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Research Article

ROBUST DELAY-DEPENDENT H_∞ CONTROL DESIGN FOR UNCERTAIN TAKAGI-SUGENO TIME-DELAY SYSTEMS

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ABSTRACT

Takagi-Sugeno (T-S) fuzzy modeling is a useful tool to represent complex nonlinear systems into a class of linear subsystems with fuzzy sets and reasoning. Presented is an extension of the T-S fuzzy modeling approach for uncertain nonlinear systems with state time-varying delay to derive robust delay-dependent H_∞ control methodology. To this end, we investigate the stability and performance conditions for uncertain T-S fuzzy systems with time-varying delay by the Lyapunov-Krasovskii functional. Then, the stabilization is fulfilled through a fuzzy state-feedback controller. For the synthesis condition, one of the recently developed methods is utilized, and that the solution is dependent on the size and change rate of the delay. The formulations are performed based on the solution of linear matrix inequalities (LMIs). Finally, two numerical examples are presented to validate the effectiveness of the proposed design.

Keywords: Robust control**,** time-delay systems, Takagi-Sugeno fuzzy systems.

1. INTRODUCTION

Time-delays exist in many real-world systems due to the lags in transmission and transport, in general, they have a negative effect on the stability and control design. The stability and stabilization of these systems have been extensively studied in control literature [1-3]. To this end, stabilization and control results are categorized into two main parts: delay-independent [4] and delay-dependent [5] stabilizations. In the first case, the delay is not dependent on the delay size and holds for all positive time delays. In this case, the designed controller remains stable against all variations of the delay. However, the delay-dependent case holds for all magnitude of the delay smaller than a given bound. There are two main approaches to the stability of a delay-dependent system. The first is Razumikhin's theory, which is known as a way of solving continuous uniformly bounded delays. The second is Lyapunov-Krasovskii functionals (LKFs) that can tackle both differentiable uniformly bounded delays with delay derivatives bounded, and continuous uniformly bounded delays. In this work, the latter approach is used.

Since stability is the main consideration for any dynamical system, how to establish a less conservative stability condition is vital. To this end, the synthesis involves selecting an appropriate LKF based upon the Lyapunov stability theory. In [6], stabilization of T-S fuzzy systems with time-varying delay is investigated with augmented LKFs. In [7], delay-partitioning

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LKF is studied for the stability of T-S fuzzy systems with time-varying delays. Improvement in conservativeness is achieved in both cases with the price of heavy computation. In [8-9], delay-product-type functional approaches are proposed in the construction of LKFs. The main idea is to multiple delay terms with nonintegral terms, which presents some advantages. The crucial step in the analysis is not only the selection of LKFs but also to use accurate integral bounding techniques to obtain linear matrix inequalities. Last decades witness many efforts to derive tighter inequalities by various integral bounding techniques with the goal of finding a better way to deal with integral terms appearing in the derivatives of LKFs. Jensen's inequality [10] approach is a commonly used integral bounding method because it requires fewer decision variables than existing ones as well as offering good performance behavior**.** In [11], Wirtinger-based inequality conditions are proposed to handle single integral terms of quadratic functions. Most recently, Bessel inequality with Legendre polynomials bounding technique is investigated to reduce conservatism of the stability of the systems with time-varying delays [12]. Although conservatism is substantially reduced with derived LMIs, the computational complexity may need further analysis. In [13], a class of integral inequalities for quadratic functions via auxiliary functions is proposed. They investigate the stability analysis of time-varying delay systems using their proposed method. It is noted that these aforementioned researches mainly consist of deriving extended-like Jensen inequalities by introducing additional quadratic terms.

The T-S fuzzy control technique has offered a tool for modeling of nonlinear systems via a set of local linear models [14-15]. With this in mind for Takagi-Sugeno fuzzy modeling, a local linear controller in a specific operation region is designed, and then fuzzy interpolation incorporates the nonlinear system, defined for each sub-model by weighting functions. This modeling is simple and the system dynamics is characterized in the state-space. In this paper, the linear controller for each local sub-system that includes delay is designed. The parallel distributed compensation (PDC) scheme facilities the control design. The resulting controller is nonlinear via the fuzzy blending operation of linear controllers. This feature enables researchers to apply the well-established modern control methods [16-17]. These control methods have been utilized for linear and nonlinear systems successfully. For instance, state-feedback control [18] and output-feedback control [19] are designed for nonlinear systems without time delays. Moreover, several successful application results have been reported in the literature. In [20], a T-S approach is utilized to capture the nonlinear behavior of an electronic power steering system. Constrained and saturated control input cases are studied based on the LMI solution. Reference [21] proposes Takagi-Sugeno-Kang models to model a nonlinear anti-lock braking system (ABS) by the idea of nature-inspired optimal tuning of the membership functions. In [22], the authors investigate a hierarchical identification and robust control via the T-S fuzzy-neural model of the ABS with an active suspension system. [23] models nonlinear tire forces with membership functions, and a fuzzy state-feedback controller is computed in terms of LMIs solution for delay-free vehicle lateral dynamics. Since uncertainties exist in different application areas, robust control design is introduced for both delayed and delay-free systems in [24-25]. Moreover, designs in [26-27] investigate robust stabilization and control for uncertain fuzzy systems with time-varying delay by LKFs. The work [28] addresses the design method of delay-dependent robust control for uncertain fuzzy systems with a constant time delay. In [29], robust delay-dependent control of uncertain T-S systems with interval time-varying delay is studied. In [30], robust control problem for uncertain T–S fuzzy systems with time-varying delay in a range is studied by introducing integral inequalities. In [31], a robust control method is proposed with an LKF involving triple integral terms for interval time-varying delays. Based on the mentioned literature, research in robust control for uncertain T-S fuzzy systems with time-varying delay presents a gap to reduce the conservatism as well as the number of the decision variables in the analysis.

In the presented study, we investigate the stability, performance, and control design challenges for uncertain nonlinear time-varying delay systems. The main contribution of this work is to design a less conservative delay-dependent H_∞ fuzzy state-feedback controllers for uncertain nonlinear time-delay systems by the selection of Lyapunov-Krasovskii functional, which ensures reduced conservativeness by avoiding any model transformation or bounding of cross terms, and a relaxation method using slack variables on the basis of the work presented in [32]. Unlike the aforementioned methods, this paper deal with fast time-varying delay meaning that the upper bound of delay variation rate is greater than one. Formulations are performed based on the solution of a set of linear matrix inequalities. The proposed LMIs are solved numerically using available convex programming toolbox *cvx route* [33] to obtain the gains of fuzzy state-feedback controllers. The structure of this paper is presented as follows. Section 2 introduces the background information on T-S fuzzy modeling, Section 3 illustrates the main results and the final LMIs for controller computation, and Section 4 introduces two numerical examples to validate the findings. Lastly, conclusions are drawn in Section 5.

2. PRELIMINARIES AND BACKGROUND

2.1. Preliminaries and Notations

We use standard notations throughout the paper as follows. $\mathbb{R}$ is for the set of real numbers, $\mathbb{R}^{m\times n}$ denotes the real set of matrices with dimension of $m \times n$. Real symmetric and positive real symmetric of $n \times n$ matrices are denoted by $\mathbb{S}^{n\times n}$, $\mathbb{S}_{+}^{n\times n}$, respectively. Transpose of an element (i, j) is denoted by the star $\star$, corresponds to (j, i) in a symmetric matrix.

