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Research Article

COMPARISON OF PARAMETER ESTIMATION METHODS IN WEIBULL DISTRIBUTION

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ABSTRACT

The main objective of this study is to compare the parameter estimation methods for Weibull distribution. We consider maximum likelihood and Bayes estimation methods for the scale and shape parameters of Weibull distribution. While computing the Bayes estimates for a Weibull distribution, the continuous conjugate joint prior distribution of the shape and scale parameters does not exist and the closed form expressions of the Bayes estimators cannot be obtained.
In this study, we assume that the scale and shape parameters have the exponential prior and they are independently distributed. We use the Lindley approximation and the Markov Chain Monte Carlo (MCMC) method to obtain the approximate Bayes estimators. In simulation study we compare the effectiveness of the parameter estimation methods with Monte Carlo simulations.
Keywords: Weibull distribution, bayes estimator, lindley approximation, Monte Carlo simulation, MCMC.

1. INTRODUCTION

There are many applications for the Weibull distribution in statistics. It was first introduced by Waloddi Weibull in 1951 to predict the life span of machines. This distribution can be applied with two or three parameters depending on the field of use. This distribution is used in quality control, modeling of deterioration periods, analysis of life tables, availability of epidemic disease, determination of earthquake risk, definition of wind speed distribution and financial applications. Weibull distribution is commonly used in data sets related to failure rates. It is a continuous and at the same time flexible distribution in this sense. Nowadays, this distribution is widely used in biology, engineering, quality control, seismic risk analysis, meteorological weather prediction models, radar systems modeling areas, wind speed distribution definition and many other field experiments.

Because of the wide applications area, it is very important to determine the best parameter estimation method for this distribution. Many authors have proposed various estimation methods for Weibull parameters. The least squares method, maximum likelihood method, moments method and Bayesian methods are used to estimate the parameters of the Weibull distribution. The maximum likelihood method is the most popular method. The efficiency of the maximum

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likelihood estimation method makes it popular. The least square method is computationally easier to handle and provides simple closed form solutions for the estimates. Hossain and Howlader (1996), made comparisons among several least squares and maximum likelihood estimator for complete samples and the shape parameter [1]. Ahmed et al. (2010) proposed Bayesian estimation with the Jeffreys prior and extension of the Jeffreys prior information for the Weibull parameters [2]. Hossain and Zimmer (2003) compared the maximum likelihood estimator to the least square estimator based on complete and censored samples [3]. Cox (1984) in [4] and Lawless (1982) in [5] made comparisons for censored data. Soland et al. (1969) introduce Bayesian analysis of the Weibull Process with unknown scale and shape parameters [6]. Guure and Ibrahim (2013) made comparisons for type-1 censored data [7].

While computing the Bayes estimates for the Weibull distribution, the continuous conjugate joint prior distribution of the shape and scale parameters does not exist and the closed form expressions of the Bayes estimators cannot be obtained. We must use approximation methods for this computation. In Bayesian approximation, the choice of prior distribution is very important. In this study, we assume that the scale parameter and the shape parameter both have the Exponential prior and they are independently distributed. We use the Lindley approximation and the Metropolis-Hasting algorithm, which is a method of Markov Chain Monte Carlo (MCMC), to obtain the approximate Bayes estimators. In simulation study we compare the effectiveness of the parameter estimation methods with Monte Carlo simulations.

In this study, we make comparison between the maximum likelihood and Bayes estimation of the standard parameterization form of Weibull distribution for the case of complete data.

The rest of the paper is organized as follows. In section 2, Weibull distribution is given. Section 3, maksimum likelihood method is given to estimate the unknown parameters for Weibull Distribution. In section 4, Bayesian estimation method is investigated. Section 4.1, estimations of the unknown Weibull parameters are obtained by using Lindley approximation. In Section 4.2, the MCMC method is explained and in subsection 4.2.1, the Metropolis-Hasting algorithm is given. In Section 5, a simulation study is presented to evaluate the performances of the estimators. Section 6, we use real data set to illustrate the estimation procedure developed in section 3-4. The last section, we make some conclusion about parameter estimation methods for Weibull distribution.

2. WEIBULL DISTRIBUTION

The Weibull distribution is a two-parameter (standard) distribution, generally a scale and a shape parameter. If a random variable X~Weibull(β,γ) then its probability density function is defined as,

$$f(x|\gamma,\beta) = \begin{cases} \beta\gamma x^{\beta-1} e^{-\gamma x^{\beta}} & , \quad x > 0 \\ 0 & , \quad x \le 0 \end{cases} \tag{1}$$

where β is the shape and γ is the scale parameter. γ is also known as the characteristic life parameter [8]. The expected value and the variance of the Weibull distribution are given,

$$E(X) = \gamma\Gamma\left(1 + \frac{1}{\beta}\right) \tag{2}$$

$$\mathrm{Var(X)} = \gamma^2\left[\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)\right] \tag{3}$$

respectively, where Γ is the gamma function. Cumulative distribution function can be derived and is defined as,

$$F(x|\gamma,\beta) = P(X \le x) = \begin{cases} 0 & , \quad x < 0 \\ 1 - e^{-\gamma x^{\beta}} & , \quad x \ge 0 \end{cases} \tag{4}$$

3. MAXIMUM LIKELIHOOD METHOD

Maximum-likelihood estimation (MLE) is one of the most common parameter estimation methods for statistical models.

Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed $Weibull(\beta, \gamma)$ random variables, where the parameters are assumed unknown. To estimate the parameters β and γ the maximum likelihood method is employed. The likelihood function of $X_1, X_2, \ldots, X_n$ can be constructed from equation (1) as

$$L(\gamma, \beta | x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i|\gamma, \beta) = \prod_{i=1}^{n} \beta\gamma {x_i}^{\beta-1} e^{-\gamma {x_i}^{\beta}} \quad (5)$$
$$= \beta^n \gamma^n \prod_{i=1}^{n} x_i^{\beta-1}\, exp\left\{-\gamma\ \sum_{i=1}^{n} x_i^{\beta}\right\}.$$

The log-likelihood function can be written as

$$log(L) = nlog(\beta) + nlog(\gamma) + (\beta - 1)\sum_{i=1}^{n} log(x_i) - \gamma \sum_{i=1}^{n} {x_i}^{\beta} \quad (6)$$

Differentiating with respect to γ and β and equating to zero, the estimating equations are obtained

$$\frac{\partial log(L)}{\partial \beta} = \left(\frac{n}{\gamma}\right) - \sum_{i=1}^{n} {x_i}^{\beta}\ = 0 \quad (7)$$

$$\frac{\partial log(L)}{\partial \gamma} = \left(\frac{n}{\beta}\right) + \sum_{i=1}^{n}(log x_i)\ - \gamma \sum_{i=1}^{n} {x_i}^{\beta}\, log x_i = 0. \quad (8)$$

The MLE of parameters are obtained by solving the above nonlinear systems of equations. It is usually more convenient to use nonlinear optimization algorithms such as Newton Raphson to numerically maximize the log-likelihood function in equation (6). In this study, we used multivariate Newton Raphson method to solve the equations (7)-(8).

4. BAYESIAN ESTIMATION METHOD

Bayesian estimation method has received a lot of attention in recent times for analysing failure time data, which has mostly been proposed as an alternative to that of the traditional methods. The Bayesian approach is based on Bayes' theorem, which was put forward by Thomas Bayes. In Bayesian method, it is desirable to estimate the θ parameter using the $\boldsymbol{x} = x_1, x_2, \ldots, x_n$ data for the statistical model defined by the probability (density) function $p(\boldsymbol{x}|\theta)$. In this method, the parameter is also considered as a random variable and therefore has its own distribution. If a prior knowledge about the parameter is not available, it is possible to make use of a non-informative prior distribution in Bayesian analysis.

When both scale and shape parameters of the Weibull distribution are unknown and considered as random variables, Soland (1969) states that the Weibull distribution does not have a conjugate continuous joint prior distribution [6]. He suggests use of mixed prior distributions, discrete for the shape parameter, continious for the scale parameter. In [9], Uniform prior for the shape parameter and Inverted Gamma prior for the scale parameter are proposed. In [10], many different prior distributions are proposed for the shape and scale parameters such as Inverted Gamma- Compound Inverted Gamma, Discrete mass function-Compound Inverted Gamma, Uniform distribution-Compound Inverted Gamma, respectively. The Gamma prior on both the scale and shape parameters are considered in [11]. In [12] a simulation study is conducted for the both Gamma priors. In [13] a Gamma prior on the scale parameter and no specific prior on the shape parameter is assumed..

In this study we assume that, both the shape and scale parameters are unknown. Suppose the prior distribution $\beta \sim Exponential(1/a)$ for the shape parameter, the prior distribution $\gamma \sim Exponential(1/b)$ for the scale parameter, and suppose that two parameters are independent of each other. Accordingly, the prior probability density functions for the parameters β and γ are,

$$\pi_1(\beta|\mathrm{a}) = ae^{-a\beta}, \quad \beta > 0 \tag{9}$$

$$\pi_2(\gamma|\mathrm{b}) = be^{-b\gamma}, \quad \gamma > 0. \tag{10}$$

Here, the hyper parameters a and b are assumed to be known real numbers. If hyper-parameters of independent exponential priors are define $a = b = 0$ note that these are non-informative priors.

The likelihood function of the Weibull distribution is obtained as,

$$L\big(f(x|\beta,\gamma)\big) = \beta^n\gamma^n \left(\prod_{i=1}^n x_i^{\beta-1}\right) e^{-\gamma\sum_{i=1}^n x_i^\beta} \tag{11}$$

To obtain the Bayes estimator, first the likelihood function and the prior distributions are multiplied,

$$P(x,\beta,\gamma)=L(f(x|\beta,\gamma)).\pi(\beta|a).\pi(\gamma|b) \tag{12}$$
$$=.a\, b\beta^n\gamma^n \left(\prod_{i=1}^n x_i^{\beta-1}\right) e^{-\gamma(\sum_{i=1}^n x_i^\beta+b)}e^{-a\beta}$$

Then, the marginal distribution of X is found by taking the integral of both parameters,

$$P(x)=\iint P(x,\beta,\gamma)\, d\gamma d\beta = \iint a\, b\beta^n\gamma^n \left(\prod_{i=1}^n x_i^{\beta-1}\right) e^{-\gamma\left(\sum_{i=1}^n x_i^\beta+b\right)}e^{-a\beta}d\gamma d\beta$$
$$=\int a\, b\beta^n \left(\prod_{i=1}^n x_i^{\beta-1}\right) e^{-a\beta}\left(\int \gamma^n e^{-\gamma\left(\sum_{i=1}^n x_i^\beta+b\right)} d\gamma\right)d\beta \tag{13}$$
$$=ab\, \Gamma\,(n+1)\int \beta^n \left(\prod_{i=1}^n x_i^{\beta-1}\right) e^{-a\beta}\,(1/(\textstyle\sum_{i=1}^n x_i^\beta + b))^{n+1}d\beta.$$

