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Technical Note

# On the exponential stability of a flexible structure in thermo-elasticity with micro-temperature effects

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## ABSTRACT

In this paper, we consider a non-uniform flexible structure with micro-temperature effect. We prove the well-posed of the problem using semi-group theory, as well as an exponential stability using the multiplier method without any restriction or relation on the coefficients of the system.

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## INTRODUCTION

In this paper, we aim to study the following inhomogeneous flexible structure system with micro-temperature effects:

$$\begin{cases} m(x)u_{tt} - \left(p(x)u_x + 2\delta(x)u_{xt}\right)_x + dw_x + \eta\theta_x = 0 \\ c\theta_t - k\theta_{xx} + \eta u_{xt} + k_1 w_x = 0 \\ \tau w_t - k_3 w_{xx} + k_2 w + k_1\theta_x + du_{xt} = 0, \end{cases} \tag{1}$$

where $u(x, t)$ is the displacement of a particle at position $x \in (0, L)$ and time $t > 0$, $\theta$ and $w$ are the temperature of the body and the micro-temperature vector respectively. $\eta > 0$ is the coupling constant, that accounts for the heating effect, and $k, k_1, k_2, k_3, c, d, \tau > 0$. $m(x)$, $\delta(x)$ and $p(x)$ are responsible for the non-uniform structure of the body, and, respectively, denote mass per unit length of structure, coefficient of internal material damping and a positive function related to the stress acting on the body at a point $x$. We consider the following initial and boundary conditions:

$$\begin{aligned} &u(.,0) = u_0(x), u_t(.,0) = u_1(x), \theta(.,0) \\ &\quad = \theta(x), w(.,0) = w_0(x), \\ &u(0,t) = u(L,t) = \theta(0,t) = \theta(L,t) = w_x(0,t) \\ &\quad = w_x(L,t) = 0, \forall t \geq 0 \end{aligned} \tag{2}$$

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The issue of existence and stability of flexible structure system has attracted a great deal of attention in the last years. Misra et al. [20] considered the vibrations of a cantilever structure modeled by the standard linear flexible model of visco-elasticity coupled to an expectedly dissipative effect through heat conduction

$$\begin{cases} m(x)u_{tt} - \left(p(x)u_x + 2\delta(x)u_{xt}\right)_x - k\theta_x = f \\ \theta_t - \theta_{xx} - ku_{xt} = 0, \end{cases}$$

By using semigroups theory and multiplier technique, they established the well-posedness and an exponential stability of the system when the disturbing force is insignificant. In the presence of second sound, Alves et al. [2] concerned with the system;

$$\begin{cases} m(x)u_{tt} - \left(p(x)u_x + 2\delta(x)u_{xt}\right)_x + \eta\theta_x = 0 \\ \theta_t + kq_x + \eta u_{xt} = 0 \\ \tau q_t + \beta q + k\theta_x = 0, \end{cases}$$

They established the well-posedness of the system and proved its stability exponential and polynomial under suitable boundary conditions. Li et al. [18] considered this last with a delay term of the form $\mu u_t(x, t - \tau_0)$ in its first equation, they proved that the system is exponential decay under a "small" condition on time delay. For more details discussion on the subject see [1, 10] and the references therein.

Historically, the linear theory of thermo-elasticity with micro-temperatures for materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess micro-temperatures was constructed by Iesan and Quintanilla [15, 17]. The work is motivated by increasing use of materials which possess thermal variation at a microstructure level. The same authors proved an existence theorem and established the continuous dependence of solutions of the initial data and body loads. We note that the concept of micro-temperature was just used in the theory of thermodynamics for elastic materials with microstructure. In addition to micro-deformations of the string, the micro-elements of the continuum possess micro-temperatures which represent the variation of the temperature within a micro-volume. Originally, Grot [11] was the first to take into consideration the inner structure of a body in order to develop a thermodynamic theory for thermo-elastic materials where micro-elements, in addition to classic micro-deformations, possess micro-temperatures. While, the fundamental solution of the equations of the theory of thermo-elasticity with micro-temperatures was constructed by Svanadze [27]. Riha [23, 24] developed a further study concerning heat conduction in thermo-elastic materials with inner structure. It is shown that the experimental data for the silicone rubber containing spherical aluminum particles and for human blood are conform closely to the predicted theoretical model of thermo-elasticity with micro-temperatures. We refer the interested readers to [3, 5, 6, 7, 8, 9, 12, 13, 14, 16, 19, 25, 26] for details discussion on the theory.

