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Technical Note

Lifts of connections to the bundle of (1,1) type tensor frames

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ABSTRACT

In this paper we consider the bundle of (1,1) type tensor frames over a smooth manifold, define the horizontal and complete lifts of symmetric linear connection into this bundle. Also we study the properties of the geodesic lines corresponding to the complete lift of the linear connection and investigate the relations between Sasaki metric and lifted connections on the bundle of (1,1) type tensor frames.

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INTRODUCTION

Let M be an $n-$dimensional manifold of class C^{∞}. The problem of extending differential-geometrical structures on M to its fiber bundles has been the subject of a number of papers. An account of these can be found in Yano and Ishihara [15] (see, also [1]). Yano, Kobayashi and Ishihara [16, 17] have defined the complete and horizontal lifts of linear connections on M to tangent bundle $T(M)$. On the other hand, using the Riemannian extension, Yano and Patterson [18, 19] have investigated the complete and horizontal lifts of linear connections on M to cotangent bundle ${}^{C}T(M)$. The relations between various metrics and connections on the ${}^{C}T(M)$ have been studied by Mok [8]. Similar studies for linear frame, coframe and tensor bundles were carried out in [2, 3, 6, 7, 11].

In the present paper, we shall define the complete and horizontal lifts of a symmetric linear connections from a manifold M to the bundle of (1,1) type tensor frames $L_1^1(M)$. In 2 we briefly describe the definitions and results that are needed later, after which the horizontal lift of a symmetric linear connection is defined in 3. The complete lift of a symmetric linear connection is investigated in 4. In 5 we study the properties of the geodesic line of the complete lift of the linear connection. The relations between Sasaki metric and lifted connections on the $L_1^1(M)$ are determined in 6.

PRELIMINARIES

In this paper all manifolds, mappings, tensor fields and connections are assumed to be differentiable of class C^{∞} Let M an $n-$ dimensional differentiable manifold and $L_1^1(M)$ the bundle of (1,1) type tensor frames of M (see, [4]). The bundle $L_1^1(M)$ consists of all pairs (x, A_x), where x is a point of M and A_x is a basis ((1,1) type tensor frame) for the linear space $T_1^1(x)$ of all (1,1) type tensors at a point x. We denote by $\pi : L_1^1(M) \to M$ the projection

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map defined by $\pi(x, A_x) = x$. For the coordinate system (U, x^i) in M, we put $L_1^1(U) = \pi^{-1}(U)$ and a (1,1) type tensor X_β^α of the frame A_x can be uniquely expressed in the form

$$X_\beta^\alpha = X_{\beta i}^{\alpha j}(\tfrac{\partial}{\partial x^j})_x \otimes (dx^i)_x,$$

so that $\left\{L_1^1(U), (x^i, X_{\beta i}^{\alpha j})\right\}$ is a coordinate system in $L_1^1(M)$ [4]. Indices $i, j, k, ..., \alpha, \beta, \gamma, ...$ have range in $\{1,2,...,n\}$, while indices $A, B, C, ...$ have range in $\{1,...,n, n+1,...,n+n^4\}$ and indices $i_{\alpha\beta}, j_{\gamma\delta}, k_{\sigma\tau}, ...$ have range in $\{n+1,...,n+n^4\}$. Summation over repeated indices is always implied.

We denote by $\Im_s^r(M)$ the set of all differentiable tensor fields of type (r,s) on M. Let ∇ be a linear connection, $V \in \Im_0^1(M)$ a vector field and $B \in \Im_1^1(M)$ a (1,1) type tensor field on M with local components Γ_{ij}^k, V^i and B_i^j, respectively. Then there are exactly one vector field ${}^H V$ on M, called the horizontal lift of V, and exactly one vector field ${}^{V_{\alpha\beta}}B$ on $L_1^1(M_n)$ for each pair $\alpha, \beta = 1,2,...,n$, called the $\alpha\beta-$vertical lift of B, that are known to be defined in $L_1^1(U)$ (see, [4]) by

$$ {}^H V = V^i \frac{\partial}{\partial x^i} + V^k (X_{\beta m}^{\alpha j}\Gamma_{ki}^m - X_{\beta i}^{\alpha m}\Gamma_{km}^j)\frac{\partial}{\partial X_{\beta i}^{\alpha j}}, \qquad (1)$$

$$ {}^{V_{\alpha\beta}}B = \delta_\alpha^\gamma \delta_\sigma^\beta B_i^j \frac{\partial}{\partial X_{\sigma i}^{\gamma j}} \qquad (2)$$

with respect to the natural frame $\{\partial_i, \partial_{i_{\alpha\beta}}\} = \left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial X_{\beta i}^{\alpha j}}\right\}$ in $L_1^1(M)$, where δ_α^γ is the Kronecker delta. If f is a differentiable function on M, ${}^V f = f \circ \pi$ denotes its canonical vertical lift to $L_1^1(M)$.