Then the likelihood function is proportioned to the marginal function and the joint posterior distribution of the two parameters is obtained as follows,

$$P(\beta,\gamma|x) = \frac{P(x,\beta,\gamma)}{P(x)} = \frac{a\, b\beta^n\gamma^n \left(\prod_{i=1}^n x_i^{\beta-1}\right) e^{-\gamma(\sum_{i=1}^n x_i^\beta+b)}e^{-a\beta}}{ab\,\Gamma\,(n+1)\int \beta^n\left(\prod_{i=1}^n x_i^{\beta-1}\right)e^{-a\beta}(1/(\sum_{i=1}^n x_i^\beta+b))^{n+1}d\beta}. \tag{14}$$

In order to obtain the marginal posterior distributions for the parameters, the parameters are integrated in turn and the marginal posterior distributions are obtained as follows,

$$P_1(\beta\,|x) = \int P(\beta,\gamma|x)\, d\gamma = \frac{a\, b\beta^n\left(\prod_{i=1}^n x_i^{\beta-1}\right)e^{-a\beta}\left(\int \boldsymbol{\gamma^n}\, \boldsymbol{e}^{-\gamma\left(\sum_{i=1}^n x_i^\beta+b\right)}\boldsymbol{d\gamma}\right)}{ab\,\Gamma\,(n+1)\int \beta^n\left(\prod_{i=1}^n x_i^{\beta-1}\right)e^{-a\beta}(1/(\sum_{i=1}^n x_i^\beta+b))^{n+1}d\beta} \tag{15}$$
$$= \frac{\beta^n\left(\prod_{i=1}^n x_i^{\beta-1}\right)e^{-a\beta}(1/(\sum_{i=1}^n x_i^\beta+b))^{n+1}}{\int \beta^n\left(\prod_{i=1}^n x_i^{\beta-1}\right)e^{-a\beta}(1/(\sum_{i=1}^n x_i^\beta+b))^{n+1}d\beta}$$

$$P_2(\gamma\,|x) = \int P(\beta,\gamma|x)\, d\beta = \int \frac{\beta^n\gamma^n \left(\prod_{i=1}^n x_i^{\beta-1}\right) e^{-\gamma(\sum_{i=1}^n x_i^\beta+b)}e^{-a\beta}}{\Gamma\,(n+1)\int \beta^n\left(\prod_{i=1}^n x_i^{\beta-1}\right)e^{-a\beta}(1/(\sum_{i=1}^n x_i^\beta+b))^{n+1}d\beta}d\beta\,. \tag{16}$$

As can be seen, the distributions obtained in both parameters are not similar to the known distributions and their closed form can not be obtained. The estimations of parameters under the quadratic loss function are the expected values of these distributions and these are obtained as follows,

$$\widehat{\beta} = E(\beta) = \frac{\int \beta^{n+1}\left(\prod_{i=1}^n x_i^{\beta-1}\right)e^{-a\beta}\left((1/(\sum_{i=1}^n x_i^\beta+b))^{n+1}\right)d\beta}{\int \beta^n\left(\prod_{i=1}^n x_i^{\beta-1}\right)e^{-a\beta}(1/\,(\sum_{i=1}^n x_i^\beta+b))^{n+1}\,d\beta} \tag{17}$$

$$\widehat{\gamma} = E(\gamma) = \frac{\int\int \beta^n\left(\prod_{i=1}^n x_i^{\beta-1}\right)e^{-a\beta}\ \gamma^{n+1}\, e^{-\gamma\left(\sum_{i=1}^n x_i^\beta+b\right)}d\gamma d\beta}{\Gamma\,(n+1)\ \int \beta^n\left(\prod_{i=1}^n x_i^{\beta-1}\right)e^{-a\beta}(1/\,(\sum_{i=1}^n x_i^\beta+b))^{n+1}\,d\beta} \tag{18}$$
$$= \frac{(n+1)\int \beta^n\left(\prod_{i=1}^n x_i^{\beta-1}\right)e^{-a\beta}\,(1/\,(\sum_{i=1}^n x_i^\beta+b))^{n+2}\boldsymbol{d}\beta}{\int \beta^n\left(\prod_{i=1}^n x_i^{\beta-1}\right)e^{-a\beta}(1/\,(\sum_{i=1}^n x_i^\beta+b))^{n+1}\,d\beta}.$$

It can be seen that (17)-(18) cannot be reduced to a closed form and numerical approximations are needed. There exist many tecniques to produce such approximations. In this study, we used

Lindley's approximation and Metropolis-Hasting algorithm, which is a Markov Chain Monte Carlo (MCMC) method, to construct Bayes estimates.

4.1. Lindley Approximation

Obtaining the Bayes estimator, expressed as the ratio of the two integrals, usually presents difficulties. Lindley (1980) developed the Lindley approximation method for the approximate solution of integrals forced in multi-parameter distributions when n is sufficiently large.

Lindley (1980) considered the ratio of the following integrals and proposed an approximate result for the solution of this ratio,

$$I = \frac{\int w(\theta)exp\{L(\theta)\}d\theta}{\int \pi(\theta)exp\{L(\theta)\}d\theta} \tag{19}$$

[14]. Here, $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ is the parameter vector, $L(\theta)$ is the logarithm of the likelihood function, $w(\theta)$ and $\pi(\theta)$ are arbitrary functions of θ. Let, $\pi(\theta)$ be the joint prior distribution of θ and $w(\theta) = u(\theta)\pi(\theta)$. The expected value of the posterior distribution is expressed as follows,

$$I = E(\, u(\theta)|x_1, x_2, \dots, x_n) = \frac{\int u(\theta)exp\{L(\theta)+G(\theta)\}d\theta}{\int exp\{L(\theta)+G(\theta)\}d\theta}$$