◻ Schur complement formula is defined for a given any symmetric matrix,

$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$, then the following conditions are equivalent, $M \geq 0, C \geq 0, A - BC^{-1}B \geq 0$.

◻ For a positive definite symmetric matrix $P > 0$ and a differentiable signal x in $[a, b] \to \mathbb{R}^n$, the following is defined in the context of Jensen's inequality:

$$\int_a^b \dot{x}(u)P\dot{x}(u) \geq \frac{1}{b-a}(x(b) - x(a))^T P(x(b) - x(a)).$$

2.2. Problem Statement

In this article, we study a nonlinear system with a time-varying delay, which is represented by the following T-S fuzzy model that is composed of r plant rules:

♦ *Plant rule i:*

IF $p_1(t)$ is M_{i1} and…. and $p_l(t)$ is M_{il} THEN

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + A_{\tau i} x(t - \tau(t)) + B_{1i} w(t) + B_{2i} u(t) \\ z(t) &= C_{1i} x(t) + C_{\tau i} x(t - \tau(t)) + D_{1i} w(t) + D_{2i} u(t), \end{aligned} \quad (1)$$

$$i = 1, 2, \dots \dots, r,$$

$x(\theta) = \phi(\cdot),\ \forall t \in [-\tau_{max}, 0]$,

where $x(t) \in \mathbb{R}^n$ is the system vector, $w(t) \in \mathbb{R}^{n_w}$ is the exogenous disturbance with finite energy in the space $l_2[0\ \infty), u(t) \in \mathbb{R}^{n_u}$ is the control input vector, $z(t) = \mathbb{R}^{n_z}$ is the controlled outputs, $\phi(\cdot)$ denotes the initial system condition, and $\tau(t)$ is a differentiable scalar function representing time-delay with bounded variation. Initial condition function ϕ is a given function in $\mathfrak{L}\,([-\tau_{max}\ 0], \mathbb{R}^n)$, r is the number of IF-THEN rules; A_i , $A_{\tau i}, B_{1i}$, B_{2i}, C_{1i}, $C_{\tau i}$, D_{1i}, and D_{2i} are real-valued constant matrices with appropriate dimensions, the premise variables and the fuzzy membership function grades are $p_j(t)$ and M_{i1} ($j = 1, \dots \dots, l, i = 1, \dots \dots, r$), respectively.

Given a pair of $x(t), u(t)$), the global model outputs are expressed:

$$\dot{x}(t) = \frac{\sum_{i=1}^{r} \omega_i\big(p(t)\big)\,[A_i x(t) + A_{\tau i} x(t-\tau(t)) + B_{1i} w(t) + B_{2i} u(t)]}{\sum_{i=1}^{r} \omega_i\big(p(t)\big)}, \tag{2}$$

$$= \sum_{i=1}^{r} h_i\big(p(t)\big)[A_i x(t) + A_{\tau i} x(t-\tau(t)) + B_{1i} w(t) + B_{2i} u(t)],$$

$$z(t) = \frac{\sum_{i=1}^{r} \omega_i\big(p(t)\big)[C_{1i} x(t) + C_{\tau i} x(t-\tau(t)) + D_{1i} w(t) + D_{2i} u(t)]}{\sum_{i=1}^{r} \omega_i\big(p(t)\big)},$$

$$= \sum_{i=1}^{r} h_i\big(p(t)\big)[C_{1i} x(t) + C_{\tau i} x(t-\tau(t)) + D_{1i} w(t) + D_{2i} u(t)],$$

$$x(\theta) = \phi(\cdot)\,, \qquad \forall t \in [-\tau_{max}, 0]$$

The truth value for the *i*-th rule is defined as

$$\omega_i\big(p(t)\big) = \prod_{j=1}^{l} M_{ij}\left(p_j(t)\right).$$

$$\omega_i\big(p(t)\big) \geq 0, \quad i = 1,2, \dots\dots, r, \qquad \sum_{i=1}^{r} \omega_i\big(p(t)\big) > 0.$$

where $M_{ij}\left(p_j(t)\right)$ is the grade of membership and the weighting function for the *i*-th rule is

$$h_i\big(p(t)\big) = \frac{\omega_i\big(p(t)\big)}{\sum_{i=1}^{r} \omega_i\big(p(t)\big)}.$$

Moreover, the fuzzy weighting functions $h_i\big(p(t)\big)$ satisfy

$$\sum_{i=1}^{r} h_i\big(p(t)\big) = 1, \;\; h_i\big(p(t)\big) \geq 0.$$

All membership functions are continuous as well as the deffuzification method is.

3. STABILITY AND PERFORMANCE OF T-S FUZZY TIME-DELAY SYSTEMS

3.1. Stability Analysis

The analysis is initiated by defining an unforced model below:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(p(t))\, A_i x(t) + A_{\tau i} x(t-\tau(t)). \tag{3}$$

The following theorem construct the sufficient condition for asymptotic stability of the unforced model.

Theorem 1: The time-delay fuzzy system presented in (3) is asymptotically stable for all 0< $\tau(t)$ $\leq \tau_{max}$ if there exist constant matrices $P, Q, R \in \mathbb{S}_{+}^{n\times n}$ such that the following LMI is feasible for $i = 1, \dots\dots, r,$

$$\begin{bmatrix} \Sigma_{1,1} & PA_{\tau i} + R & \tau_{max} A_i^T R \\ \star & -(1-\dot{\tau}(t))Q - R & \tau_{max} {A_{\tau i}}^T R \\ \star & \star & -R \end{bmatrix} < 0. \tag{4}$$

with $\sum_{1,1} = A_i^T P + PA_i + Q - R$.

Proof: We consider the following Lyapunov-Kasovskii functional

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)), \tag{5}$$
$$V_1(x(t)) = x^T(t)Px(t),$$
$$V_2(x(t)) = \int_{t-\tau(t)}^{t} x^T(\xi)Qx(\xi)\, d\xi,$$
$$V_3(x(t)) = \int_{-\tau_{max}}^{0}\int_{t+\theta}^{t} \dot{x}^T(\xi)\tau_{max}R\dot{x}(\xi)d\xi d\theta.$$

It is clear to see that $V(x(t))$ is positive definite with infinitesimal upper bound functional. In order (3) to be asymptotically stable, it is necessary and sufficient the time derivative of (5) is negative definite along the system trajectory. Then we have

$$\dot{V}_1(x(t)) = \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t),$$
$$\dot{V}_2(x(t)) = x^T(t)Qx(t) - \big(1-\dot{\tau}(t)\big)x^T\big(t-\tau(t)\big)Qx(t-\tau(t)),$$
$$\dot{V}_3(x(t)) = \tau_{max}^2\dot{x}^T(t)R\dot{x}(t) - \int_{t-\tau_{max}}^{t} \dot{x}^T(\theta)\tau_{max}R\dot{x}(\theta)d\,\theta,$$

Since $\tau(t) \le \tau_{max}$, then

$$-\int_{t-\tau_{max}}^{t} \dot{x}^T(\theta)\tau_{max}R\dot{x}(\theta)d\,\theta \;\le\; -\int_{t-\tau(t)}^{t} \dot{x}^T(\theta)\tau_{max}R\dot{x}(\theta)d\,\theta.$$

Using Jensen's inequality, we can bound the integral term in $\dot{V}_3(x(t))$

$$\int_{t-\tau(t)}^{t} \dot{x}^T(\theta)\tau_{max}R\dot{x}(\theta)d\,\theta \le -\frac{\tau_{max}}{\tau(t)}\left(\int_{t-\tau(t)}^{t}\dot{x}^T(\theta)\,d\theta\right)^T R\left(\int_{t-\tau(t)}^{t}\dot{x}^T(\theta)\,d\theta\right) =$$
$$-\frac{\tau_{max}}{\tau(t)}[x(t)-x(t-\tau(t))]^T R[x(t)-x(t-\tau(t))].$$