Let (U, x^i) be a coordinate system in M. In $U \subset M$, we put

$$X_{(i)} = \frac{\partial}{\partial x^i} = \delta_i^h \frac{\partial}{\partial x^h} \in \Im_0^1(M),$$

$$\Lambda_i^j = \frac{\partial}{\partial x^i} \otimes dx^j = \delta_i^h \delta_k^j \partial_h \otimes dx^k \in \Im_1^1(M), i, j = 1,2,...,n.$$

From (1) and (2), we have

$$ {}^H X_{(i)} = \delta_i^h \partial_h + (X_{\beta m}^{\alpha k}\Gamma_{hi}^m - X_{\beta i}^{\alpha m}\Gamma_{hm}^k)\frac{\partial}{\partial X_{\beta h}^{\alpha k}}, \qquad (3)$$

$$ {}^{V_{\alpha\beta}}\Lambda_i^j = \delta_\gamma^\alpha \delta_\beta^\sigma \delta_i^h \delta_k^j \frac{\partial}{\partial X_{\sigma h}^{\gamma k}} \qquad (4)$$

with respect to the natural frame $\{\partial_i, \partial_{i_{\alpha\beta}}\}$ in $L_1^1(M)$. These $n+n^4$ vector fields are linearly independent and generate, respectively, the horizontal distribution of linear connection ∇ and the vertical distribution of $L_1^1(M)$. We call the set $\left\{{}^H X_{(i)}, {}^{V_{\alpha\beta}}\Lambda_i^j\right\}$ the frame adapted to the linear connection ∇ on $\pi^{-1}(U) \subset L_1^1(M)$. Putting

$$D_i = {}^H X_{(i)}, \; D_{i_{\alpha\beta}} = {}^{V_{\alpha\beta}}\Lambda_i^j,$$

we write the adapted frame as $\{D_I\} = \{D_i, D_{i_{\alpha\beta}}\}$. From (3) and (4) we see that ${}^H V$ and ${}^{V_{\alpha\beta}}B$ have respectively, components

$$ {}^H V = V^i D_i = ({}^H V^I) = \begin{pmatrix} V^i \\ 0 \end{pmatrix}, \qquad (5)$$

$$ {}^{V_{\alpha\beta}}B = B_i^j \delta_\alpha^\gamma \delta_\sigma^\beta D_{i_{\gamma\sigma}} = ({}^{V_{\alpha\beta}}B^I) = \begin{pmatrix} 0 \\ \delta_\alpha^\gamma \delta_\sigma^\beta B_i^j \end{pmatrix} \qquad (6)$$

with respect to the adapted frame $\{D_I\}$, V^i and B_i^j are the local components of V and B on M, respectively.

Let $B \in \Im_1^1(M)$, which is locally represented by $B = B_i^j \frac{\partial}{\partial x^j} \otimes dx^i$. We define the vector fields γB and $\tilde{\gamma} B$ on $L_1^1(M)$ by

$$\begin{cases} \gamma B = (X_{\beta i}^{\alpha m} B_m^j)\dfrac{\partial}{\partial X_{\beta i}^{\alpha j}}, \\ \tilde{\gamma} B = (X_{\beta m}^{\alpha j} B_i^m)\dfrac{\partial}{\partial X_{\beta i}^{\alpha j}} \end{cases}$$

with respect to the natural frame $\{\partial_i, \partial_{i_{\alpha\beta}}\}$ in $L_1^1(M)$.

HORIZONTAL LIFTS OF LINEAR CONNECTIONS

Let ∇ be the symmetric linear connection on M and Γ_{ij}^k its components.

Definition 3.1 A horizontal lift of the symmetric linear connection ∇ on M to the bundle of (1,1) type tensor frames $L_1^1(M)$ is the linear connection ${}^H\nabla$ defined by

$$ {}^H\nabla_{{}^H X} {}^H Y = {}^H(\nabla_X Y), \; {}^H\nabla_{{}^H X} {}^{V_{\beta\sigma}}B = {}^{V_{\beta\sigma}}(\nabla_X B),$$

$$ {}^H\nabla_{{}^{V_{\alpha\gamma}}A} {}^H Y = 0, \; {}^H\nabla_{{}^{V_{\alpha\gamma}}A} {}^{V_{\beta\sigma}}B = 0 \qquad (7)$$

for all $X, Y \in \Im_0^1(M)$ and $A, B \in \Im_1^1(M)$.

The components of the horizontal lift ${}^H\nabla$ of ∇ on M with components Γ_{ij}^k in the natural frame $\left\{\frac{\partial}{\partial x^i}\right\}$, are defined in the adapted frame $\{D_I\}$ by decomposition

$$ {}^H\nabla_{D_I} D_J = {}^H\Gamma_{IJ}^K D_K. \qquad (8)$$

From (7) and (8), by using of (5) and (6), we get.

Theorem 3.1 The horizontal lift ${}^{H}\nabla$ of the symmetric linear connection ∇ given on M, to the bundle of (1,1) type tensor frames $L_1^1(M)$ have the components

$$
\begin{aligned}
&{}^{H}\Gamma^{p}_{i_{\alpha\gamma}k_{\beta\sigma}} = 0, \quad {}^{H}\Gamma^{p_{\eta\varepsilon}}_{i_{\alpha\gamma}k_{\beta\sigma}} = {}^{H}\Gamma^{p}_{i_{\alpha\gamma}k} = {}^{H}\Gamma^{p_{\eta\varepsilon}}_{i_{\tau\gamma}k} = 0,\\
&{}^{H}\Gamma^{p}_{ik} = \Gamma^{p}_{ik},\ {}^{H}\Gamma^{p_{\eta\varepsilon}}_{ik} = {}^{H}\Gamma^{p}_{ik_{\beta\sigma}} = 0,\\
&{}^{H}\Gamma^{p_{\eta\varepsilon}}_{ik_{\beta\sigma}} = \delta^{\eta}_{\beta}\delta^{\sigma}_{\varepsilon}\delta^{k}_{p}\Gamma^{q}_{il} - \delta^{\eta}_{\beta}\delta^{\sigma}_{\varepsilon}\delta^{q}_{l}\Gamma^{k}_{ip}
\end{aligned}
\tag{9}
$$

with respect to the adapted frame $\{D_I\}$.