Here, $G(\theta) = log\pi(\theta)$. Accordingly, the Lindley approximation is given in

$$I = E(\, u(\theta)|x_1, x_2, \dots, x_n) \approx$$
$$\left\{u + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}(u_{ij} + 2u_i g_j)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} l_{ijk}\,\sigma_{ij}\sigma_{kl}u_l\right\}$$

form. Here, if

$$l_{ijk} = \frac{d^3 l}{d\theta_i d\theta_j d\theta_k}, i = 1,2, \dots, p, j = 1,2, \dots, p, k = 1,2, \dots, p$$
$$l_{ij} = \frac{d^2 l}{d\theta_i d\theta_j}, i = 1,2, \dots, p, j = 1,2, \dots, p$$
$$u_i = \frac{du(\theta)}{d\theta_i}, i = 1,2, \dots, p$$
$$u_{ij} = \frac{d^2 u(\theta)}{d\theta_i d\theta_j}, i = 1,2, \dots, p, j = 1,2, \dots, p$$
$$\sigma_{ij} = \left[-l_{ij}\right]^{-1}, i = 1,2, \dots, p, j = 1,2, \dots, p$$

then, Lindley approximation for the $p = 2$ parameter can be written as,

$$u(\hat{\theta})_{Bayes} = E(\, u(\theta)|x_1, x_2, \dots, x_n)$$
$$\approx u(\hat{u}_1, \hat{u}_2) + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}(u_{ij} + 2u_i g_j)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} l_{ijk}\,\sigma_{ij}\sigma_{kl}u_l$$

here $\hat{u}_1$ and $\hat{u}_2$, expresses the maximum likelihood estimators of u_1 and u_2.

The approximate solutions of integrals given by (17) and (18) using the Lindley approximation are given in Appendix-1 [14], [15], [16].

4.2. MCMC Method

Following Bayes' rule

$$p(\theta|y) \propto L(\theta)\Pi(\theta) \tag{20}$$

for estimation of posterior distribution of Weibull distribution with standard parameterization, we write the joint posterior distribution as

$$P(\beta, \gamma|x) \propto \beta^n \gamma^n \left(\prod_{i=1}^{n} x_i^{\beta-1}\right) e^{-\gamma \sum_{i=1}^{n} x_i^{\beta}} a e^{-a\beta}\, b e^{-b\gamma} \tag{21}$$

We can use the MCMC to get the Bayesian estimates for Weibull parameters. MCMC is a general simulation method that replaces analytic integration computations by summation over samples generated from iterative algorithms. The Metropolis-Hastings algorithm and the Gibbs algorithm are the two most popular example of a MCMC method [17]. In this study we used Metropolis-Hastings algorithm for multivariate distributions to obtain Bayes estimates. We used component-wise updating approach. This approach is given below.

4.2.1. Metropolis-Hastings Algorithm:

We have a bivariate distribution $\theta = (\beta, \gamma)$. First we initialize the sampler with some suitable values for $\beta^{(1)}$ and $\gamma^{(1)}$. At each iteration t, we first make a proposal β^* depending on the last state $\beta^{(t-1)}$. We then evaluate the acceptance ratio comparing the likelihood of $(\beta^*, \gamma^{(t-1)})$ against $(\beta^{(t-1)}, \gamma^{(t-1)})$. In the next step, we make proposal γ^* depending on the last state $\gamma^{(t-1)}$. We then evaluate the acceptance ratio comparing the likelihood of $(\beta^{(t)}, \gamma^*)$ against $(\beta^{(t)}, \gamma^{(t-1)})$.

Step by step component-wise Metropolis-Hasting algorithm is given below [17].

Step 1. Set $t = 1$
Step 2. Generate an initial value $u = (u_1, u_2, \dots, u_n)$ and set $\theta^{(t)} = u$
Step 3. Repeat

$$t = t + 1$$

Generate a proposal β^* from $q(\beta|\beta^{(t-1)})$
Evaluate the acceptance probability $\alpha = min\left(1, \frac{p(\beta^*, \gamma^{(t-1)})}{p(\beta^{(t-1)}, \gamma^{(t-1)})}\right) \frac{q(\beta^{(t-1)}|\beta^*)}{q(\beta^*|\beta^{(t-1)})}$
Generate a u from a $Uniform(0,1)$ distribution
If $u \le \alpha$, accept the proposal and set $\beta^{(t)} = \beta^*$, else set $\beta^{(t)} = \beta^{(t-1)}$
Generate a proposal γ^* from $q(\gamma|\gamma^{(t-1)})$
Evaluate the acceptance probability $\alpha = min\left(1, \frac{p(\gamma, \beta^*)}{p(\beta^{(t)}, \gamma^{(t-1)})}\right) \frac{q(\gamma^{(t-1)}|\gamma^*)}{q(\gamma^*|\gamma^{(t-1)})}$
Generate a u from a $Uniform(0,1)$ distribution
If $u \le \alpha$, accept the proposal and set $\gamma^{(t)} = \gamma^*$, else set $\gamma^{(t)} = \gamma^{(t-1)}$.

Step 4. Until $t = T$

5. SIMULATION STUDY

In this section, we conduct a Monte-Carlo simulation study to compare the performance of the classical and Bayesian estimation methods for Weibull distribution. For Bayesian estimation, we assume that both the shape and scale parameters have independent Exponential priors. We compute Bayesian estimates using Lindley's approximation and Metropolis-Hasting algorithm. We generated 1000 realizations of the Markov chains using Metropolis–Hastings algorithms. The convergence of the sequences of parameters for their stationary distributions is checked through different starting values. It is observed that after 100 burn-in periods, all the Markov chains reach their stationary condition. For all the numerical computations, we develop a program using Matlab 7 (R-2013). We also compute the maximum likelihood estimates and we compare these estimates results.