Bounding the $-\frac{\tau_{max}}{\tau(t)} \le$ -1, the expression for $\dot{V}_3(x(t))$

$$\dot{V}_3(x(t)) \le \tau_{max}^2\dot{x}^T(t)R\dot{x}(t) - [x(t)-x(t-\tau(t))]^T R[x(t)-x(t-\tau(t))].$$

Substituting the derivative term gives

$$\dot{V}(x(t)) \le \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) + x^T(t)Qx(t) - \big(1-\dot{\tau}(t)\big)x^T(t-\tau(t))$$
$$Qx(t-\tau_{max}) + \tau_{max}^2\dot{x}^T(t)R\dot{x}(t) - [x(t)-x(t-\tau(t))]^T R[x(t)-x(t-\tau(t))]\,. \tag{6}$$

This inequality provides the stability condition for (3). To derive the matrix form of the equation, we replace $\dot{x}$ term in (6). The stability condition for each sub-system is

$$\dot{V}(x(t)) \le \sum_{i=1}^{r} h_i(p(t))\Big[x^T(t)\big(A_i^T P + PA_i + Q - R + A_i^T\tau_{max}^2 RA_i\big)x(t) + x^T(t)\big(PA_{\tau i} + R +$$
$$A_i^T\tau_{max}^2 RA_{\tau i}\big)x(t-\tau(t)) + x^T(t-\tau(t))\big(A_{\tau i}^T P + R + A_{\tau i}^T\tau_{max}^2 RA_i\big)^T x(t) + x^T(t-$$
$$\tau(t))\big(-(1-\dot{\tau}(t))Q - R + A_{\tau i}^T\tau_{max}^2 RA_{\tau i}\big)x(t-\tau(t))\Big],$$

This leads to

$$\dot{V}(x(t)) \le \sum_{i=1}^{r} h_i(p(t))\begin{bmatrix} x(t) \\ x(t-\tau(t))\end{bmatrix}^T \begin{bmatrix} A_i^T P + PA_i + Q - R & PA_{\tau i} + R \\ \star & -(1-\dot{\tau}(t))Q - R\end{bmatrix} +$$
$$\begin{bmatrix} A^T \\ A_{\tau i}^T\end{bmatrix}\tau_{max}^2 R\begin{bmatrix} A_i^T \\ A_{\tau i}^T\end{bmatrix}^T \begin{bmatrix} x(t) \\ x(t-\tau(t))\end{bmatrix} < 0.$$

This equation turns to (4) by applying Schur complement formula. ■

3.2. Performance Analysis

Theorem 2: The time-delay fuzzy system (1) is asymptotically stable for all $0 < \tau(t) \leq \tau_{max}$ and satisfies the condition $\|z\|_2 \leq \gamma\|w\|_2$ if there exist constant matrices $P, Q, R \in \mathbb{S}_+^{n\times n}$ and a scalar γ>0 such that the following LMI holds

for all $i = 1,2, \dots \dots, r,$

$$\begin{bmatrix} \Sigma_{1,1} & PA_{\tau i} + R & PB_{1i} & C_{1i}^T & \tau_{max}A_i^T R \\ \star & -(1-\dot{\tau}(t))Q - R & 0 & C_{\tau i}^T & \tau_{max}A_{\tau i}^T R \\ \star & \star & -\gamma I & D_{1i}^T & \tau_{max}B_{1i}^T R \\ \star & \star & \star & -\gamma I & 0 \\ & & \star & \star & -R \end{bmatrix} < 0. \tag{7}$$

Proof: Applying the Schur complement formula on (7) results in

$$\begin{bmatrix} \Sigma_{1,1} & PA_{\tau i} + R & PB_{1i} & C_{1i}^T \\ \star & -(1-\dot{\tau}(t))Q - R & 0 & C_{\tau i}^T \\ \star & \star & -\gamma I & D_{1i}^T \\ \star & \star & \star & -\gamma I \end{bmatrix} - \begin{bmatrix} \tau_{max}A_i^T R \\ \tau_{max}A_{\tau i}^T R \\ \tau_{max}B_{1i}^T R \\ 0 \end{bmatrix} (-R)^{-1} * [\tau_{max}RA_i \quad \tau_{max}RA_{\tau i} \quad \tau_{max}RB_{1i} \quad 0],$$

and

$$\begin{bmatrix} \Sigma_{1,1} & PA_{\tau i} + R & PB_{1i} & C_{1i}^T \\ \star & -(1-\dot{\tau}(t))Q - R & 0 & C_{\tau i}^T \\ \star & \star & -\gamma I & D_{1i}^T \\ \star & \star & \star & -\gamma I \end{bmatrix} + \begin{bmatrix} A_i^T \\ A_{\tau i}^T \\ B_{1i}^T \\ 0 \end{bmatrix} \tau_{max}^2 R \begin{bmatrix} A_i^T \\ A_{\tau i}^T \\ B_{1i}^T \\ 0 \end{bmatrix}^T < 0.$$

Applying the second schur complement again leads us to the compact form of

$$\begin{bmatrix} \Sigma_{1,1} & PA_{\tau i} + R & PB_{1i} \\ \star & -(1-\dot{\tau}(t))Q - R & 0 \\ \star & \star & -\gamma I \end{bmatrix} + \begin{bmatrix} C_{1i}^T \\ C_{\tau i}^T \\ D_{1i}^T \end{bmatrix} \gamma^{-1} \begin{bmatrix} C_{1i}^T \\ C_{\tau i}^T \\ D_{1i}^T \end{bmatrix}^T + \begin{bmatrix} A_i^T \\ A_{\tau i}^T \\ B_{1i}^T \end{bmatrix} \tau_{max}^2 R \begin{bmatrix} A_i^T \\ A_{\tau i}^T \\ B_{1i}^T \end{bmatrix}^T < 0.$$

Further arrangements on the matrix inequality gives the final expression

$$\dot{V}(x(t)) \leq$$
$$\sum_{i=1}^r h_i(p(t)) \begin{bmatrix} x(t) \\ x(t-\tau(t)) \\ w(t) \end{bmatrix}^T \left[\begin{bmatrix} \Sigma_{1,1} & PA_{\tau i} + R & PB_{1i} \\ \star & -(1-\dot{\tau}(t))Q - R & 0 \\ \star & \star & -\gamma I \end{bmatrix} + \begin{bmatrix} C_{1i}^T \\ C_{\tau i}^T \\ D_{1i}^T \end{bmatrix} \gamma^{-1} \begin{bmatrix} C_{1i}^T \\ C_{\tau i}^T \\ D_{1i}^T \end{bmatrix}^T + \right.$$
$$\left. \begin{bmatrix} A_i^T \\ A_{\tau i}^T \\ B_{1i}^T \end{bmatrix} \tau_{max}^2 R \begin{bmatrix} A_i^T \\ A_{\tau i}^T \\ B_{1i}^T \end{bmatrix}^T \right] \begin{bmatrix} x(t) \\ x(t-\tau(t)) \\ w(t) \end{bmatrix} < 0.$$

It is important to note that this expression is equivalent to

$$\begin{aligned} \dot{V}(x(t)) \leq\ & \dot{x}^T P x + x^T P \dot{x} + x(t)^T Q x(t) - (1-\dot{\tau}(t)) x^T(t-\tau(t)) Q x(t-\tau(t)) \\ & + \tau_{max}^2 \dot{x}^T(t) R \dot{x}(t) - [x(t) - x(t-\tau(t))]^T R [x(t) - x(t-\tau(t))] \\ & - \gamma w^T(t) w(t) + \frac{1}{\gamma} z^T(t) z(t) < 0. \end{aligned}$$

In the above equation, integrating the both sides from 0 to ∞ leads to H_∞ performance index $\|z\|_{L_2} \leq \gamma\|w\|_{L_2}$ [34].