Now let us consider the following transformation of frames on $L_1^1(M)$:

$$
\{D_i, D_{i_{\alpha\gamma}}\} = \left\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial X^{\beta k}_{\sigma j}}\right\}
\begin{pmatrix} \delta^j_i & 0 \\ -X^{\beta m}_{\sigma j}\Gamma^{k}_{im} + X^{\beta k}_{\sigma m}\Gamma^{m}_{ij} & \delta^{\beta}_{\alpha}\delta^{\gamma}_{\sigma}\delta^{k}_{h}\delta^{i}_{j} \end{pmatrix},
$$

i.e.

$D_I = A_I{}^{J}\partial_J$.

We note that matrix $(A_I{}^{J})$ and its inverse matrix $(\tilde{A}^K{}_J)$ are defined of the form

$$
A = (A_L{}^{J}) = \begin{pmatrix} A_l{}^{j} & A_{l_{\tau\lambda}}{}^{j} \\ A_l{}^{j_{\beta\sigma}} & A_{l_{\tau\lambda}}{}^{j_{\beta\sigma}} \end{pmatrix}
= \begin{pmatrix} \delta^j_l & 0 \\ -X^{\beta m}_{\sigma j}\Gamma^{k}_{lm} + X^{\beta k}_{\sigma m}\Gamma^{m}_{lj} & \delta^{\beta}_{\tau}\delta^{\lambda}_{\sigma}\delta^{k}_{r}\delta^{l}_{j} \end{pmatrix}
\tag{10}
$$

and

$$
A^{-1} = (\tilde{A}^I{}_J) = \begin{pmatrix} \tilde{A}^i{}_j & \tilde{A}^i{}_{j_{\beta\sigma}} \\ \tilde{A}^{i_{\alpha\gamma}}{}_j & \tilde{A}^{i_{\alpha\gamma}}{}_{j_{\beta\sigma}} \end{pmatrix}
= \begin{pmatrix} \delta^i_j & 0 \\ X^{\alpha m}_{\gamma i}\Gamma^{h}_{jm} - X^{\alpha h}_{\gamma m}\Gamma^{m}_{ji} & \delta^{\alpha}_{\beta}\delta^{\sigma}_{\gamma}\delta^{h}_{k}\delta^{j}_{i} \end{pmatrix}, \text{ (11) respectively.}
$$

We denote the components of the linear connection ${}^{H}\nabla$ with respect to the natural frame $\{\partial_I\}$ by ${}^{H}\bar{\Gamma}^{P}_{IK}$, i.e.

$$
{}^{H}\nabla_{\partial_I}\partial_K = {}^{H}\bar{\Gamma}^{P}_{IK}\partial_P.
$$

Then

$$
{}^{H}\bar{\Gamma}^{S}_{JL} = A_P{}^{S}\,{}^{H}\Gamma^{P}_{IK}\tilde{A}^I{}_J\tilde{A}^K{}_L - (D_I A_K{}^{S})\tilde{A}^I{}_J\tilde{A}^K{}_L. \tag{12}
$$

Using (9), (10) and (11), from (12) we have

Theorem 3.2 The horizontal lift ${}^{H}\nabla$ of a symmetric linear connection ∇ given on M, to the bundle of (1,1) type tensor frames $L_1^1(M)$ have the components

$$
\begin{aligned}
&{}^{H}\bar{\Gamma}^{s}_{jl} = \Gamma^{s}_{jl},\ {}^{H}\bar{\Gamma}^{s_{\phi\psi}}_{jl} = X^{\phi b}_{\psi s}(\partial_j\Gamma^{r}_{lb} - \Gamma^{r}_{pb}\Gamma^{p}_{jl} + \Gamma^{r}_{jp}\Gamma^{p}_{lb})\\
&+X^{\phi r}_{\psi a}(-\partial_j\Gamma^{a}_{ls} + \Gamma^{p}_{js}\Gamma^{a}_{lp} + \Gamma^{a}_{ps}\Gamma^{p}_{jl}) - X^{\phi b}_{\psi a}(\Gamma^{r}_{lb}\Gamma^{a}_{sj} + \Gamma^{a}_{ls}\Gamma^{r}_{bj}),\\
&{}^{H}\bar{\Gamma}^{s}_{j_{\tau\nu}l} = {}^{H}\bar{\Gamma}^{s}_{jl_{\omega\upsilon}} = {}^{H}\bar{\Gamma}^{s}_{j_{\tau\nu}l_{\omega\upsilon}} = {}^{H}\bar{\Gamma}^{s_{\phi\psi}}_{j_{\tau\nu}l_{\omega\upsilon}} = 0,\\
&{}^{H}\bar{\Gamma}^{s_{\phi\psi}}_{j_{\tau\nu}l} = \delta^{\phi}_{\tau}\delta^{\nu}_{\psi}\delta^{j}_{s}\Gamma^{r}_{lq} - \delta^{\phi}_{\tau}\delta^{\nu}_{\psi}\delta^{r}_{q}\Gamma^{j}_{ls},\ {}^{H}\bar{\Gamma}^{s_{\phi\psi}}_{jl_{\omega\upsilon}} =\\
&\qquad \delta^{\phi}_{\omega}\delta^{\upsilon}_{\psi}\delta^{l}_{s}\Gamma^{r}_{jm} - \delta^{\phi}_{\omega}\delta^{\upsilon}_{\psi}\delta^{r}_{m}\Gamma^{l}_{js}
\end{aligned}
\tag{13}
$$

with respect to the natural frame $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial X^{\alpha h}_{\gamma i}}\right\}$.

We note that the horizontal lifts of linear connections to tangent, cotangent, linear frame and linear coframe bundles were investigated in the [2, 6, 11, 17, 19].