In simulation study, we generate random data from Weibull distribution. The sample size is chosen as $n = 20, 50, 80, 110$. For each sample size, samples with $\beta = 2$ and $\gamma = 2,3,4$ values are generated by simulation. The mean square error for both parameters is chosen as the criterion to compare the performance of the maximum likelihood method and the estimation results obtained with Bayesian methods. We use 1000 trial for simulation. Accordingly, the selected criterion, as an average measure of errors come from both parameters, is calculated as,

$$\boldsymbol{MSE} = \frac{\sum_{i=1}^{1000}(\beta_i-\hat{\beta}_i)^2+(\gamma_i-\hat{\gamma}_i)^2}{1000}.$$

Simulation results are given in Table 1. The Figures, from 1 to 4, represent the Weibull probability density function with the parameters of which we generate the data and with the parameters estimate via MLE, Lindley and MCMC procedures. Here we only report four group of density curve plots for $\boldsymbol{n = 20, 110}$ for parameter values $\boldsymbol{\beta = 2}$, $\boldsymbol{\gamma = 2}$ and $\boldsymbol{\beta = 2}$, $\boldsymbol{\gamma = 3}$.

Table 1. Estimation results and MSE values

Parameter Values	Maximum Likelihood Method				Bayes Method: Lindley Approximation			Bayes Method: MCMC		
	n	$\hat{\beta}$	$\hat{\gamma}$	**MSE**	$\hat{\beta}$	$\hat{\gamma}$	**MSE**	$\hat{\beta}$	$\hat{\gamma}$	**MSE**
$\beta = 2$ $\gamma = 2$	**20**	1.984113	2.148155	0.247663	2.051442	2.028711	0.274917	2.020395	2.087577	0.216158
	50	1.995733	2.057807	0.079207	2.016556	2.022375	0.074327	2.008877	2.035714	0.075657
	80	2.000999	2.033285	0.049009	2.013351	2.012337	0.047336	2.009365	2.019934	0.048222
	110	1.994418	2.025135	0.031421	2.003414	2.009948	0.030370	2.000835	2.015510	0.030826
$\beta = 2$ $\gamma = 3$	**20**	1.987253	3.206099	0.428115	2.056101	3.059351	0.376660	2.003225	3.080299	0.342021
	50	1.998278	3.084720	0.139830	2.023278	3.032213	0.131522	2.004611	3.040264	0.128670
	80	1.999780	3.066825	0.085441	2.015161	3.034573	0.081446	2.003028	3.036147	0.080368
	110	1.996473	3.025229	0.062545	2.007619	3.002079	0.061401	1.998681	3.002910	0.060378
$\beta = 2$ $\gamma = 4$	**20**	1.992641	4.342519	0.844570	2.061743	4.157718	0.723571	2.000871	4.125299	0.616358
	50	1.997864	4.124483	0.234210	2.024203	4.055863	0.217619	2.000572	4.042864	0.205615
	80	1.998724	4.072200	0.138386	2.015006	4.030102	0.132553	2.000463	4.022842	0.128528
	110	1.997488	4.049150	0.107878	2.009338	4.018666	0.104926	1.998562	4.013163	0.102924

According to the simulation study results obtained in Table 1, it can be said that the results obtained with the maximum likelihood method and the Bayesian methods are similar. But for small sample size, the estimates using with Bayesian methods are better than the MLE. When sample size increases, the maximum likelihood method and the estimates obtained by Lindley approximation and Metropolis Hasting algorithm for the Bayesian methods are close to each other. When the number of sample size increases the Mean Square Error (MSE) decrease in all cases.

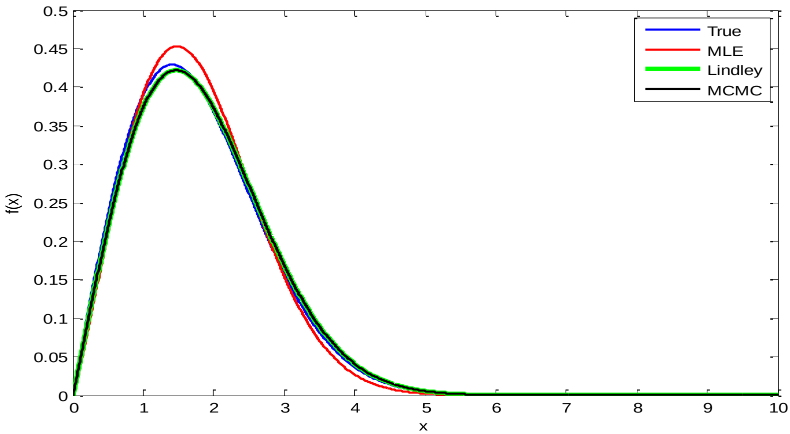


Figure 1. Weibull density curves ($n = 20, \beta = 2, \gamma = 2$)

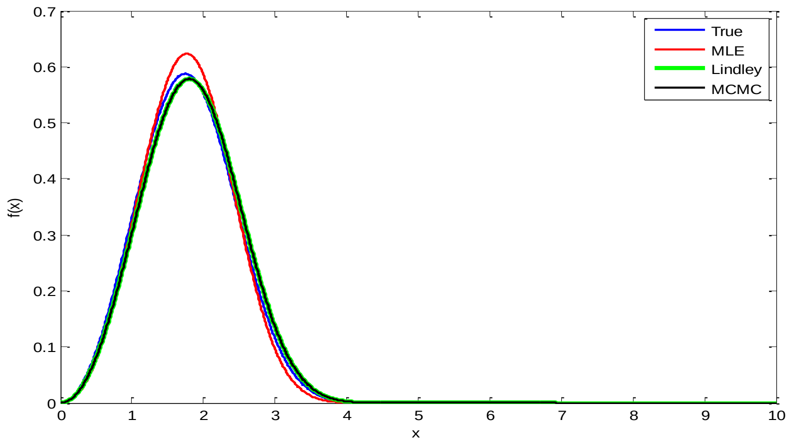


Figure 2. Weibull density curves ($n = 20, \beta = 2, \gamma = 3$)