Motivated by works mentioned above, we investigate (1)–(2) under suitable condition and establish the well-posedness of the problem using semi-group theory, as well as the stability result of the solution using the multiplier method. We should mention here that, to the best of our knowledge, there is no result concerning flexible structure system with micro-temperature effect. Our purpose in the present manuscript is to obtain an exponential decay rate estimates of the energy function of (1) without any restriction or relation on the coefficients of the system.

This paper is organized as follows; In the second section, we introduce some assumptions needed in our work then prove the well-posedness of the system (1) – (2). In the last section we state and prove our stability result.

## WELL-POSEDNESS OF THE PROBLEM

In this section, we present some assumptions and give the existence and uniqueness result of system (1) – (2) using the semigroup theory. Throughout this paper, $c'$ represents ageneric positive constant and is different in various occurrences.

The aim of this section is to prove that system (1) – (2) is well-posed. From Equation $(1)_3$ and the boundary conditions (2), we have

$$\frac{d}{dt}\int_0^L w(x,t)\,dx + \frac{k_2}{\tau}\int_0^L w(x,t)\,dx = 0, \forall t \geq 0,$$

Thus

$$\int_0^L w(x,t)\,dx = \left(\int_0^L w_0\,dx\right)\exp\left(\frac{-t}{\tau}k_2\right), \ \forall t \geq 0.$$

So, if we set

$$\tilde{w}(x,t) = w(x,t) - \frac{1}{L}\left(\int_0^L w_0\,dx\right)\exp\left(\frac{-t}{\tau}k_2\right), \ \forall t \geq 0, x \in [0,L].$$

Then, $(u, u_t, \theta, \tilde{w})$ satisfies Equations (1), and

$$\int_0^L \tilde{w}(x,t)\,dx = 0,$$

for all $t \geq 0$. In the sequel, we shall work with $\tilde{w}$ but we write $w$ for simplicity. The energy functional associated to (1) – (2), namely,

$$E(t,u,u_t,\theta,w)=\frac{1}{2}\int_0^L\left\{p(x)u_x^2+m(x)u_t^2+c\theta^2+\tau w^2\right\}dx, \quad (3)$$

we denote $E(t) = E(t, u, u_t, \theta, w)$ and $E(0) = E(0, u_0, u_1, \theta_0, w_0)$ for simplicity of notations. Then the energy $E$ is decreasing function and satisfies, for all $t \geq 0$.

$$\begin{aligned} E'(t) &= -2\int_0^L\delta(x)u_{xt}^2dx-k_2\int_0^L w^2dx-k_3\int_0^L w_x^2dx-k\int_0^L\theta_x^2dx \\ &\leq -c'\int_0^L u_t^2dx-k_2\int_0^L w^2dx-k_3\int_0^L w_x^2dx-k\int_0^L\theta_x^2dx\leq 0. \end{aligned} \quad (4)$$

To obtain precise decay rates of $E(t)$ as $t \to +\infty$, we assume that

$m, \delta, p \in W^{1,\infty}(0,L)$, $m(x), p(x)$ and $\delta(x) > 0$, $\forall x \in [0,L]$. (5)