COMPLETE LIFTS OF LINEAR CONNECTIONS

Now we consider the torsion-free (or the symmetric) linear connection ∇ on the differentiable manifold M, i.e. $\Gamma^k_{ij} = \Gamma^k_{ji}$. We determine the new linear connection $\breve{\nabla}$ on the bundle of (1,1) type tensor frames $L_1^1(M)$ by following manner:

$$
\breve{\Gamma}^{P}_{JL} = {}^{H}\bar{\Gamma} + K^{P}_{JL}, \tag{14}
$$

where K^{P}_{JL} is the (1,2) type tensor field on the $L_1^1(M)$ with unique non zero components

$$
K^{p_{\phi\varphi}}_{jl} = X^{\phi r}_{\varphi m}R^{m}_{lpj} - X^{\phi m}_{\varphi p}R^{r}_{mjl} \tag{15}
$$

with respect to the natural frame $\left\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial X^{\beta q}_{\sigma j}}\right\}$ and R is the curvature tensor of ∇.

By using of (13), (14) and (15), we obtain the non zero components of the linear connection $\breve{\nabla}$ in the induced natural frame $\left\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial X^{\beta q}_{\sigma j}}\right\}$:

$$
\begin{aligned}
&\breve{\Gamma}^{p}_{jl} = \Gamma^{p}_{jl},\ \breve{\Gamma}^{p_{\phi\psi}}_{j_{\tau\nu}l} = \delta^{\phi}_{\tau}\delta^{\nu}_{\psi}\delta^{j}_{p}\Gamma^{r}_{lq} - \delta^{\phi}_{\tau}\delta^{\nu}_{\psi}\delta^{r}_{q}\Gamma^{j}_{lp},\\
&\breve{\Gamma}^{p_{\phi\psi}}_{jl_{\omega\sigma}} = \delta^{\phi}_{\omega}\delta^{\sigma}_{\psi}\delta^{l}_{p}\Gamma^{r}_{js} - \delta^{\phi}_{\omega}\delta^{\sigma}_{\psi}\delta^{r}_{s}\Gamma^{l}_{jp},\\
&\breve{\Gamma}^{p_{\phi\psi}}_{jl} = X^{\phi b}_{\psi p}\partial_b\Gamma^{r}_{jl} + X^{\phi r}_{\psi a}(\partial_p\Gamma^{a}_{lj} - \partial_l\Gamma^{a}_{pj} - \partial_j\Gamma^{a}_{lp} +\\
&2\Gamma^{a}_{mp}\Gamma^{m}_{jl}) - X^{\phi b}_{\psi a}(\Gamma^{r}_{lb}\Gamma^{a}_{pj} + \Gamma^{a}_{lp}\Gamma^{r}_{bj}).
\end{aligned}
\tag{16}
$$

We have

Theorem 4.1 Covariant differentiation with respect to the linear connection $\breve{\nabla}$ has the following property:

$$\hat{\nabla}_{{}^C X}\, {}^C Y = {}^C(\nabla_X Y) + \gamma(Q(X,Y))$$

for all $X, Y \in \Im_0^1(M)$, where $\gamma(Q(X,Y)) = \begin{pmatrix} 0 \\ F^{k_{\beta\sigma}} \end{pmatrix}$ is the vertical vector field on the bundle of $(1,1)$ type tensor frames $L_1^1(M)$ such that

$$F^{k_{\beta\sigma}} = X_{\sigma a}^{\beta l}(\nabla_k X^m \nabla_m Y^a + \nabla_k Y^m \nabla_m X^a$$

$$-X^i Y^j (R_{jki}^a + R_{ikj}^a)) - X_{\sigma a}^{\beta b}(\nabla_b X^l \nabla_k Y^a + \nabla_k X^a \nabla_b Y^l),$$

R is the curvature tensor field of linear connection ∇ and ${}^C X, {}^C Y$ are complete lifts of vector fields X, Y from a manifold M to $L_1^1(M)$, respectively.

Proof. Let us consider the vector fields $X, Y \in \Im_0^1(M)$. The complete lift ${}^C X$ of vector field X from a manifold to the bundle of $(1,1)$ type frames $L_1^1(M)$ defined by [4]

$$ {}^C X^i = X^i, \quad {}^C X^{i_{\alpha\beta}} = X_{\beta i}^{\alpha m}\partial_m X^j - X_{\beta m}^{\alpha j}\partial_i X^m \qquad (17)$$

with respect to the induced frame $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial X_{\beta i}^{\alpha j}} \right\}$.

1) If $K = k$, then by using (16) and (17), we have

$$(\breve{\nabla}_{{}^C X}\, {}^C Y)^k = {}^C X^I(\partial_I\, {}^C Y^k + \breve{\Gamma}_{IJ}^k\, {}^C Y^J)$$

$$= X^i(\partial_i Y^k + \Gamma_{ij}^k Y^j) = (\nabla_X Y)^k = {}^C(\nabla_X Y)^k;$$

2) In the case $K = k_{\beta\sigma}$ by the same way, we obtain:

$$(\breve{\nabla}_{{}^C X}\, {}^C Y)^{k_{\beta\sigma}} = {}^C X^I(\partial_I\, {}^C Y^{k_{\beta\sigma}} + \breve{\Gamma}_{IJ}^{k_{\beta\sigma}}\, {}^C Y^J)$$

$$= {}^C X^i \partial_i\, {}^C Y^{k_{\beta\sigma}} + {}^C X^{i_{\alpha\gamma}} \partial_{i_{\alpha\gamma}}\, {}^C Y^{k_{\beta\sigma}} + {}^C X^i \breve{\Gamma}_{ij}^{k_{\beta\sigma}}\, {}^C Y^j$$

$$+ {}^C X^{i_{\alpha\gamma}} \breve{\Gamma}_{i_{\alpha\gamma} j}^{k_{\beta\sigma}}\, {}^C Y^j + {}^C X^i \breve{\Gamma}_{i j_{\omega\tau}}^{k_{\beta\sigma}}\, {}^C Y^{j_{\omega\tau}}$$