■

3.3. Employing Slack Variable Approach

The standard LMI characterization approach may cause bilinear matrix inequality due to the multiplication of the terms PA_i and RA_i unknown matrix functions are not suitable for the synthesis of finding the feasible solutions. One way of dealing with this problem is Projection Lemma. This approach does not only provide a flexible way in the synthesis, but also introduce a far less conservative condition.

Lemma 1: The time-delayed fuzzy system presented by (1) is asymptotically stable for all $0 < \tau(t) \le \tau_{max}$ and satisfies the condition $\|z\|_2 \le \gamma\|w\|_2$ if there exist constant matrices $P, Q, R, V_1, V_2, V_3 \in \mathbb{S}_+^{n\times n}$, and a scalar γ>0 such that the following LMI holds

for all $i = 1,2,\dots\dots,r,$

$$\begin{bmatrix} -V_1 - V_1^T & P - V_2^T + V_1A_i & -V_3^T + V_1A_{\tau i} & V_1B_{1i} & 0 & V_1 + \tau_{max}R \\ \star & \Psi_{22} + A_i^TV_2^T + V_2A_i & R + A_i^TV_3^T + V_2A_{\tau i} & V_2B_{1i} & C_{1i}^T & V_2 - P \\ \star & \star & \Psi_{33} + A_{\tau i}^TV_3^T + V_3A_{\tau i} & V_3B_{1i} & C_{\tau i}^T & V_3 \\ \star & \star & \star & -\gamma I & D_{1i}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & (-1-2\tau_{max})R \end{bmatrix} < 0. \tag{8}$$

with $\Psi_{22} = Q - R$ and $\Psi_{33} = -(1-\dot{\tau}(t))Q - R$.

Proof. The proof is inspired from [35]. First, we write the definition of Projection Lemma defining the following linear matrix inequality with a symmetric matrix Φ and appropriately dimensioned two matrices Λ and Γ:

$$\Phi + \Lambda^T\Theta^T\Gamma + \Gamma^T\Theta\Lambda < 0. \tag{9}$$

has a feasible solution in terms of Θ if and only if

$$\mathcal{N}_\Lambda^{\mathrm{T}}\Phi\mathcal{N}_\Lambda < 0, \text{ and} \tag{10}$$

$$\mathcal{N}_\Gamma^{\mathrm{T}}\Phi\mathcal{N}_\Gamma < 0, \tag{11}$$

where $\mathcal{N}_\Lambda$ and $\mathcal{N}_\Gamma$ are any basis of the null spaces of Λ and Γ, respectively. Writing (8) in the format (9), we have

$$\Phi = \begin{bmatrix} 0 & P & 0 & 0 & 0 & \tau_{max}R \\ \star & \Psi_{22} & R & 0 & C_{1i}^T & -P \\ \star & \star & \Psi_{33} & 0 & C_{\tau i}^T & 0 \\ \star & \star & \star & -\gamma I & D_{1i}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & (-1-2\tau_{max})R \end{bmatrix}, \Lambda = [-I \;\; A_i \;\; A_{\tau i} \;\; B_{1i} \;\; 0 \;\; I],$$

$$\Gamma = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}, \Theta = [V_1^T V_2^T \; V_3^T]^{\mathrm{T}}.$$

and the null spaces of Λ and Γ are shown as;

$$\mathcal{N}_\Lambda = \begin{bmatrix} A_i & A_{\tau i} & B_{1i} & 0 & I \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \text{ and } \mathcal{N}_\Gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

The first solvability condition in (10) leads to the LMI (7) and the following is obtained by the second solvability condition (11).

$$\begin{bmatrix} -\gamma I & D_{1i}^T & 0 \\ \star & -\gamma I & 0 \\ \star & \star & (-1-2\tau_{max})R \end{bmatrix} < 0. \tag{12}$$

■

3.4. State-Feedback H_∞ Synthesis

The design of a fuzzy controller is sought in this section. The designed controller not only provides the asymptotic stability but also possesses an energy-to-energy norm, $\mathcal{L}_2$, less than γ. The control law is given for each i-th rule as

$$u(t) = \sum_{i=1}^{r} h_i(p(t))K_i x(t). \tag{13}$$

Substituting (13) into (2), we obtain the corresponding closed-loop system

$$\dot{x}_{cl}(t) = \sum_{i=1}^{r} h_i^2(p(t))\{G_{ii}x(t) + A_{\tau i}x(t-\tau(t)) + B_{1i}w(t) + 2\sum_{i<j} h_i(p(t))h_j(p(t)) \left\{\frac{G_{ij}+G_{ji}}{2}x(t) + \frac{A_{\tau i}x(t-\tau(t)) + A_{\tau j}x(t-\tau(t))}{2} + \frac{B_{1i}w(t)+B_{1j}w(t)}{2}\right\},$$

$$z_{cl}(t) = \sum_{i=1}^{r} h_i^2(p(t))\{J_{ii}x(t) + C_{\tau i}x(t-\tau(t)) + D_{1i}w(t) + 2\sum_{i<j} h_i(p(t))h_j(p(t)) \left\{\frac{J_{ij}+J_{ji}}{2}x(t) + \frac{C_{\tau i}x(t-\tau(t)) + C_{\tau j}x(t-\tau(t))}{2} + \frac{D_{1i}w(t)+D_{1j}w(t)}{2}\right\},$$

where $G_{ij} = A_i + B_{2i}K_j$ and $J_{ij} = C_{1i} + D_{2i}K_j$.

Theorem 2: Consider the fuzzy time-delay system (1), there exist a state-feedback controller of the form (13) such that the closed-loop system is asymptotically stable for all $0 < \tau(t) \le \tau_{max}$, and satisfies the condition $\|z\|_2 \le \gamma\|w\|_2$ if there exist constant matrices $\tilde{P}, \widetilde{Q}, \tilde{R}$ and $U \in \mathbb{S}_+^{n\times n}$, two given scalars λ_2 and λ_3, matrix $Y_i \in \mathbb{S}^{n_u\times n}$, and a scalar γ>0 such that the following LMIs hold

for all $i = 1,2,\dots\dots,r$,

$$\begin{bmatrix} -2U & \tilde{P}-\lambda_2 U + A_i U + B_{2i}Y_i & -\lambda_3 U + A_{\tau i}U & B_{1i} & 0 & U+\tau_{max}\tilde{R} \\ \star & \widetilde{\Psi}_{22} & \widetilde{\Psi}_{23} & \lambda_2 B_{1i} & UC_{1i}^T + Y_i^T D_{2i}^T & \lambda_2 U - \tilde{P} \\ \star & \star & \widetilde{\Psi}_{33} & \lambda_3 B_{1i} & UC_{\tau i}^T & \lambda_3 U \\ \star & \star & \star & -\gamma I & D_{1i}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & (-1-2\tau_{max})\tilde{R} \end{bmatrix} < 0, \tag{14}$$

where $\widetilde{\Psi}_{22} = \Psi_{22} + \lambda_2(A_i U + B_{2i}Y_i + UA_i^T + Y_i^T B_{2i}^T)$, $\widetilde{\Psi}_{23} = \tilde{R} + \lambda_2 A_{\tau i}U + \lambda_3(UA_i^T + Y_i^T B_{2i}^T)$ and $\widetilde{\Psi}_{33} = \Psi_{33} + \lambda_3(A_{\tau i}U + UA_{\tau i}^T)$. Ψ_{22} and Ψ_{33} as defined earlier hold.