Let us introducing the vector function $U = (u, v, \theta, w)^T$, where $v = u_t$, using the standard Lebesgue space $L^2(0,L)$ and the Sobolev space $H_0^1(0,L)$ with their usual scalar products and norms for define the spaces;

$$\breve{H} = H_0^1(0,L) \times [L^2(0,L)]^2 \times L_*^2(0,L),$$

And

$$H_*^2(0,L) = \{w \in H^2(0,L): w_x(L) = w_x(0) = 0\},$$

Where

$$L_*^2(0,L)=\left\{\omega\in L^2(0,L):\int_0^L\omega(s)ds=0\right\}.$$

We equip $\breve{H}$ with the inner product:

$$\left(U,\tilde{U}\right)_{\breve{H}}=\int_0^L p(x)u_x\tilde{u}_xdx+\int_0^L m(x)v\tilde{v}dx+c\int_0^L\theta\tilde{\theta}dx+\tau\int_0^L w\tilde{w}dx.$$

Next, the system (1) – (2) can be reduced to the following abstract Cauchy problem;

$$\begin{cases} U'(t)+AU(t)=0 & t>0 \\ U(0)=U_0=(u_0,u_1,\theta_0,w_0)^T, \end{cases} \quad (6)$$

where the operator $A:D(A) \to \breve{H}$ is defined by

$$AU=\begin{pmatrix} -v \\ -\frac{1}{m(x)}\left(p(x)u_x+2\delta(x)v_x-dw-\eta\theta\right)_x \\ \frac{1}{c}\left(-k\theta_{xx}+\eta u_{xt}+k_1w_x\right) \\ \frac{1}{\tau}\left(-k_3w_{xx}+k_2w+k_1\theta_x+du_{xt}\right) \end{pmatrix}.$$

The domain of $A$ is then

$$AU=\left\{\begin{matrix} U\in\breve{H}:u\in H^2(0,L)\cap H_0^1(0,L), \\ v\in H_0^1(0,L),\theta\in H^2(0,L) \\ w\in L_*^2(0,L)\cap H_*^2(0,L) \end{matrix}\right\},$$

which is dense in $\breve{H}$.

**Proposition 2.1.** *Let $U_0 \in \breve{H}$ be given. Problem (6) possesses then a unique solution satisfying $U \in C(\mathbb{R}^+, \breve{H})$. If $U_0 \in D(A)$ then $U \in C^1(R^+,\breve{H}) \cap C(R^+,D(A))$.*

**Proof.** For any $U \in D(A)$, we have

$$\begin{aligned} \left(AU,U\right)_{\breve{H}} &= 2\int_0^L\delta(x)v_x^2dx+k\int_0^L\theta_x^2dx \\ &+k_2\int_0^L w^2dx+k_3\int_0^L w_x^2dx\geq 0. \end{aligned}$$

Hence, $A$ is monotone. Next, we prove that the operator $I + A$ is surjective.

Given $G = (g_1, g_2, g_3, g_4)^T \in \breve{H}$, we prove that there exists $U \in D(A)$ satisfying

$$(I + A)U = G, \quad (7)$$

which gives

$$\begin{aligned} -v+u&=g_1\in H_0^1(0,L), \\ -\left(p(x)u_x+2\delta(x)v_x-dw-\eta\theta\right)_x&+m(x)v \\ &=m(x)g_2\in L^2(0,L), \\ -k\theta_{xx}+\eta v_x+k_1w_x+c\theta&=cg_3\in L^2(0,L), \\ -k_3w_{xx}+k_2w+k_1\theta_x+dv_x+\tau w&=\tau g_4\in L_*^2(0,L). \end{aligned} \quad (8)$$

Inserting $v = u - g_1$, in $(8)_2$, $(8)_3$ and $(8)_4$ we obtain

$$\begin{aligned} -\left(p(x)u_x+2\delta(x)u_x-dw-\eta\theta\right)_x&+m(x)u \\ =m(x)(g_1+g_2)-2\delta(x)g_{1xx}&=f_1\in L^2(0,L), \\ -k\theta_{xx}+\eta u_x+k_1w_x+c\theta&=cg_3+\eta g_{1x} \\ &=f_2\in L^2(0,L), \\ -k_3w_{xx}+k_2w+k_1\theta_x+du_x+\tau w&=\tau g_4+dg_{1x} \\ &=f_3\in L_*^2(0,L). \end{aligned} \quad (9)$$