$$= X^i \partial_i (X_{\sigma k}^{\beta m}\partial_m Y^l - X_{\sigma m}^{\beta l}\partial_k Y^m)$$

$$+(X_{\gamma i}^{\alpha m}\partial_m X^r - X_{\gamma m}^{\alpha r}\partial_i X^m)\partial_{i_{\alpha\gamma}}(X_{\sigma k}^{\beta s}\partial_s Y^l - X_{\sigma s}^{\beta l}\partial_k Y^s)$$

$$+X^i Y^j \Big[X_{\sigma k}^{\beta b}\partial_b \Gamma_{ij}^l + X_{\sigma a}^{\beta l}(\partial_k \Gamma_{ji}^a - \partial_j \Gamma_{ki}^a - \partial_i \Gamma_{jk}^a + 2\Gamma_{mk}^a \Gamma_{ij}^m)$$

$$-X_{\sigma a}^{\beta b}(\Gamma_{jb}^l \Gamma_{ki}^a + \Gamma_{jk}^a \Gamma_{bi}^l)\Big]$$

$$+(X_{\gamma i}^{\alpha m}\partial_m X^r - X_{\gamma m}^{\alpha r}\partial_i X^m) Y^j (\delta_\alpha^\beta \delta_\sigma^\gamma \delta_k^i \Gamma_{jr}^l - \delta_\alpha^\beta \delta_\sigma^\gamma \delta_r^l \Gamma_{jk}^i)$$

$$+X^i (X_{\tau j}^{\omega m}\partial_m Y^p - X_{\tau m}^{\omega p}\partial_j Y^m)(\delta_\omega^\beta \delta_\sigma^\tau \delta_k^j \Gamma_{ip}^l - \delta_\omega^\beta \delta_\sigma^\tau \delta_p^l \Gamma_{ik}^j)$$

$$= X_{\sigma k}^{\beta m} X^i \partial_i \partial_m Y^l - X_{\sigma m}^{\beta l} X^i \partial_i \partial_k Y^m + X_{\tau i}^{\beta m}\partial_m X^r \delta_r^s \delta_k^i \partial_s Y^l$$

$$-X_{\tau i}^{\beta m}\partial_m X^r \delta_r^l \delta_s^i \partial_k Y^s - X_{\tau m}^{\beta r}\partial_i X^m \delta_r^s \delta_k^i \partial_s Y^l$$

$$+X_{\tau m}^{\beta r}\partial_i X^m \delta_r^l \delta_s^i \partial_k Y^s + X^i Y^j X_{\sigma k}^{\beta b}\partial_b \Gamma_{ij}^l + X^i Y^j X_{\sigma a}^{\beta l}\partial_k \Gamma_{ji}^a$$

$$-X^i Y^j X_{\sigma a}^{\beta l}\partial_j \Gamma_{ki}^a - X^i Y^j X_{\sigma a}^{\beta l}\partial_i \Gamma_{jk}^a + 2X^i Y^j X_{\sigma a}^{\beta l}\Gamma_{mk}^a \Gamma_{ij}^m$$

$$-X^i Y^j X_{\sigma a}^{\beta b}\Gamma_{jb}^l \Gamma_{ki}^a - X^i Y^j X_{\sigma a}^{\beta b}\Gamma_{jk}^a \Gamma_{bi}^l + X_{\sigma k}^{\beta m}\partial_m X^r \Gamma_{jr}^l Y^j$$

$$-X_{\sigma i}^{\beta m}\partial_m X^l \Gamma_{jk}^i Y^j - X_{\sigma m}^{\beta r}\partial_k X^m \Gamma_{jr}^l Y^j + X_{\sigma m}^{\beta l}\partial_i X^m \Gamma_{jk}^i Y^j$$

$$+X^i X_{\sigma k}^{\beta m}\partial_m Y^p \Gamma_{ip}^l - X^i X_{\sigma j}^{\beta m}\partial_m Y^l \Gamma_{ik}^j - X^i X_{\sigma m}^{\beta p}\partial_k Y^m \Gamma_{ip}^l$$

$$+X^i X_{\sigma m}^{\beta l}\partial_j Y^m \Gamma_{ik}^j = {}^C(\nabla_X Y)^{k_{\beta\sigma}} + X_{\sigma a}^{\beta l}(\nabla_k X^m \nabla_m Y^a$$

$$+\nabla_k Y^m \nabla_m X^a - X^i Y^j (R_{jki}^a + R_{ikj}^a)) - X_{\sigma a}^{\beta b}(\nabla_b X^l \nabla_k Y^a +$$

$$+\nabla_k X^a \nabla_b Y^l).$$

Thus, we have shown that

$$\hat{\nabla}_{{}^C X}\, {}^C Y = {}^C(\nabla_X Y) + \gamma(Q(X,Y)) \qquad (18)$$

for all $X, Y \in \Im_0^1(M)$, where $\gamma(Q(X,Y)) = \begin{pmatrix} 0 \\ F^{k_{\beta\sigma}} \end{pmatrix}$ is the vertical vector field on $L_1^1(M)$, moreover

$$F^{k_{\beta\sigma}} = X_{\sigma a}^{\beta l}(\nabla_k X^m \nabla_m Y^a + \nabla_k Y^m \nabla_m X^a$$

$$-X^i Y^j (R_{jki}^a + R_{ikj}^a)) - X_{\sigma a}^{\beta b}(\nabla_b X^l \nabla_k Y^a + \nabla_k X^a \nabla_b Y^l).$$

This means that the Theorem 4.1 is proved.