for all $1 \le i < j \le r$,

$$\begin{bmatrix} -2U & \tilde{P}-\lambda_2 U + \Phi_{ij} & -\lambda_3 U + \frac{A_{\tau i}U + A_{\tau j}U}{2} \\ \star & \widetilde{¥}_{22} & \widetilde{¥}_{23} \\ \star & \star & \widetilde{¥}_{33} \\ \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix} \tag{15}$$

$$
\left.\begin{matrix}
\frac{B_{1i}+B_{1j}}{2} & 0 & U+\tau_{max}\tilde{R} \\
\lambda_2\left(\frac{B_{1i}+B_{1j}}{2}\right) & \Xi_{ij} & \lambda_2 U-\tilde{P} \\
\lambda_3\left(\frac{B_{1i}+B_{1j}}{2}\right) & U\left(\frac{C_{\tau i}^T+C_{\tau j}^T}{2}\right) & \lambda_3 U \\
-\gamma I & \left(\frac{D_{1i}^T+D_{1j}^T}{2}\right) & 0 \\
\star & -\gamma I & 0 \\
\star & \star & (-1-2\tau_{max})\tilde{R}
\end{matrix}\right] < 0.
$$

where

$$
\Phi_{ij} = \frac{A_i U + B_{2i}Y_j + A_j U + B_{2j}Y_i}{2}, \Xi_{ij} = \frac{UC_{1i}^T + Y_i^T D_{2j}^T + UC_{1j}^T + Y_j^T D_{2i}^T}{2},
$$

$$
\widetilde{¥}_{22} = \Psi_{22} + \lambda_2\left(\frac{A_i U + B_{2i}Y_j + UA_i^T + Y_i^T B_{2j}^T + A_j U + B_{2j}Y_i + UA_j^T + Y_j^T B_{2i}^T}{2}\right),
$$

$$
\widetilde{¥}_{23} = \tilde{R} + \lambda_2\left(\frac{A_{\tau i}U + A_{\tau j}U}{2}\right) + \lambda_3\left(\frac{UA_i^T + Y_i^T B_{2j}^T + UA_j^T + Y_j^T B_{2i}^T}{2}\right),
$$

$$
\widetilde{¥}_{33} = \Psi_{33} + \lambda_3\left(\frac{A_{\tau i}U + UA_{\tau i}^T + A_{\tau j}U + UA_{\tau j}^T}{2}\right).
$$

Then the corresponding control law is given by $K_i = Y_i U^{-1}$.

Proof. First we substitute the closed-loop matrices into (8) to derive the synthesis condition. Three different slack variable matrices are chosen to analyse the condition in (8) as $V_1 = V, V_2 = \lambda_2 V$ and $V_3 = \lambda_3 V$ where λ_2 and λ_3 are given scalars. Note that the above LMI is an inequality only with fixed λ_2 and λ_3. Defining the new variables $U = V^{-1}$ and $Y_i = K_i U$ by applying the congruence transformation using matrix diagram (U, U, U, I, I, U) to LMI (8) gives the result of (14) with $\tilde{P} = U^T P U$, $\tilde{R} = U^T R U$, $\tilde{Q} = U^T Q U, \tilde{\Psi}_{22} = U^T \Psi_{22} U$ and $\tilde{\Psi}_{33} = U^T \Psi_{33} U$ for $1 \le i \le r$, and the result of (15) with $\widetilde{¥}_{22} = U^T \Psi_{22} U$ and $\widetilde{¥}_{33} = U^T \Psi_{33} U$ for $1 \le i < j \le r$. This completes the proof. ■

Remark 1: We have two LMIs conditions to be addressed to reach a feasible solution. The condition (14) seeks a feasible solution for each-subsequent rules, and the condition (15) for the cross terms. To capture the nonlinear behavior of a given system, the derived LMIs have to be solved simultaneously for each rule. Then fuzzy blending control law is employed to the nonlinear model to achieve the prescribed performance requirements.

3.5. Robust H_∞ Controller Synthesis for Uncertain T-S Fuzzy Systems with Time-Delay

A T-S fuzzy time-delay model with norm-bounded uncertainties in the state and control that is composed of r plant rules:

♦ *Plant rule i:*

IF $p_1(t)$ is M_{i1} and….. and $p_l(t)$ is M_{il} THEN

$$
\dot{x}(t) = (A_i + \Delta A_i)x(t) + (A_{\tau i} + \Delta A_{\tau i})x(t-\tau(t)) + B_{1i}w(t) + (B_{2i} + \Delta B_{2i})u(t) \tag{16}
$$

$$
z(t) = C_{1i}x(t) + C_{\tau i}x(t-\tau(t)) + D_{1i}w(t) + D_{2i}u(t), \qquad i = 1, 2, \dots\dots, r,
$$

$$
x(\theta) = \phi(\cdot),\ \forall t \in [-\tau_{max}, 0].
$$

where the matrices ΔA_i, $\Delta A_{\tau i}$, and ΔB_{2i} denote the uncertainties in the system in the form of

$$[\Delta A_i, \Delta A_{\tau i}, \Delta B_{2i}] = H_i \Delta(t) [E_i, E_{\tau i}, E_{2i}], \tag{17}$$

where $H_i, E_i, E_{\tau i}, E_{2i}$ are known constant matrices and $\Delta(t)$ is an unknown time-varying matrix function satisfying

$$\Delta(t)^T \Delta(t) \leq I. \tag{18}$$

The following condition provides necessary and sufficient condition for the synthesis of a state-feedback H_∞ controller, which guarantees the asymptotic stability, and provides a prescribed disturbance attenuation level in the sense of $\mathcal{L}_2$ energy norm of the uncertain T-S fuzzy system, presented in (16).

Theorem 3: Consider the uncertain fuzzy time-delay system in (16), there exist a state-feedback controller of the form (13) with all admissible uncertainties of the form (17) and satisfying (18) such that the closed-loop system is asymptotically stable for all $0 < \tau(t) \leq \tau_{max}$, and satisfies the condition $\|z\|_2 \leq \gamma \|w\|_2$ if there exist constant matrices $\tilde{P}, \tilde{Q}, \tilde{R}$ and $U \in \mathbb{S}_+^{n\times n}$, two given scalars λ_2 *and* λ_3, matrix $Y_i \in \mathbb{S}^{n_u \times n}$, and a scalar γ>0 such that the following LMIs hold

