n, $\in K(X)$. If $A_n \xrightarrow{r-H} A$ and $B_n \xrightarrow{r-H} B$, then $A_n + B_n \xrightarrow{2r-H} A + B$.

Proposition 3.3. Let A, A_n, B, $B_n \in K(X)$. If $A_n \overset{r-H}{\to} A$ and $B_n \overset{r-H}{\to} B$, then $\|A_n + B_n\| \overset{2r}{\to} \|A + B\|$.

Proof. Denote

$$\|A_n + B_n\| = \sup_{u \in A_n + B_n} \|u\| \text{ and } \|A + B\| = \sup_{v \in A+B} \|v\|.$$

Let $\varepsilon > 0$. Since $A_n + B_n \overset{2r-H}{\to} A + B$ from Proposition 3.2, there exists an $n_0(\varepsilon) \in \mathbb{N}$ such that

$$h(A_n + B_n, A + B) < 2r + \varepsilon \text{ and}$$
$$h(A + B, A_n + B_n) < 2r + \varepsilon \text{ for all } n \geq n_0. \quad (3.2)$$

Now we assume that $\|A + B\| < \infty$. If we get the infimum for all $v \in A + B$ and then the supremum for all $u \in A_n + B_n$ on both sides of the inequality

$$\|u\| - \|v\| \leq \|u - v\|,$$

we obtain

$$\sup_{u \in A_n + B_n} \inf_{v \in A+B} \left(\|u\| - \|v\|\right) \leq \sup_{u \in A_n + B_n} \inf_{v \in A+B} \|u - v\|$$
$$\sup_{u \in A_n + B_n} \|u\| - \sup_{v \in A+B} \|v\| \leq \sup_{u \in A_n + B_n} \inf_{v \in A+B} \|u - v\|.$$

Thus, we have

$$\|A_n + B_n\| - \|A + B\| \leq h(A_n + B_n, A + B).$$

Similarly,

$$\|A + B\| - \|A_n + B_n\| \leq h(A + B, A_n + B_n)$$

can be easily calculated. From (3.2), we obtain

$$\left|\|A_n + B_n\| - \|A + B\|\right| < 2r + \varepsilon.$$

This proves that $\|A_n + B_n\| \overset{2r}{\to} \|A + B\|$.

Example 3.2. Define

$$A_n := \left[1, 3 - \frac{1}{n}\right] \times [0,1] \subset \mathbb{R}^2 \text{ and}$$

$$B_n := \left[8, 10 - \frac{1}{n}\right] \times [0,1] \subset \mathbb{R}^2.$$

Then we have

$$A_n \overset{r-H}{\to} A = [5,6] \text{ and } B_n \overset{r-H}{\to} B = [12,14] \text{ for } r = 4.$$

Since

$$A_n + B_n = \left[9, 13 - \frac{2}{n}\right] \text{ and } A + B = [17, 20],$$

we also have

$$A_n + B_n \overset{2r-H}{\to} A + B.$$

Moreover,

$$\|A_n + B_n\| = \sup_{u \in A_n + B_n} \|u\| = \|(13,1)\| = \sqrt{170}$$
$$\|A + B\| = \sup_{v \in A+B} \|v\| = \|(20,1)\| = \sqrt{401}$$
$$\left|\|A_n + B_n\| - \|A + B\|\right| < 8 + \varepsilon.$$

Thus, we obtain $\|A_n + B_n\| \overset{2r}{\to} \|A + B\|$ for $r = 4$.

Theorem 3.1. Let $A_n, A \in P(X)$. If

$$\|A_n - A\| \overset{r}{\to} 0,$$

then for every $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that

$$A_n \subset S(A, r + \varepsilon) \text{ for all } n \geq n_0.$$

Proof. The condition $\|A_n - A\| \overset{r}{\to} 0$ is equivalent with: for every $\varepsilon > 0$ there exists $n_0(\varepsilon) \in N$ such that

$$\sup_{\substack{a \in A \\ a_n \in A_n}} \|a_n - a\| < r + \varepsilon$$

for all $n \geq n_0$. This implies that $A_n \subset S(A, r + \varepsilon)$ for all $n \geq n_0$.

Apreutesei [1, Theorem 4.11] proved that A is a singletone $\{a\}$ by assuming the similar condition of the Theorem 3.1. However, this situation does not hold for the theory of rough convergence as can be seen following example.

Example 3.3. Define

$$A_n = \{(-1)^n\}\mathbb{R} \text{ and } A = \{3,4\} \subset \mathbb{R}.$$

Then we have

$$\|A_n - A\| \overset{r}{\to} 0 \text{ for } r = 5.$$

Here A is not a singletone set.

Since there is no inverse element property with respect to the Minkowski sum in almost linear spaces, now we give a necessary condition to be the sequence $||A_n - A||$ rough convergent to zero.

Theorem 3.2. Let $A_n, A \in P(X)$. If A is a singletone $\{a\}$ and for every $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathrm{N}$ such that

$$A_n \subset S(a, r + \varepsilon) \text{ for all } n \geq n_0$$

then

$$\left\|A_n - A\right\| \xrightarrow{r} 0.$$

Proof. Given $\varepsilon > 0$. Assume that A is singleton $\{a\}$ and $A_n \subset S(a, r + \varepsilon)$ for all $n \geq n_0$. Then we have $\sup_{a_n \in A_n} \left\|a_n - a\right\| < r + \varepsilon$. This is equivalent to $\left\|A_n - A\right\| \xrightarrow{r} 0$. So, the proof is completed.

CONCLUSION

Although the norm definition given on the $Pb(X)$ class satisfies the norm axioms, $S(A,\varepsilon)$ does not specify a neighborhood for A since there is no inverse element property in the almost linear space. For this reason, while expressing convergence according to this norm, it is studied on sufficiently small sets. In this paper, it has been shown that in the rough convergence theory, contrary to Apreutesei [1, Theorem 4.11], the limit set does not need to be a singleton for convergence with respect to the norm.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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