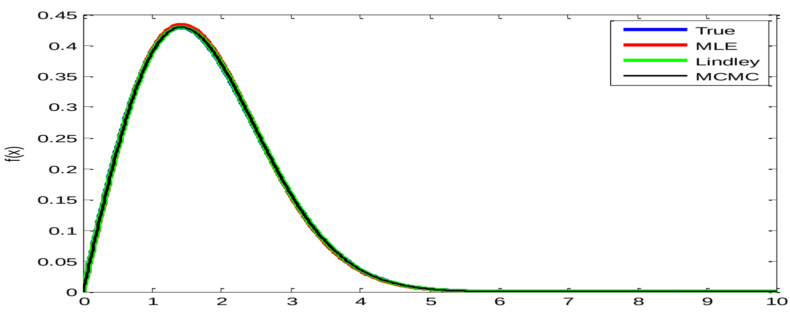


Figure 3. Weibull density curves ($n = 110, \beta = 2, \gamma = 2$)

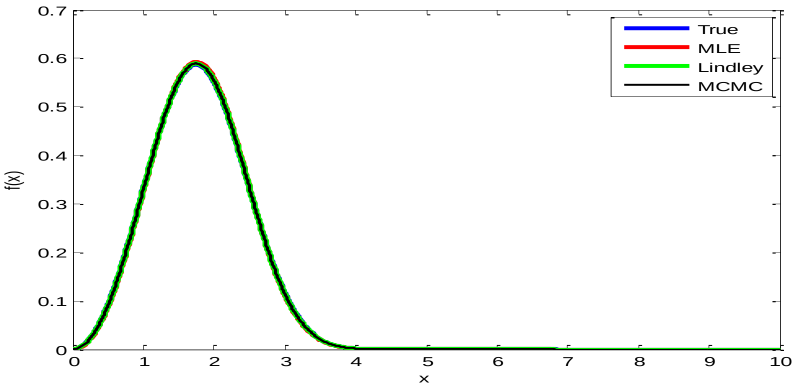


Figure 4. Weibull density curves ($n = 110, \beta = 2, \gamma = 3$)

From the density curves, for small sample size, the estimates with Bayesian methods are better fit than MLE. But, for big sample size all estimates are good fit to the real values.

6. APPLICATION (Kevlar 49/Epoxy Strands Failure At 90% Stress Level)

In this section, an actual data set is used to illustrate the estimation procedure developed in section 3-4. The following 101 data points represent the stress-rupture life of kevlar 49/epoxy strands which were subjected to constant sustained pressure at the 90% stress level until all had failed, so that we have complete data with exact times of failure. This data set was studied by Andrews and Herzberg (1985), Cooray and Ananda (2008) and Paraniaba et al. (2013).

The failure times in hours are shown below:

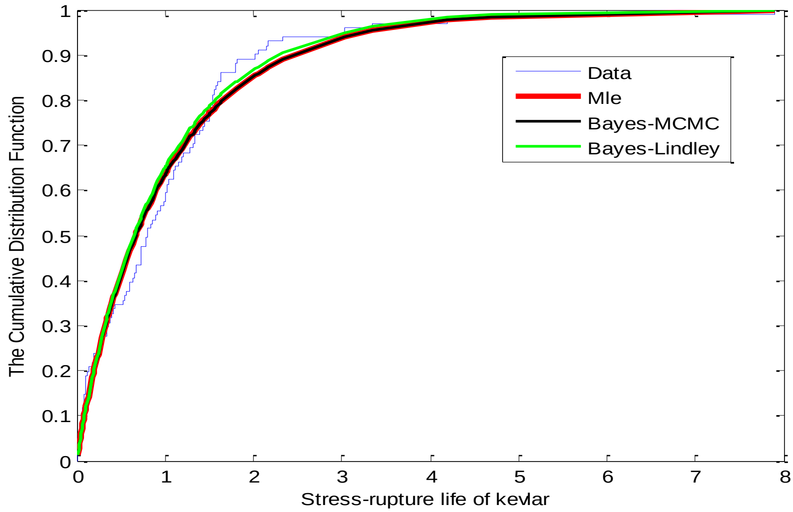
0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07,0.08, 0.09, 0.09, 0.1, 0.1, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.2, 0.23, 0.24, 0.24, 0.29 , 0.34, 0.35, 0.36, 0.38, 0.4, 0.42, 0.43, 0.52, 0.54, 0.56, 0.6, 0.6, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.8, 0.8, 0.83, 0.85, 0.9, 0.92, 0.95, 0.99, 1, 1.01, 1.02, 1.03, 1.05, 1.1, 1.1, 1.11, 1.15, 1.18, 1.2, 1.29, 1.31, 1.33, 1.34, 1.4, 1.43, 1.45, 1.5, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.8, 1.8, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.2, 4.69, 7.89.

As mentioned earlier, Weibull distribution is widely used in modeling the failure time data. In this study, two-parameter Weibull distribution is used for modelling the stress-rupture life of kevlar data. We estimated the parameters by using the MLE and Bayesian methods which are given in Section 3-4.

The estimations of the β and γ parameters obtained by using MLE and Bayesian methods are given in Table 2.