The variational formulation corresponding to Equation (9) takes the form

$$B\left((u,\theta,w),(\tilde{u},\tilde{\theta},\tilde{w})\right)=F\left((\tilde{u},\tilde{\theta},\tilde{w})\right), \quad (10)$$

where $B:[H_0^1(0,L)\times L^2(0,L)\times L_*^2(0,L)]^2\to\mathbb{R}$ is the bilinear form defined by

$$B\big((u,\theta,w),(\tilde{u},\tilde{\theta},\tilde{w})\big)= \int_0^L[(p(x)+2\delta(x))u_x-dw-\eta\theta]\tilde{u}_x dx +\int_0^L m(x)u\tilde{u}dx+k\int_0^L\theta_x\tilde{\theta}_x dx -\eta\int_0^L u\tilde{\theta}_x dx-k_1\int_0^L w\tilde{\theta}_x dx+k_3\int_0^L w_x\tilde{w}_x dx +(k_2+\tau)\int_0^L w\tilde{w}dx+k_1\int_0^L\theta_x\tilde{w}dx -d\int_0^L u\tilde{w}_x dx,$$

and $F:H_0^1(0,L)\times L^2(0,L)\times L_*^2(0,L)\to\mathbb{R}$ is the linear form defined by

$$F\big((\tilde{u},\tilde{\theta},\tilde{w})\big)=\int_0^L f_1\tilde{u}dx+\int_0^L f_2\tilde{\theta}dx+\int_0^L f_3\tilde{w}dx.$$

For $V=H_0^1(0,L)\times L^2(0,L)\times L_*^2(0,L)$ equipped with the norm

$$\|(u,\theta,w)\|_V^2=\|u\|_2^2+\|u_x\|_2^2+\|w\|_2^2+\theta_x\|_2^2,$$

where $\|\cdot\|_2$ is the usual norm.

One can easily see that $B$ and $F$ are bounded. Also, we get

$$B\big((u,\theta,w),(u,\theta,w)\big)=\int_0^L\big(p(x)+2\delta(x)\big)u_x^2dx +\int_0^L m(x)u^2dx+k\int_0^L\theta_x^2dx + k_3\int_0^L w_x^2dx+k_2\int_0^L w^2dx \geq c\big(u,\theta,w\big)_V^2.$$

Then, $B$ is coercive. Consequently, by the Lax-Milgram lemma (see [4] Corollary 5.8), system (9) has a unique solution

$$u\in H_0^1(0,L),\ \theta\in L^2(0,L),\ w\in L_*^2(0,L).$$

From $(8)_1$, we infer that

$$v\in H_0^1(0,L),$$

Moreover, if $(\tilde{\theta},\tilde{w})\equiv(0,0)\in L^2(0,L)\times L_*^2(0,L)$ then Equation (10) reduces to

$$-\int_0^L[(p(x)+2\delta(x))u_x-dw-\eta\theta]_x\tilde{u}dx +\int_0^L m(x)u\tilde{u}dx=\int_0^L f_1\tilde{u}dx,$$

That is

$$-[(p(x)+2\delta(x))u_x]_x=dw_x+\eta\theta_x-m(x)u+f_1\in L^2(0,L).$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$u\in H^2(0,L)\cap H_0^1(0,L).$$

Similary, if $(\tilde{u},\tilde{\theta})\equiv(0,0)\in H_0^1(0,L)\times L^2(0,L)$ then Equation (10) reduces to

$$k_3\int_0^L w_x\tilde{w}_x dx+(k_2+\tau)\int_0^L w\tilde{w}dx+k_1\int_0^L\theta_x\tilde{w}dx-d\int_0^L u\tilde{w}_x dx =\int_0^L f_3\tilde{w}dx,\forall\tilde{w}\in L_*^2(0,L). \tag{11}$$