for all $i = 1,2, \dots \dots, r$,

$$\begin{bmatrix} -2U & \tilde{P} - \lambda_2 U + A_i U + B_{2i} Y_i & -\lambda_3 U + A_{\tau i} U & B_{1i} & 0 & U + \tau_{max} \tilde{R} & H_i & 0 \\ \star & \tilde{\Psi}_{22} & \tilde{\Psi}_{23} & \lambda_2 B_{1i} & U C_{1i}^T + Y_i^T D_{2i}^T & \lambda_2 U - \tilde{P} & \lambda_2 H_i & U E_i^T + Y_i^T E_{2i}^T \\ \star & \star & \tilde{\Psi}_{33} & \lambda_3 B_{1i} & U C_{\tau i}^T & \lambda_3 U & \lambda_3 H_i & U E_{\tau i}^T \\ \star & \star & \star & -\gamma I & D_{1i}^T & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & (-1 - 2\tau_{max}) \tilde{R} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\epsilon_{ii} I & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\alpha_{ii} I \end{bmatrix} < 0, \tag{19}$$

where $\alpha_{ii} = \epsilon_{ii}{}^{-1}$, and $\tilde{\Psi}_{22}$, $\tilde{\Psi}_{23}$, and $\tilde{\Psi}_{33}$ as defined earlier hold. And

for all $1 \leq i < j \leq r$,

$$\begin{bmatrix} -2U & \tilde{P} - \lambda_2 U + \Phi_{ij} & -\lambda_3 U + \dfrac{A_{\tau i} U + A_{\tau j} U}{2} & \dfrac{B_{1i} + B_{1j}}{2} \\ \star & \tilde{¥}_{22} & \tilde{¥}_{23} & \lambda_2 \left(\dfrac{B_{1i} + B_{1j}}{2} \right) \\ \star & \star & \tilde{¥}_{33} & \lambda_3 \left(\dfrac{B_{1i} + B_{1j}}{2} \right) \\ \star & \star & \star & -\gamma I \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{matrix} \tag{20}$$

$$
\left.\begin{array}{cccc}
0 & U+\tau_{max}\tilde{R} & H_i & 0 \\
\Xi_{ij} & \lambda_2 U-\tilde{P} & \lambda_2 H_i & \frac{UE_i^T+Y_i^T E_{2j}^T+UE_j^T+Y_j^T E_{2i}^T}{2} \\
U\left(\frac{C_{\tau i}^T+C_{\tau j}^T}{2}\right) & \lambda_3 U & \lambda_3 H_i & \frac{UE_{\tau i}^T+UE_{\tau j}^T}{2} \\
\frac{D_{1i}^T+D_{1j}^T}{2} & 0 & 0 & 0 \\
-\gamma I & 0 & 0 & 0 \\
\star & (-1-2\tau_{max})\tilde{R} & 0 & 0 \\
\star & \star & -\epsilon_{ij} I & 0 \\
\star & \star & \star & -\alpha_{ij} I
\end{array}\right] < 0.
$$

where $\alpha_{ij} = {\epsilon_{ij}}^{-1}$, and Φ_{ij}, $\widetilde{\yen}_{22}$, $\widetilde{\yen}_{23}$, $\widetilde{\yen}_{33}$, Ξ_{ij} as defined earlier hold.

Proof. We first substitute the norm-bounded uncertainties into the LMI conditions (14-15) of Theorem 2, a new set of LMI conditions is obtained by summation of initial LMIs (14-15) and corresponding uncertain part as shown below,

for all $i = 1,2, \dots \dots, r,$

$$
(21) = (14) + \begin{bmatrix}
0 & \Delta A_i U + \Delta B_{2i} Y_i & \Delta A_{\tau i} U & 0 & 0 & 0 \\
\star & (2,2)_{ii} & (2,3)_{ii} & 0 & 0 & 0 \\
\star & \star & (3,3)_{ii} & 0 & 0 & 0 \\
\star & \star & \star & 0 & 0 & 0 \\
\star & \star & \star & \star & 0 & 0 \\
\star & \star & \star & \star & \star & 0
\end{bmatrix} < 0,
$$

where $(2,2)_{ii} = \lambda_2\left(\Delta A_i U + \Delta B_{2i} Y_i + U \Delta A_i^T + Y_i^T \Delta B_{2i}^T\right)$,
$(2,3)_{ii} = \lambda_2 \Delta A_{\tau i} U + \lambda_3\left(U \Delta A_i^T + Y_i^T \Delta B_{2i}^T\right)$,
$(3,3)_{ii} = \lambda_3\left(\Delta A_{\tau i} U + U \Delta A_{\tau i}^T\right)$. And

for all $1 \le i < j \le r,$

$$
(22) = (15) + \begin{bmatrix}
0 & \frac{\Delta A_i U + \Delta B_{2i} Y_j + \Delta A_j U + \Delta B_{2j} Y_i}{2} & \frac{\Delta A_{\tau i} U + \Delta A_{\tau j} U}{2} & 0 & 0 & 0 \\
\star & (2,2)_{ij} & (2,3)_{ij} & 0 & 0 & 0 \\
\star & \star & (3,3)_{ij} & 0 & 0 & 0 \\
\star & \star & \star & 0 & 0 & 0 \\
\star & \star & \star & \star & 0 & 0 \\
\star & \star & \star & \star & \star & 0
\end{bmatrix} < 0,
$$

where

$$
(2,2)_{ij} = \frac{\lambda_2\left(\Delta A_i U + \Delta B_{2i} Y_j + U \Delta A_i^T + Y_j^T \Delta B_{2i}^T + \Delta A_j U + \Delta B_{2j} Y_i + U \Delta A_j^T + Y_i^T \Delta B_{2j}^T\right)}{2},
$$

$$
(2,3)_{ij} = \frac{\lambda_2(\Delta A_{\tau i} U + \Delta A_{\tau j} U) + \lambda_3\left(U \Delta A_i^T + Y_j^T \Delta B_{2i}^T + U \Delta A_j^T + Y_i^T \Delta B_{2j}^T\right)}{2},
$$

$$
(3,3)_{ij} = \frac{\lambda_3\left(\Delta A_{\tau i} U + U \Delta A_{\tau i}^T + \Delta A_{\tau j} U + U \Delta A_{\tau j}^T\right)}{2}.
$$

These conditions are equivalent to

for all $i = 1,2, \dots \dots, r,$

(21)= (14)+

$$He(\begin{bmatrix} H_i \\ \lambda_2 H_i \\ \lambda_3 H_i \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta(t)[0 \quad E_i U + E_{2i} Y_i \quad E_{\tau i} U \quad 0 \quad 0 \quad 0]) < 0,$$

for all $1 \leq i < j \leq r$,

(22)= (15)+

$$He(\begin{bmatrix} H_i \\ \lambda_2 H_i \\ \lambda_3 H_i \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta(t) \left[0 \quad \frac{E_i U + E_{2i} Y_j + E_j U + E_{2j} Y_i}{2} \quad \frac{E_{\tau i} U + E_{\tau j} U}{2} \quad 0 \quad 0 \quad 0\right]) < 0.$$

where $He(G)$ Hermitian operator is defined as $He(G) = G + G^T$. Finally, the following inequality is employed [36]

$$\Omega \Delta(t) ₣ + ₣^T \Delta(t)^T \Omega^T \leq {\epsilon_{ii}}^{-1} \Omega \Omega^T + \epsilon_{ii} ₣^T ₣.$$

which holds for all scalars ϵ_{ii}, $i = j$, and ϵ_{ij}, $i < j$, and all constant matrices Ω and ₣ of appropriate dimension. Schur complement formula is employed to (14-15) to finally compute the LMIs in (19-20). ■

4. NUMERICAL EXAMPLES

Example 1: The following is adopted and modified from [26]. Uncertain T-S fuzzy time-delay model is:

Rule 1: IF $x_2(t)$ is M_1 THEN

$\dot{x}(t) = (A_1 + H\Delta(t)E)x(t) + (A_{\tau 1} + H\Delta(t)E)x(t - \tau(t)) + B_{11}w(t) + (B_{21} + H\Delta(t)E)u(t)$,
$z(t) = C_{11}x(t) + D_{21}u(t)$,

Rule 2: IF $x_2(t)$ is M_2 THEN

$\dot{x}(t) = (A_2 + H\Delta(t)E)x(t) + (A_{h2} + H\Delta(t)E)x(t - \tau(t)) + B_{12}w(t) + (B_{22} + H\Delta(t)E)u(t)$,
$z(t) = C_{12}x(t) + D_{22}u(t)$.

where

$A_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.2 \end{bmatrix}$, $A_{\tau 1} = \begin{bmatrix} -1 & 0.4 \\ 0.2 & 0.1 \end{bmatrix}$, $B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B_{21} = \begin{bmatrix} 0.1 & 0.4 \\ 0 & -1 \end{bmatrix}$, $C_{11} = \begin{bmatrix} 0.1 & 1 \\ 0 & 1 \end{bmatrix}$,
$D_{21} = \begin{bmatrix} 0.3 & -0.3 \\ 0 & 0.2 \end{bmatrix}$,
$A_2 = \begin{bmatrix} 0.1 & 0.3 \\ 0.7 & 0.1 \end{bmatrix}$, $A_{\tau 2} = \begin{bmatrix} 0.1 & 0 \\ 0.5 & 0.4 \end{bmatrix}$, $B_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B_{22} = \begin{bmatrix} 0.2 & 1 \\ 0.4 & -0.3 \end{bmatrix}$, $C_{12} = \begin{bmatrix} 0.1 & 0.4 \\ 0 & 0.1 \end{bmatrix}$,
$D_{22} = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.1 \end{bmatrix}$, $H = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.1 \end{bmatrix}$, $E = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$, $\Delta(t) = \begin{bmatrix} cost & 0 \\ 0 & sint \end{bmatrix}$

To present a comparison study, the disturbance attenuation level is chosen as $\gamma = 2$. According to the Theorem 3 in this paper, Table 1 provides maximum allowable time delay bounds for different delay variation rates below

Table 1. Comparison of maximum allowable time delay

$\dot{\tau}$	0.1	0.5	0.9	1.1	5×10[5]
τ_{max} [26]	0.9103	0.8763	0.7832	0.7454	0.7454
τ_{max} [27]	0.9450	0.8997	0.7941	0.7480	Not reported
τ_{max} of Theorem 3	0.9667	0.9238	0.8322	0.7959	0.7959

LMIs (19-20) are simultaneously solved to compute the maximum time-delay allowing for controller synthesis with respect to different delay variation rates $\dot{\tau}$ using *cvx route* semidefinite programming mode. The solution is sensitive to the selections of the λ_2 and λ_3. After an 2-D search, λ_2=2.72 and λ_3=-0.5 are obtained. As observed from the upper bounds on the allowable delay in Table 1, the method proposed in this paper exhibits improved performance.

Example 2: This example is to control of a truck trailer [2]. The uncertain time-delay state-space model is given as

$$\dot{x}_1(t) = -a\frac{v\bar{t}}{(L+\Delta L(t))t_0}x_1(t) - (1-a)\frac{v\bar{t}}{(L+\Delta L(t))t_0}x_1\big(t-\tau(t)\big) + w(t) + \frac{v\bar{t}}{lt_0}u(t),$$
$$\dot{x}_2(t) = a\frac{v\bar{t}}{(L+\Delta L(t))t_0}x_1(t) + (1-a)\frac{v\bar{t}}{(L+\Delta L(t))t_0}x_1(t-\tau(t)),$$
$$\dot{x}_3(t) = \frac{v\bar{t}}{t_0}\sin\left[x_2(t) + a\frac{v\bar{t}}{2(L+\Delta L(t))t_0}x_1(t) + (1-a)\frac{v\bar{t}}{2(L+\Delta L(t))t_0}x_1(t-\tau(t))\right].$$

The values are stated as a=0.7, $l = 2.8$, L = 5.5, v = -1.0, $\bar{t}$ = 2.0, t_0 = 0.5, and −0.2619 $\le \Delta L(t) \le$ 0.2895 same as [28].

The following fuzzy rules describe the behavior of the local dynamics of the fuzzy uncertain system

Rule 1: IF $\theta(t) = x_2(t) + a\frac{v\bar{t}}{2L}x_1(t) + (1-a)x_1(t-\tau(t))$ is about 0,

THEN $\dot{x}(t) = (A_1 + \Delta A_1)x(t) + (A_{\tau 1} + \Delta A_{\tau 1})x(t-\tau(t)) + B_{11}w(t) + B_{21}u(t)$,

Rule 2: IF $\theta(t) = x_2(t) + a\frac{v\bar{t}}{2L}x_1(t) + (1-a)x_1(t-\tau(t))$ is about π or $-\pi$,

THEN $\dot{x}(t) = (A_2 + \Delta A_2)x(t) + (A_{\tau 2} + \Delta A_{\tau 2})x(t-\tau(t)) + B_{12}w(t) + B_{22}u(t)$,

where

$$A_1 = \begin{bmatrix} -a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{v^2\bar{t}^2}{2Lt_0} & \frac{v\bar{t}}{t_0} & 0 \end{bmatrix}, A_{\tau 1} = \begin{bmatrix} -(1-a)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{v^2\bar{t}^2}{2Lt_0} & 0 & 0 \end{bmatrix}, B_{11} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, B_{21} = \begin{bmatrix} \frac{v\bar{t}}{lt_0} \\ 0 \\ 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{dv^2\bar{t}^2}{2Lt_0} & \frac{v\bar{t}}{t_0} & 0 \end{bmatrix}, A_{\tau 2} = \begin{bmatrix} -(1-a)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{dv^2\bar{t}^2}{2Lt_0} & 0 & 0 \end{bmatrix}, B_{12} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, B_{22} = \begin{bmatrix} \frac{v\bar{t}}{lt_0} \\ 0 \\ 0 \end{bmatrix}.$$

$\Delta A_1 = \Delta A_{\tau 1} = \Delta A_2 = \Delta A_{\tau 2} = H\Delta(t)E$ with $H = [0.255\ 0.255\ 0.255]^T$, $E = [0.1\ 0\ 0]$, and $\Delta(t) = \sin(t)$.

The value for $d = 10 * t_0/\pi$ and the following is the membership functions

$$h_1(\theta(t)) = \left(1 - \frac{1}{1+\exp(-3(\theta(t)-0.5\pi))}\right)\left(\frac{1}{1+\exp(-3(\theta(t)+0.5\pi))}\right), h_2(\theta(t)) = 1 - h_1(\theta(t)).$$

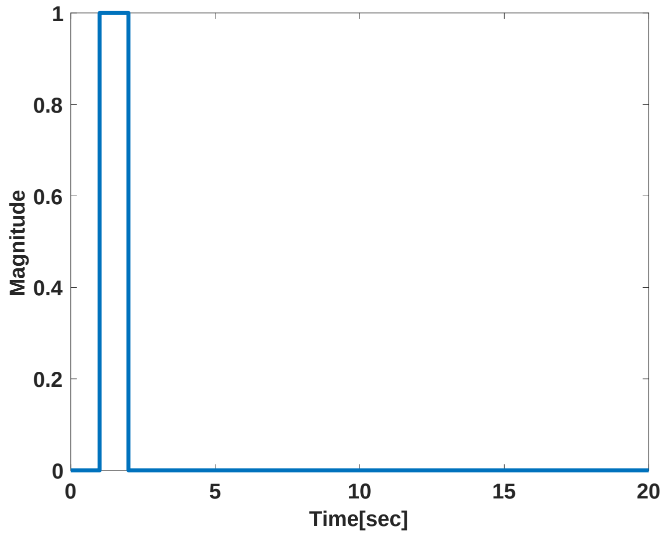

Figure 1. Disturbance profile.