Table-2. Parameter estimates for the Stress-rupture life of kevlar data

	$\hat{\beta}$	$\hat{\gamma}$
Mle	0.925888	1.009400
Bayes-MCMC	0.927481	1.004326
Bayes-Lindley	0.947262	1.049954



Figue 5. The cumulative distribution function for stress-rupture life of kevlar data

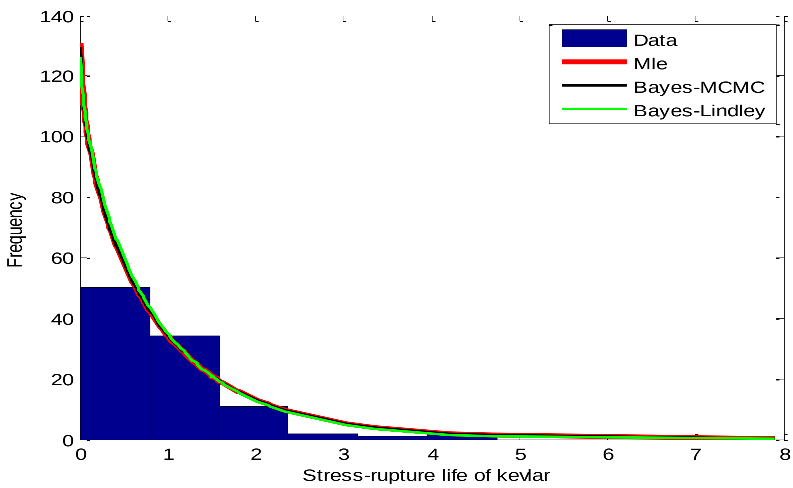


Figure 6. The histogram and fitted densities for stress-rupture life of kevlar data

It is clear that the results in Figure 5-6 are consistent with Table 2. Because of the sample size is big, parameter estimations are similar for classical and Bayesian methods. All estimated densities are good fitted to data. So, we say that we can use Weibull distribution to model the stress-rupture life of kevlar data.

7. CONCLUSION

In this paper, we use maximum likelihood estimation and Bayesian estimation for the two parameter Weibull distribution. MLE is one of the most frequently used parameter estimation methods. Newton-Raphson is one of the widely used methods for solving the system of equations especially in maximum likelihood estimation.

Bayesian estimation method receives a lot of attention in recent times. When we want to make conclusion via Bayesian method, if there isn't conjugate prior distribution, to get the posterior distribution has many difficulties. In this case, we need to use a numerical method or a MCMC method.

In this paper, Bayesian estimations are first obtained using Lindley approximation under the assumption of exponential priors while MLE are obtained using Newton-Raphson method. Second, Bayesian estimations are obtained using Metropolis-Hasting algorithm, which is a MCMC method.

A simulation study is conducted to examine and compare the performance of the estimates for different sample sizes with different values for parameters.

As a result of study, we can say that, Bayesian methods can be used as an alternative to the maximum likelihood method when the two parameters of the Weibull distribution are estimated. Especially, if sample size is small we can prefer to use the Bayesian estimation method.

Appendix 1.

Using the Lindley Approximation given in Section 3.3.2.1, the approximate solution of the integrals obtained by (17) and (18) is found as follows.

$$g(\beta,\gamma)= ae^{-a\beta} \times be^{-b\gamma}$$
$$=abe^{-(a\beta+b\gamma)}$$
$$G(\beta,\gamma)=\log g(\beta,\gamma) =\log a + \log b + \log e^{-(a\beta+b\gamma)}$$
$$=\log a + \log b - (a\beta + b\gamma)$$
$$g_1 = \frac{dG(\beta,\gamma)}{d\beta} = -a$$
$$g_2 = \frac{dG(\beta,\gamma)}{d\gamma} = -b$$
$$L(\beta,\gamma) = \beta^n\gamma^n \left(\prod_{i=1}^{n} x_i^{\beta-1}\right) e^{-\gamma\sum_{i=1}^{n} x_i^{\beta}}$$
$$\log L(\beta,\gamma) = n\log\beta + n\log\gamma + (\beta+1)\log\left(\prod_{i=1}^{n} x_i\right) - \gamma\sum_{i=1}^{n} x_i^{\beta}$$
$$L_1 = \frac{d\log L(\beta,\gamma)}{d\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log x_i - \gamma\sum_{i=1}^{n} x_i^{\beta}\log x_i$$
$$L_{12} = \frac{d^2\log L(\beta,\gamma)}{d\beta d\gamma} = \frac{d}{d\gamma}\left(\frac{n}{\beta} + \sum_{i=1}^{n} log x_i x_i - \gamma\sum_{i=1}^{n} x_i^{\beta}\log x_i\right) = -\sum_{i=1}^{n} x_i^{\beta}\log x_i$$
$$L_{112} = \frac{d^3\log L(\beta,\gamma)}{d\beta^2 d\gamma} = \frac{d}{d\beta}\left(-\sum_{i=1}^{n} x_i^{\beta}\log x_i\right) = -\sum_{i=1}^{n} x_i^{\beta}(\log x_i)^2$$
$$L_{121} = \frac{d^3\log L(\beta,\gamma)}{d\beta d\gamma d\beta} = \frac{d}{d\beta}\left(-\sum_{i=1}^{n} x_i^{\beta}\log x_i\right) = -\sum_{i=1}^{n} x_i^{\beta}(\log x_i)^2$$
$$L_{122} = \frac{d^3\log L(\beta,\gamma)}{d\beta d\gamma^2} = \frac{d^2}{d\beta d\gamma}\left(\frac{d}{d\gamma}\left(\frac{n}{\gamma} - \sum_{i=1}^{n} x_i^{\beta}\right)\right) = 0$$
$$L_2 = \frac{d\log L(\beta,\gamma)}{d\gamma} = \frac{n}{\gamma} - \sum_{i=1}^{n} x_i^{\beta}$$
$$L_{21} = \frac{d^2\log L(\beta,\gamma)}{d\gamma d\beta} = -\sum_{i=1}^{n} x_i^{\beta}\log x_i$$
$$L_{221} = \frac{d^3\log L(\beta,\gamma)}{d\gamma^2 d\beta} = 0$$
$$L_{212} = \frac{d^3\log L(\beta,\gamma)}{d\gamma d\beta d\gamma} = 0$$
$$L_{211} = \frac{d^3\log L(\beta,\gamma)}{d\gamma d\beta^2} = \frac{d}{d\gamma}\left(-\frac{n}{\beta^2} - \gamma\sum_{i=1}^{n} x_i^{\beta}(\log x_i)^2\right) = -\sum_{i=1}^{n} x_i^{\beta}(\ln x_i)^2$$
$$L_{222} = \frac{d^3\log L(\beta,\gamma)}{d\gamma^3} = \frac{2n}{\gamma^3}$$
$$L_{11} = \frac{d^2\log L(\beta,\gamma)}{d\beta^2} = -\frac{n}{\beta^2} - \gamma\sum_{i=1}^{n} x_i^{\beta}(\log x_i)^2$$