The complete lifts of the symmetric linear connections in the cotangent and coframe bundles satisfies relations analogously to (18) (see, [11, 18]). Therefore, the linear connection $\breve{\nabla}$ defined by formula (4.1) and satisfying the relation (18) is called the complete lift of the symmetric linear connection ∇ on M to the bundle of $(1,1)$ type tensor frames $L_1^1(M)$ and denoted by ${}^C\nabla$. By using the transformation

$$ {}^C\tilde{\Gamma}_{JL}^S = \tilde{A}^S{}_P\, {}^C\Gamma_{IK}^P A_J{}^I A_L{}^K - (\partial_I \tilde{A}^S{}_K) A_J{}^I A_L{}^K,$$

it is easy to establish that a complete lift ${}^C\nabla$ of a symmetric linear connection ∇ defined on M to the bundle of $(1,1)$ type tensor frames $L_1^1(M)$ has nonzero components in the form

$$ {}^C\tilde{\Gamma}_{jl}^s = \Gamma_{jl}^s, \quad {}^C\tilde{\Gamma}_{jl}^{s_{\phi\psi}} = X_{\psi m}^{\phi r} R_{lsj}^m - X_{\psi s}^{\phi m} R_{mjl}^r,$$

$$ {}^C\tilde{\Gamma}_{j l_{\omega\sigma}}^{s_{\phi\psi}} = \delta_\omega^\phi \delta_\psi^\sigma \delta_s^l \Gamma_{jq}^r - \delta_\omega^\phi \delta_\psi^\sigma \delta_q^r \Gamma_{js}^l \qquad (19)$$

with respect to the adapted frame $\{D_I\}$, where R is the curvature tensor of ∇.

GEODESICS OF THE COMPLETE LIFTS

Different problems of geodesics in fiber bundles has been very well investigated (see, for example [10, 12, 13]). Geodesics of complete lifts of linear connections in tangent, cotangent, tensor, linear frame and linear coframe bundles has been studied in [5, 7,9,11,16,18]. In the present section we will investigate geodesics of the complete lifts of linear connections in the bundle of (1,1) type tensor frames.

Let $\tilde{C}$ be a geodesic curve on the bundle of (1,1) type tensor frames $L_1^1(M)$ with respect to the complete lift ${}^C\nabla$ of the symmetric linear connection ∇ on M. In induced coordinates $(\pi^{-1}(U), x^i, X^{\alpha h}_{\gamma i})$ the equation of the geodesic curve

$$\tilde{C}: I \to L_1^1(M), \tilde{C}: t \to \tilde{C}(t) = (x^i(t), X^{\alpha h}_{\gamma i}(t)) = (x^I(t))$$

are of the form

$$\frac{d^2 x^K}{dt^2} + {}^C\Gamma^K_{IJ}\frac{dx^I}{dt}\frac{dx^J}{dt} = 0, \quad I,J,K = 1,2,...,n+n^4. \tag{20}$$

By using of formulas (17) for ${}^C\Gamma^K_{IJ}$, from (20) we obtain:

$$\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0, \tag{21}$$

$$\frac{d^2 X^{\beta l}_{\sigma k}}{dt^2} + \Big[X^{\beta m}_{\sigma k}\partial_m\Gamma^l_{ij} + X^{\beta l}_{\sigma a}(\partial_k\Gamma^a_{ji} - \partial_j\Gamma^a_{ki} - \partial_i\Gamma^a_{jk}$$
$$+2\Gamma^a_{mk}\Gamma^m_{ij}) - X^{\beta b}_{\sigma a}(\Gamma^l_{jb}\Gamma^a_{ki} + \Gamma^a_{jk}\Gamma^l_{bi})\Big]\frac{dx^i}{dt}\frac{dx^j}{dt}$$
$$+2(\Gamma^l_{jb}\delta^i_k - \Gamma^i_{jk}\delta^l_b)\frac{dX^{\beta b}_{\sigma i}}{dt}\frac{dx^j}{dt} \tag{22}$$

Let us consider the covariant differentiation of $X^{\beta l}_{\sigma k}(t)$:

$$\frac{\delta}{dt}(X^{\beta l}_{\sigma k}(t)) = \frac{dX^{\beta l}_{\sigma k}}{dt} + \Gamma^l_{pb}X^{\beta b}_{\sigma k}\frac{dx^p}{dt} - \Gamma^a_{pk}X^{\beta l}_{\sigma a}\frac{dx^m}{dt}. \tag{23}$$

Now taking into account the equality (21) and symmetry of the linear connection ∇ given on M, from (23) we obtain:

$$\frac{\delta^2 X^{\beta l}_{\sigma k}}{dt^2} = \frac{\delta}{dt}\left(\frac{dX^{\beta l}_{\sigma k}}{dt} + \Gamma^l_{pb}X^{\beta b}_{\sigma k}\frac{dx^p}{dt} - \Gamma^a_{pk}X^{\beta l}_{\sigma a}\frac{dx^p}{dt}\right)$$

$$= \frac{d}{dt}\left(\frac{dX^{\beta l}_{\sigma k}}{dt} + \Gamma^l_{pb}X^{\beta b}_{\sigma k}\frac{dx^p}{dt} - \Gamma^a_{pk}X^{\beta l}_{\sigma a}\frac{dx^p}{dt}\right)$$
$$+\Gamma^l_{ib}\left(\frac{dX^{\beta b}_{\sigma k}}{dt} + \Gamma^b_{pa}X^{\beta a}_{\sigma k}\frac{dx^p}{dt}\right.$$
$$\left.-\Gamma^b_{pk}X^{\beta b}_{\sigma a}\frac{dx^i}{dt}\right)\frac{dx^i}{dt}$$
$$-\Gamma^b_{ik}\left(\frac{dX^{\beta l}_{\sigma b}}{dt} + \Gamma^j_{pa}X^{\beta a}_{\sigma b}\frac{dx^p}{dt} - \Gamma^a_{pb}X^{\beta l}_{\sigma a}\frac{dx^p}{dt}\right)\frac{dx^i}{dt}$$