That is

$$k_3w_{xx}=(k_2+\tau)w+k_1\theta_x+du_x-f_3\in L^2(0,L). \tag{12}$$

then, it follows that $\int_0^L w\,dx=0$, and we get

$$w\in L_*^2(0,L)\cap H^2(0,L).$$

Moreover, (11) is also true for any $\varphi\in C^1([0,L])\subset L_*^2(0,L)$. Hence, we have

$$k_3\int_0^L w_x\varphi_x dx+(k_2+\tau)\int_0^L w\varphi dx+k_1\int_0^L\theta_x\varphi dx-d\int_0^L u\varphi_x =\int_0^L f_3\varphi dx,$$

for all $\varphi\in C^1([0,L])$. Thus, using integration by parts and bearing in mind (12), we obtain

$$w_x(L)\varphi(L)-w_x(0)\varphi(0)=0,\ \forall\varphi\in C^1([0,L]).$$

Therefore, $w_x(L)=w_x(0)=0$, consequently, we have

$$w\in L_*^2(0,L)\cap H_*^2(0,L).$$

Now, if $(\tilde{u},\tilde{w})\equiv(0,0)\in H_0^1(0,L)\times L_*^2(0,L)$, then Equation (10) reduces to

$$k\int_0^L\theta_x\tilde{\theta}_x dx-\eta\int_0^L u\tilde{\theta}_x dx-k_1\int_0^L w\tilde{\theta}_x dx=\int_0^L f_2\tilde{\theta}dx.$$

That is

$$-k\theta_{xx} = f_2 - \eta u_x - k_1 w_x \in L^2(0,L),$$

then, we get

$$\theta \in H^2(0,L).$$

Hence, there exists a unique $U \in D(A)$ such that Equation (7) is satisfied. Consequently, $A$ is a maximal monotone operator. Then, $D(A)$ is dense in $\breve{H}$ (see Proposition 7.1 in [4]) and the result of Proposition 2.1 follows from Lumer-Phillips theorem (see [22]).

## EXPONENTIAL STABILITY

In this section, we introduce some lemmas allow us to achieve our goal, which is the proof of the stability result.

**Lemma 3.1. [21]** *(Poincaré type Scheeffer's inequality) Let $h \in H_0^1(0,L)$. Then it holds*

$$\int_0^L |h|^2\, dx \le l \int_0^L |h_x|^2\, dx,\ l = \frac{L^2}{\pi^2}. \tag{13}$$

**Lemma 3.2. [2, 20]** *Let $(u, u_t, \theta, w)$ be the solution to system* (1) – (2), *with an initial datum in* $D(A)$. *Then, for any* $t > 0$, *there exists a sequence of real numbers (depending on t), denoted by* $\xi_i \in [0,L]$, $(i = 1,\ldots, 6)$, *such that:*

$$\int_0^L p(x)u_x^2 dx = p(\xi_1)\int_0^L u_x^2 dx,\quad \int_0^L m(x)u_t^2 dx = m(\xi_2)\int_0^L u_t^2 dx,$$
$$\int_0^L m(x)u^2 dx = m(\xi_3)\int_0^L u^2 dx,\quad \int_0^L \delta(x)u^2 dx = \delta(\xi_4)\int_0^L u^2 dx,$$
$$\int_0^L \delta(x)u_x^2 dx = \delta(\xi_5)\int_0^L u_x^2 dx,\quad \int_0^L \delta(x)u_{xt}^2 dx = \delta(\xi_6)\int_0^L u_{xt}^2 dx.$$

**Lemma 3.3.** *Let $(u, u_t, \theta, w)$ be the solution to system* (1) – (2), *then the energy E is non-increasing function and satisfies, for all $t \ge 0$,*

$$\begin{aligned} E'(t) &= \int_0^L \delta(x)u_{xt}^2 dx - k_2\int_0^L w^2 dx - k_3\int_0^L w_x^2 dx - k\int_0^L \theta_x^2 dx \\ &\le -c'\int_0^L u_t^2 dx - k_2\int_0^L w^2 dx - k_3\int_0^L w_x^2 dx - k\int_0^L \theta_x^2 dx \le 0, \end{aligned} \tag{14}$$

*where $c' = 2\delta(\xi_6)/l$.*

**Proof.** Multiplying the equations in $(1)_1$, $(1)_2$, and $(1)_3$ by $u_t$, $\theta$ and $w$, respectively, integrate over $(0, L)$ and using (13), we obtain (14).