$$L_{111} = \frac{d^3 \log L(\beta,\gamma)}{d\beta^3} = \frac{2n}{\beta^3} - \gamma \sum_{i=1}^{n} x_i^{\beta} (\log x_i)^3$$

$$L_{21} = \frac{d^2 \log L(\beta,\gamma)}{d\gamma d\beta} = -\sum_{i=1}^{n} x_i^{\beta} \log x_i$$

$$L_{22} = \frac{d^2 \log L(\beta,\gamma)}{d\gamma^2} = -\frac{n}{\gamma^2}$$

$$G_{ij} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}^{-1}$$

$$G_{ij} = \frac{1}{\det(G_{ij})} \begin{bmatrix} L_{22} & -L_2 \\ -L_{21} & L_{11} \end{bmatrix}$$

$$= \frac{1}{\left(-\frac{n}{\gamma^2}\right)\times\left(-\frac{n}{\beta^2}-\gamma\sum_{i=1}^{n} x_i^{\beta} (\log x_i)^2\right)-\left(-\sum_{i=1}^{n} x_i^{\beta}\log x_i\right)^2} \begin{vmatrix} -\frac{n}{\gamma^2} & -\sum_{i=1}^{n} x_i^{\beta} \log x_i \\ -\sum_{i=1}^{n} x_i^{\beta} \log x_i & -\frac{n}{\beta^2} - \gamma \sum_{i=1}^{n} x_i^{\beta} (\log x_i)^2 \end{vmatrix}$$

$$T = \left(-\frac{n}{\gamma^2}\right) \times \left(-\frac{n}{\beta^2} - \gamma \sum_{i=1}^{n} x_i^{\beta} (\log x_i)^2\right) - \left(-\sum_{i=1}^{n} x_i^{\beta} \log x_i\right)^2$$

$$U = -\frac{n}{\gamma^2}$$

$$V = -\sum_{i=1}^{n} x_i^{\beta} \log x_i$$

$$W = -\frac{n}{\beta^2} - \gamma \sum_{i=1}^{n} x_i^{\beta} (\log x_i)^2$$

$$G_{ij} = \begin{bmatrix} \frac{U}{T} & \frac{V}{T} \\ \frac{V}{T} & \frac{W}{T} \end{bmatrix}$$

The approximate Bayes estimators obtained using the Lindley approximation for $p = 2$ are obtained as follows.

If $U(\beta,\gamma) = \beta$, then $U_1 = 1, U_2 = U_{12} = U_{21} = U_{11} = U_{22} = 0$. Therefore, the Bayes estimate of γ is defined as

$$\hat{\beta}_{Bayes} = \hat{\beta}_{MLE} + G_{11}g_1 + G_{12}g_2 + \frac{1}{2}(L_{111}{G_{11}}^2 + 3L_{112}G_{11}G_{12} + L_{222}G_{12}G_{22})$$

$$\hat{\beta}_{Bayes} = \hat{\beta} + \frac{U}{T}(-a) + \frac{V}{T}(-b) + \frac{1}{2}\left(\frac{2n}{\beta^3} - \gamma \sum_{i=1}^{n} x_i^{\beta} (\log x_i)^3 . \left(\frac{U}{T}\right)^2 + 3\left(-\sum_{i=1}^{n} x_i^{\beta} (\ln x_i)^2\right) \times \frac{U}{T}.\frac{V}{T} + \frac{2n}{\gamma^3}.\frac{V}{T}.\frac{W}{T}\right)$$

If $U(\beta,\gamma) = \gamma$, then $U_2 = 1, U_1 = U_{12} = U_{21} = U_{11} = U_{22} = 0$. Therefore, the Bayes estimate of γ is defined as

$$\hat{\gamma}_{Bayes} = \hat{\gamma}_{MLE} + G_{21}g_1 + G_{22}g_2 + \frac{1}{2}(L_{222}{G_{22}}^2 + L_{112}(G_{11}G_{22} + 2{G_{12}}^2) + L_{111}G_{11}G_{12})$$

$$\hat{\gamma}_{Bayes} = \hat{\gamma} + \frac{V}{T}(-a) + \frac{W}{T}(-b) + \frac{1}{2}\left(\frac{2n}{\gamma^3}.\left(\frac{W}{T}\right)^2 + \left(-\sum_{i=1}^{n} x_i^{\beta} (\log x)^2\right).\left(\frac{U}{T}.\frac{W}{T} + 2\left(\frac{V}{T}\right)^2\right) + \left(\frac{2n}{\beta^3} - \gamma \sum_{i=1}^{n} x_i^{\beta} (\log x)^3 . \frac{U}{T}.\frac{V}{T}\right)\right)$$

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