$$= \frac{d^2 X^{\beta l}_{\sigma k}}{dt^2} + \Big(X^{\beta b}_{\sigma k}(\partial_i\Gamma^l_{jb} - \Gamma^l_{pb}\Gamma^p_{ij} + \Gamma^l_{ip}\Gamma^p_{jb}) + X^{\beta l}_{\sigma a}(-\partial_i\Gamma^a_{jk}$$
$$+\Gamma^p_{ik}\Gamma^a_{jp} + \Gamma^a_{pk}\Gamma^p_{ij}) - X^{\beta b}_{\sigma a}(\Gamma^l_{jb}\Gamma^a_{ki} + \Gamma^a_{jk}\Gamma^r_{bi})\Big)\frac{dx^i}{dt}\frac{dx^j}{dt}$$
$$+2\Big(\Gamma^l_{jb}\delta^i_k - \Gamma^i_{jk}\delta^l_b\Big)\frac{dX^{\beta b}_{\sigma i}}{dt}\frac{dx^j}{dt}. \tag{24}$$

Taking into account (24), the equation (22) is written in the form:

$$\frac{\delta^2 X^{\beta l}_{\sigma k}}{dt^2} + R^l_{bij}X^{\beta b}_{\sigma k}\frac{dx^i}{dt}\frac{dx^j}{dt} - R^a_{kij}X^{\beta l}_{\sigma a}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0. \tag{25}$$

From the above we get

Theorem 5.1 Let ∇ be a symmetric linear connection on a differentiable manifold M and let $\tilde{C}(t) = (C(t), X^{\beta l}_{\sigma k}(t))$ be a curve on the bundle of (1,1) type tensor frames $L_1^1(M)$. In order for the curve $\tilde{C}(t)$ to be a geodesic line of the complete lift ${}^C\nabla$ of the ∇, it is necessary and sufficient that the following conditions be satisfied:

i) The curve $C(t)$ is a geodesic line of the linear connection ∇;
ii) The each (1,1) tensor field $X^{\beta l}_{\sigma k}(t)$ satisfies the relation (25) along the curve $C(t)$.

THE SASAKI METRIC AND THE COMPLETE LIFT

Let g be a metric and ∇ a symmetric linear connection on M. The Sasaki metric on the bundle of (1,1) type tensor frames $L_1^1(M)$ denoted by ${}^Sg$ (see, [4]). Note that metric ${}^Sg$ is an analogue of the metric introduced by Sasaki [14].

The line element of ${}^Sg$ on $\pi^{-1}(U)$ is taken to be

$${}^Sg_{IJ}dx^I dx^J = g_{ij}dx^i dx^j + \delta_{\alpha\sigma}\delta^{\beta\gamma}g_{pq}g^{ij}\delta X^{\alpha p}_{\beta i}\delta X^{\sigma q}_{\gamma j}, \tag{26}$$

where

$$\delta X^{\alpha p}_{\beta i} = dX^{\alpha p}_{\beta i} + \Gamma^p_{km}X^{\alpha m}_{\beta i}dx^k - \Gamma^m_{ki}X^{\alpha p}_{\beta m}dx^k$$

is the usual covariant differential. It is easily seen that (26) defines a global metric on $L_1^1(M)$ and that the component matrix of ${}^Sg$ with respect to the adapted frame is

$$\begin{pmatrix} g_{ij} & 0 \\ 0 & \delta_{\alpha\sigma}\delta^{\beta\gamma}g_{pq}g^{ij} \end{pmatrix}. \tag{27}$$

We would like to establish conditions for ${}^C\nabla$ to be metrical with respect to ${}^Sg$. Let us denote by $({}^Sg_{IJ})$ the matrix in (27). By a simple calculation based on (19) and (27) we determine the possible non-zero components of ${}^C\nabla{}^Sg$:

$$
\begin{aligned}
{}^{C}\nabla_k {}^{S}g_{ij} &= D_k {}^{S}g_{ij} - {}^{C}\tilde{\Gamma}^{M}_{ki} {}^{S}g_{Mj} - {}^{C}\tilde{\Gamma}^{M}_{kj} {}^{S}g_{iM} \\
&= D_k g_{ij} - {}^{C}\tilde{\Gamma}^{m}_{ki} {}^{S}g_{mj} - {}^{C}\tilde{\Gamma}^{m_{\beta\sigma}}_{ki} {}^{S}g_{m_{\beta\sigma}j} - {}^{C}\tilde{\Gamma}^{m}_{kj} {}^{S}g_{im} \\
&\quad - {}^{C}\tilde{\Gamma}^{m_{\beta\sigma}}_{kj} {}^{S}g_{im_{\beta\sigma}} = \nabla_k g_{ij}; \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
{}^{C}\nabla_k {}^{S}g_{i_{\alpha\gamma}j} &= {}^{C}\nabla_k {}^{S}g_{ji_{\alpha\gamma}} = D_k {}^{S}g_{i_{\alpha\gamma}j} - {}^{C}\tilde{\Gamma}^{M}_{ki_{\alpha\gamma}} {}^{S}g_{Mj} - {}^{C}\tilde{\Gamma}^{M}_{kj} {}^{S}g_{i_{\alpha\gamma}M} \\
&- {}^{C}\tilde{\Gamma}^{m}_{ki_{\alpha\gamma}} {}^{S}g_{mj} - {}^{C}\tilde{\Gamma}^{m_{\beta\sigma}}_{ki_{\alpha\gamma}} {}^{S}g_{m_{\beta\sigma}j} - {}^{C}\tilde{\Gamma}^{m}_{kj} {}^{S}g_{i_{\alpha\gamma}m} - {}^{C}\tilde{\Gamma}^{m_{\beta\sigma}}_{kj} {}^{S}g_{i_{\alpha\gamma}m_{\beta\sigma}} \\
&= -(X^{\beta r}_{\sigma l} R^{l}_{jmk} - X^{\beta l}_{\sigma m} R^{r}_{lkj})\delta_{\alpha\beta}\delta^{\gamma\sigma} g_{rl} g^{mi} \\
&= -(X^{\alpha r}_{\gamma l} R^{l}_{jmk} - X^{\alpha l}_{\gamma m} R^{r}_{lkj}) g_{rl} g^{mi}; \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
{}^{C}\nabla_k {}^{S}g_{i_{\alpha\gamma}j_{\phi\psi}} &= D_k {}^{S}g_{i_{\alpha\gamma}j_{\phi\psi}} - {}^{C}\tilde{\Gamma}^{M}_{ki_{\alpha\gamma}} {}^{S}g_{Mj_{\phi\psi}} - {}^{C}\tilde{\Gamma}^{M}_{kj_{\phi\psi}} {}^{S}g_{i_{\alpha\gamma}M} \\
&= D_k(\delta_{\alpha\phi}\delta^{\gamma\psi} g_{lq} g^{ji}) - {}^{C}\tilde{\Gamma}^{m}_{ki_{\alpha\gamma}} {}^{S}g_{mj_{\phi\psi}} - {}^{C}\tilde{\Gamma}^{m_{\beta\sigma}}_{ki_{\alpha\gamma}} {}^{S}g_{m_{\beta\sigma}j_{\phi\psi}} \\
&- {}^{C}\tilde{\Gamma}^{m}_{kj_{\phi\psi}} {}^{S}g_{i_{\alpha\gamma}m} - {}^{C}\tilde{\Gamma}^{m_{\beta\sigma}}_{kj_{\phi\psi}} {}^{S}g_{i_{\alpha\gamma}m_{\beta\sigma}} = \delta_{\alpha\phi}\delta^{\gamma\psi}(D_k g_{lq}) g^{ji} \\
&+ \delta_{\alpha\phi}\delta^{\gamma\psi} g_{lq} D_k g^{ji} - \delta^{\beta}_{\alpha}\delta^{\gamma}_{\sigma}\delta^{i}_{m}\Gamma^{r}_{kl}\delta_{\beta\phi}\delta^{\sigma\psi} g_{rq} g^{jm} \\
&+ \delta^{\beta}_{\alpha}\delta^{\gamma}_{\sigma}\delta^{r}_{l}\Gamma^{i}_{km}\delta_{\beta\phi}\delta^{\sigma\psi} g_{rq} g^{jm} - \delta^{\beta}_{\phi}\delta^{\psi}_{\sigma}\delta^{j}_{m}\Gamma^{r}_{kq}\delta_{\alpha\beta}\delta^{\sigma\gamma} g_{rl} g^{im} \\
&+ \delta^{\beta}_{\phi}\delta^{\psi}_{\sigma}\delta^{r}_{q}\Gamma^{j}_{km}\delta_{\alpha\beta}\delta^{\sigma\gamma} g_{rl} g^{im} = \delta_{\alpha\phi}\delta^{\gamma\psi}(\nabla_k g_{lq}) g^{ji} \\
&+ \delta_{\alpha\phi}\delta^{\gamma\psi} g_{lq} \nabla_k g^{ji}. \qquad (30)
\end{aligned}
$$

From (28), (29) and (30), we get

Theorem 6.1 Let g be a metric and ∇ a symmetric linear connection on M. Then, ${}^{C}\nabla$ is metrical with respect to ${}^{S}g$ if ∇ is the Riemannian connection of g and is locally flat.

Using relations (14) and (29) - (30), we find the possible non-zero components of ${}^{H}\nabla {}^{S}g$:

$$ {}^{H}\nabla_k {}^{S}g_{ij} = \nabla_k g_{ij}; $$

$$ {}^{H}\nabla_k {}^{S}g_{i_{\alpha\gamma}j_{\phi\psi}} = \delta_{\alpha\phi}\delta^{\gamma\psi}(\nabla_k g_{lq}) g^{ji} + \delta_{\alpha\phi}\delta^{\gamma\psi} g_{lq}\nabla_k g^{ji}. $$

Therefore, we have

Theorem 6.2 Let g be a metric and ∇ a symmetric linear connection on M. Then, ${}^{H}\nabla$ is metrical with respect to ${}^{S}g$ if ∇ is the Riemannian connection of g.

CONCLUSION

In this paper we study on the bundle of (1,1) type tensor frames over a smooth manifold, define the horizontal and complete lifts of symmetric linear connection into this bundle. In addition, we study the properties of the geodesic lines corresponding to the complete lift of the linear connection and investigate the relations between Sasaki metric and lifted connections on the bundle of (1,1) type tensor frames.

AUTHORSHIP CONTRIBUTIONS

Concept: H.F., H.Ç.; Design: H.F., H.Ç.; Supervision: H.F., H.Ç.; Materials: H.F., H.Ç.; Data: H.F., H.Ç.;

Analysis: H.F., H.Ç.; Literature search: H.F., H.Ç.; Writing: H.F., H.Ç.; Critical revision: H.F., H.Ç.

DATA AVAILABILITY STATEMENT

The published publication includes all graphics and data collected or developed during the study. Ana başlık olarak büyük harfle yazılıp, satır başından başlamalıdır.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article. Ana başlık olarak büyük harfle yazılıp, satır başından başlamalıdır.

ETHICS

There are no ethical issues with the publication of this manuscript. Ana başlık olarak büyük harfle yazılıp, satır başından başlamalıdır.

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