**Lemma 3.4.** *The functional*

$$I_1(t) = \int_0^L (\delta(x)u_x^2 + m(x)u_t u)dx, \tag{15}$$

*Satisfies*

$$\begin{aligned} I_1'(t) \le& -\left(p(\xi_1) - (\eta + d)\epsilon_1\right)\int_0^L u_x^2 dx + m(\xi_2) \\ & \int_0^L u_t^2 dx + \frac{\eta}{4\epsilon_1}\int_0^L \theta^2 dx + \frac{d}{4\epsilon_1}\int_0^L w^2 dx, \end{aligned} \tag{16}$$

*for any $\in_1 > 0$*

**Proof.** Differentiating Equation (15) with respect to $t$ and using Equations $(1)_1$, we get

$$I_1'(t) = -\int_0^L p(x)u_x^2 dx + m(x)\int_0^L u_t^2 dx - \eta\int_0^L \theta_x u dx - d\int_0^L u w_x dx,$$

Using Young's inequality (see [4] p. 92), we have for $\in_1 > 0$

$$-\eta\int_0^L \theta_x u dx = \eta\int_0^L \theta u_x dx \le \eta\epsilon_1\int_0^L u_x^2 dx + \frac{\eta}{4\epsilon_1}\int_0^L \theta^2 dx$$
$$-d\int_0^L u w_x dx = d\int_0^L w u_x dx \le d\epsilon_1\int_0^L u_x^2 dx + \frac{d}{4\epsilon_1}\int_0^L w^2 dx,$$

application of Lemma 3.2 and the last two inequality completes the proof.

**Lemma 3.5.** *The functional*

$$I_2(t) = \tau c\int_0^L \theta\left(\int_0^x w(y)dy\right)dx, \tag{17}$$

*satisfies*

$$\begin{aligned} I_2'(t) \le& \left(-k_1 c + 3c'\varepsilon_2\right)\int_0^L \theta^2 dx + \frac{1}{2\varepsilon_2}\int_0^L u_t^2 dx + \frac{1}{4\varepsilon_2}\int_0^L \theta_x^2 dx \\ &+ \left(k_1\tau + 2\varepsilon_2 c' + \frac{c'}{4\varepsilon_2}\right)\int_0^L w^2 dx + \frac{1}{4\varepsilon_2}\int_0^L w_x^2 dx, \end{aligned} \tag{18}$$

*for any* $\in_1 > 0$.

Proof. Taking the derivative of (17) and using $(1)_2$ and $(1)_3$ we find

$$\begin{aligned} I_2'(t) =& \tau\left(k\int_0^L \theta_{xx}\left(\int_0^x w(y)dy\right)dx - \eta\int_0^L u_{tx}\left(\int_0^x w(y)dy\right)dx\right) \\ &- k_1\tau\int_0^L w_x\left(\int_0^x w(y)dy\right)dx \\ &+ c\left(k_3\int_0^L \theta\left(\int_0^x w_{yy}(y)dy\right)dx - k_2\int_0^L \theta\left(\int_0^x w(y)dy\right)dx\right) \\ &+ c\left(-k_1\int_0^L \theta\left(\int_0^x \theta_y(y)dy\right)dx - d\int_0^L \theta\left(\int_0^x u_{ty}(y)dy\right)dx\right). \end{aligned}$$