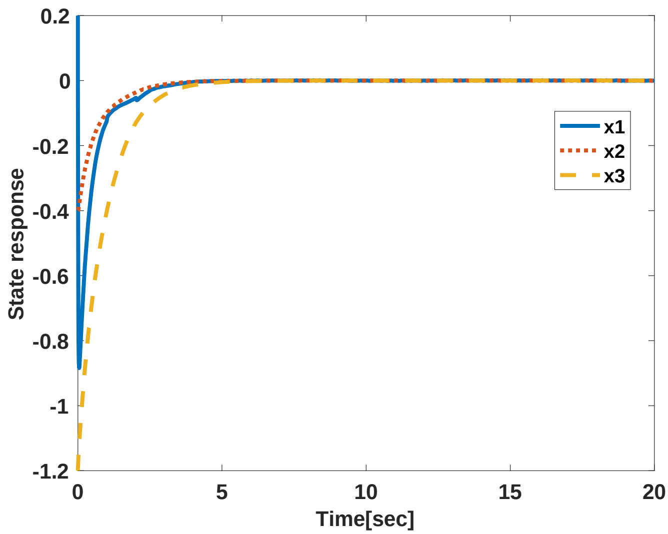

Figure 2. Initial condition response for ($\tau(t) = 0.5\sin(t)$).

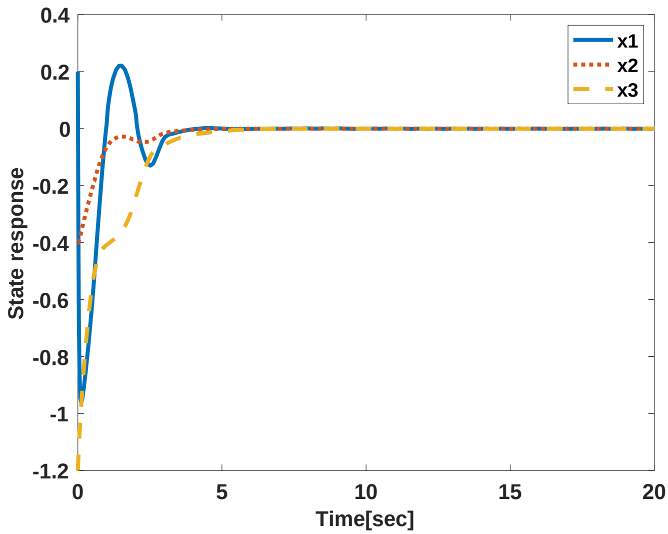

Figure 3. Initial condition response for $(\tau(t) = \sin(t))$.

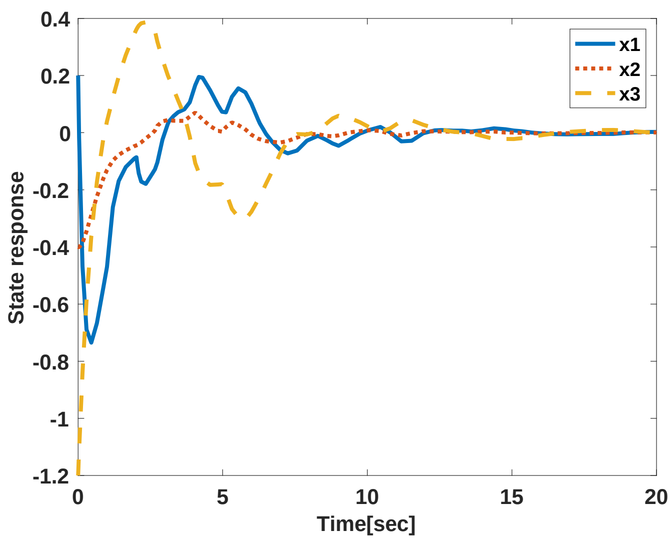

Figure 4. Initial condition response for $(\tau(t) = 5\sin(t))$.

The simulation are performed for an initial condition $\phi(\cdot)$ = [0.2 -0.4 -1.2], and time-varying delays: $\tau \in [-0.5\ 0]$, $\tau \in [-1\ 0]$, and $\tau \in [-5\ 0]$. The fuzzy control law $u(t) = \sum_{i=1}^{2} h_i K_i x(t)$ is applied to the system. The controller gains are computed using *cvx route*. The regulated output is set $z_1(t) = z_2(t)$ = [1 0 0]$x(t)$, and Figure 1 shows the applied disturbance profile to analyze the performance of the presented system. It is noted that the designed robust fuzzy H_∞ state-feedback control law seeks to minimize the induced $\mathcal{L}_2$ norm, i.e., the energy-to-energy norm from disturbance $d(t)$ to output $z(t)$. Figures 2-4 depict state responses of the truck trailer system under the designed fuzzy controller for three different levels of upper delay bound.

For the first case, the delay is bounded within 0-0.5 seconds with the highest value (τ_{max}) of 0.5 seconds. The induced $\mathcal{L}_2$ (γ disturbance attenuation level) performance level is obtained as +0.0454709, which guarantees the internal stability of the system with time-varying delay. In the second case, the delay is varied within 0-1 second with the highest value (τ_{max}) of 1 second and the induced $\mathcal{L}_2$ performance level is obtained as +0.103349. One should note that the appropriate selection of λ_2 and λ_3 is vital, and the assigned values are 15 and 0.35, respectively. The initial condition plots in Figures 2-3 ensure that output of the states converges to zero within a short period of time, as well as the fuzzy controller possesses an improved disturbance rejection. As observed in Figure 2, a better disturbance rejection and better state convergence are achieved for $\tau(t) = 0.5\sin(t)$. In addition, it is interpreted that the negative impacts of time-varying delay on the regulated output are less than that of the case in Figure 3. Generally speaking, delay bound degenerate the closed-loop stability and performance at high values. To this end, the case of $\tau(t) = 5\sin(t)$ is also plotted in Figure 4, which results in larger overshot on the state outputs along with a significant deterioration of the disturbance rejection. The induced $\mathcal{L}_2$ performance level is obtained as +1.59838. The system states converge to zero value after a reasonable time period. It is proven that the proposed design can even handle high delay bound, wherein the presented system is internally stabilized with improved disturbance rejection. The simulation results show that the uncertain systems with time-varying delays are effectively controlled using the proposed robust control strategy.

5. CONCLUSION

Delay-dependent stabilization and robust H_∞ control design for uncertain Takagi-Sugeno fuzzy systems with time-varying delay are studied in this paper. The nonlinear models are reformulated within the Takagi-Sugeno framework, and resulting individual sub-models are blended by the fuzzy operation of weighting functions. Lyapunov-Krasovskii functionals are employed to accommodate different types of delay in size and bound. Both the stability and performance conditions are realized through a set of linear matrix inequalities to derive a robust fuzzy controller. In the sequel, two numerical examples are provided. In the first example, we compute the maximum time-delay allowing for controller synthesis with respect to different delay variation rates. In the second example, we plot the time histories of the initial condition stabilization of an uncertain T-S fuzzy system for different delay ranges and bounds.

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