Integration by parts and the fact that $\int_0^L wdx = 0$, give us

$$I_2'(t) = \tau\left(k\int_0^L \theta_x wdx + \eta\int_0^L u_t wdx + k_1\int_0^L w^2dx\right) + c\left(k_3\int_0^L \theta w_x dx - k_2\int_0^L \theta\left(\int_0^x w(y)dy\right)dx - k_1\int_0^L \theta^2 dx - d\int_0^L \theta u_t dx,\right) \quad (19)$$

using Young's inequality, we get also

$$\begin{aligned}
-k\int_0^L \theta_x wdx &\le \frac{1}{4\epsilon_2}\int_0^L \theta_x^2dx + c'\varepsilon_2\int_0^L w^2dx \\
\eta\int_0^L u_t wdx &\le \frac{1}{4\epsilon_2}\int_0^L u_t^2dx + c'\varepsilon_2\int_0^L w^2dx \\
k_3\int_0^L \theta w_x dx &\le \frac{1}{4\epsilon_2}\int_0^L w_x^2dx + c'\varepsilon_2\int_0^L \theta^2dx \\
-k_2\int_0^L \theta\left(\int_0^x w(y)dy\right)dx &\le c'\varepsilon_2\int_0^L \theta^2dx + \frac{c'}{4\epsilon_2}\int_0^L w^2dx \\
-d\int_0^L \theta u_t dx &\le \frac{1}{4\epsilon_2}\int_0^L u_t^2dx + c'\varepsilon_2\int_0^L \theta^2dx
\end{aligned} \quad (20)$$

From (19) and the inequalities (20) we infer (18).

Next, we define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the energy functional.

**Lemma 3.6.** *For N sufficiently large, the functional defined by*

$$\mathcal{L}(t) = NE(t) + I_1(t) + N_1 I_2(t), \quad (21)$$

*where N and $N_1$ are positive real numbers to be chosen appropriately later, satisfies*

$$c_1'E(t) \le \mathcal{L}(t) \le c_2'E(t), \quad (22)$$

*where $c_1'$ and $c_2'$ are positive constants.*

**Proof.** Let

$$\wp(t) = I_1(t) + N_1 I_2(t),$$

then, exploiting Young's inequality, (13) and (3) we obtain

$$\begin{aligned}
|\wp(t)| &\le N_1\tau c\int_0^L\left|\theta\left(\int_0^x w(y)dy\right)\right|dx + \int_0^L(\delta(x)u_x^2 + m(x)|u_t u|)dx \\
&\le \int_0^L \delta(x)u_x^2dx + \frac{1}{2}\int_0^L m(x)u^2dx + N_1\tau c\int_0^L|\theta w|dx + \frac{1}{2}\int_0^L m(x)u_t^2dx \\
&\le \frac{1}{2}\int_0^L m(x)u_t^2dx + \frac{\|\delta(x)\|_\infty}{\lambda}\int_0^L p(x)u_x^2dx + \frac{l\|m(x)\|_\infty}{2\lambda}\int_0^L p(x)u_x^2dx \\
&\quad + \frac{N_1\tau lc}{2}\int_0^L \theta^2dx + \frac{N_1\tau lc}{2}\int_0^L w^2dx \le c'E(t)
\end{aligned}$$

where $\lambda = \inf_{(x\in[0,L])}\{p(x)\}$, and $c' > 0$. Consequently,

$$\left|\mathcal{L}(t) - NE(t)\right| \le c'E(t),$$

which yields

$$(N - c')E(t) \le \mathcal{L}(t) \le (N + c')E(t).$$

Choosing *N* large enough, we obtain estimate (22).

Now, we are ready to state and prove the main result of this section.

**Theorem 3.7.** *Let $(u, u_t, \theta, w)$ be the solution to system* (1) – (2), *then the energy E satisfies, for all $t \ge 0$,*

$$E(t) \le c_1 e^{-c_2 t},$$

*where $c_1$ and $c_2$ are positive constants.*

**Proof.** We differentiate (21), and recall (14), (16) and (18), we obtain

$$\begin{aligned}
\mathcal{L}'(t) \le\ & N\left(-c'\int_0^L u_t^2dx - k_2\int_0^L w^2dx - k_3\int_0^L w_x^2dx - k\int_0^L \theta_x^2dx\right) \\
& -\left(p(\xi_1) - (\eta+d)\varepsilon_1\right)\int_0^L u_x^2dx + m(\xi_2)\int_0^L u_t^2dx \\
& + \frac{\eta}{4\varepsilon_1}\int_0^L \theta^2dx + \frac{d}{4\varepsilon_1}\int_0^L w^2dx \\
& + N_1\left((-k_1c + 3c'\varepsilon_2)\int_0^L \theta^2dx + \frac{1}{2\varepsilon_2}\int_0^L u_t^2dx + \frac{1}{4\varepsilon_2}\int_0^L \theta_x^2dx\right) \\
& + N_1\left(\left(k_1\tau + 2\varepsilon_2c' + \frac{c}{4\epsilon_2}\right)\int_0^L w^2dx + \frac{1}{4\varepsilon_2}\int_0^L w_x^2dx\right) \\
\le\ & \left\{-Nc' + \frac{N_1}{2\varepsilon_2} + m(\xi_2)\right\}\int_0^L u_t^2dx + \left\{-p(\xi_1) + (\eta+d)\varepsilon_1\right\}\int_0^L u_x^2dx \\
& + \left\{-Nk_2 + N_1\left(k_1\tau + 2\varepsilon_2c' + \frac{c'}{4\varepsilon_2}\right) + \frac{d}{4\varepsilon_1}\right\}\int_0^L w^2dx \\
& + \left\{N_1(-k_1c + 3c'\varepsilon_2) + \frac{\eta}{4\epsilon_1}\right\}\int_0^L \theta^2dx + \left\{-Nk + \frac{N_1}{4\epsilon_2}\right\}\int_0^L \theta_x^2dx \\
& + \left\{-Nk_3 + \frac{N_1}{4\varepsilon_2}\right\}\int_0^L w_x^2dx.
\end{aligned}$$

At this point, we choose $\epsilon_1$ and $\epsilon_2$ small enough such that

$$-p(\xi_1) + (\eta + d)\,\epsilon_1 < 0, \quad -k_1 c + 3c'\epsilon_2 < 0,$$

then we choose $N_1$ large enough so that

$$N_1(-k_1c + 3c'\varepsilon_2) + \frac{\eta}{4\varepsilon_1} < 0.$$

Once $N_1$ is fixed, we then choose $N$ large enough so that

$$-Nc' + \frac{N_1}{2\dot{o}_2} + m(\xi_2) < 0,$$
$$-Nk_2 + N_1\left(k_1\tau + 2\varepsilon_2 c' + \frac{c'}{4\dot{o}_2}\right) + \frac{d}{4\varepsilon_1} < 0,$$
$$-Nk + \frac{N_1}{4\varepsilon_2} < 0,$$
$$-Nk_3 + \frac{N_1}{4\varepsilon_2} < 0.$$

Thus, using (13), we arrive at

$$\mathcal{L}(t) \le cE(t), \forall t > 0, \qquad (23)$$

A combination of (22) and (23) gives

$$\mathcal{L}'(t) \le -c_2\mathcal{L}(t), \forall t > 0, \qquad (24)$$

where $c_2 = c/c_2'$, a simple integration of (24) over $(0, t)$ yields

$$c_1'E(t) \le \mathcal{L}(t) \le \mathcal{L}(0)e^{-c_2 t}, \ \forall t > 0.$$

Taking $c_1 = \mathcal{L}(0)/c_1'$ which completes the proof.

## AUTHORSHIP CONTRIBUTIONS

Concept: M.H.; Design: M.H.; Supervision: S.Z., A.D; Data: M.H.; Analysis: M.H.; Literature Search: M.H.; Writing: M.H.; Critical Revision: A.D.

## DATA AVAILABILITY STATEMENT

No new data werecreated in this study. The published publication includes all graphics collected or developed during the study.

## CONFLICT OF INTEREST

The author declared no potential conflicts of interest-with respect to the research, authorship, and/or publication of this article.

## ETHICS

There are no ethical issues with the publication of this manuscript.

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