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Research Article

ANALYSIS OF RANDOM DISCRETE TIME LOGISTIC MODEL

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ABSTRACT

In this study, the behavior of the logistic difference model is investigated under random conditions using discrete probability distributions. The logistic difference model consists of parameters that depend on the population models to be used. For the study of random difference equation population models, the parameters are treated as random variables which constitutes the basis of the study. Random models were created using Uniform, Bernouilli, Binom, Negative Binomial (or Pascal), Geometric, Hypergeometric, Poisson distributions and their numerical characteristics are obtained through their simulations. Then, the results showing random numerical characteristics such as expected value, variance, standard deviation, coefficient of variation and confidence intervals were obtained with MATLAB package program. Analysis of random logistic difference model is given with the help of graphics and tables.
Keywords: Logistic difference equation, probability distributions, expected value, variance.

1. INTRODUCTION

While random dynamical systems are expressed as an equation mathematically, the events observed in nature also represent the dynamic process. These systems are systems that change depending on time. Difference equations arise in discretization methods of differential equations. This theory is also used in the mathematical expression of discrete events based on time. Difference equations have many applications in various fields of mathematics and science, such as economics, biology, signal processing, computer engineering, genetics, health sciences, ecology (environmental science) and fixed point theory. Mathematical modeling of population dynamics has an important place in the field of difference equations, and Thomas Malthus developed the first model in 1798. Thomas Malthus in his work " An Essay on the Principle of Population " stated that while food sources are growing linearly, the human population shows exponential growth and finally competition for food will emerge, poverty and war will be inevitable if the population is not taken into account such events as war, disease and famine. Considering that the obstacles mentioned by Thomas Malthus are proportional to the square of the population growth rate, some kind of "resistance" occurs and the population does not grow geometrically for a long time. Later in 1838, Verhulst developed the population growth model in which population size, known as the logistic equation, was limited by carrying capacity.

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To describe the population briefly, it is a community of the same species living in the same place and at the same time[33]. Population dynamics, on the other hand, are one or more population sizes, den sity, time dependent numerical changes of age distribution and an ecology branch that studies them [31].

According to the population dynamic models, discrete time models (n = 0,1,2,…) and continuous time models (n∈R) are divided into two. While creating the population model, the equations of the population model are obtained by first defining the behaviors of individuals and making use of them [30]. Difference equation of this population model

$$X(n+1) = X(n) + rX(n)\left(1 - \frac{X(n)}{K}\right) \tag{1}$$

If white noise is added to this equation, we obtain

$$X(n+1) = X(n) + rX(n)\left(1 - \frac{X(n)}{K}\right) + noise(rand - 0.5)2$$

it is obtained and r refers to the birth rate per capita as K carrying capacity. It is studied using random effect terms with some probability distributions for the study of the population model. Numerical characteristics of the model randomized using discrete uniform distribution, Bernouilli distribution, Binom distribution, Negative binomial (or Pascal) distribution, Geometric Distribution, Hypergeometric distribution and Poisson distribution will be obtained to compare the random behaviors of these distributions.[26]

While modeling the population dynamics has been used for a long time, its effect on randomness models has been studied very little. " Stochastic analogues of deterministic single-species population models", one of the major articles by Brännström and Sumpter, is the study [29]. This study shows that noise in the population process depends on whether it is environmental or demographic. If the noise is demographic, it is proportional to the variance around expectation, and when it is environmental, the variation is not trivial but somehow depends on how the variation enters the model parameter, but if the environmental population affects the product multiplied, the variance is said to be proportional to the square of the expectation. They examined the suitability of the models by making various comparisons such as parameter estimation and variance analysis using the maximum likelihood method of various models of the term added noise. Here, the demographic model is much better than the environmental model in locating the noise created by population processes where noise is predominantly demographic. [23]

In this study, random difference equations are used to obtain the solution of the logistic population model and they are randomized using random effect terms with various probability distributions. The approximate solutions obtained are given with the expected value, variance, standard deviation, coefficient of variation and confidence interval graphs and tables for results. In Chapters 2 and 3, introductory information about random difference equations and discrete probability distributions are given. In Chapter 4, the numerical characteristics of these models are shown on uniform, binomial, geometric, hypergeometric and Poisson distributions through logistic model simulations.

2. DIFFERENCE EQUATIONS

Definition: Let $n \in \mathbb{N} = \{0,1,\dots\}$ be an independent variable, then the function with unknown x such that

$$G\big(n, x(n), x(n+1), \dots, x(n+k)\big) = 0$$

is called a difference equation. This equation is also called the non-autonomous difference equation. If the equation is given as $G(x(n), x(n+1), \dots, x(n+k))$, then it is called an autonomous equation [20-27].

2.1. Random Difference Equations

Basically, using the deterministic difference equations, random difference equations can be obtained in the following three ways.

Definition: Let I be any sub-interval of real numbers and $f: I \times I \rightarrow I$ be a function that can be continuously differentiated. For each $x_{-k}, x_{-(k-1)}, \dots, x_0 \in I$ initial conditions

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \ \ n = 0,1, \dots$$

shapes a n-th order difference equation. Using this difference equation, a random difference equation can be constructed

i. With random initial values

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}, n); n = 0,1, \dots; \ x_{-k}, x_{-(k-1)}, \dots, x_0 \in I$$

Here $x_{-k}, x_{-(k-1)}, \dots, x_0$ are random variables.

ii. With random non-homogeneous term

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}, n) + Y(n); n = 0,1, \dots; x_{-k}, x_{-(k-1)}, \dots, x_0 \in I$$

using a random $Y(t)$ process and

iii. With random coefficients

$$x_{n+1} = A(n) f(x_n, x_{n-1}, \dots, x_{n-k}, n) + Y(n); n = 0,1, \dots; x_{-k}, x_{-(k-1)}, \dots, x_0 \in I.$$

Hence, we can make a random difference equation using a random process $A(n)$ [10-19].

2.2. Random Stochastic Difference Equations

Consider the first order difference equation

$$x_{n+1} = F(n, x_n), n \in N \tag{2}$$

It can be used to describe facts that develop in a separate time, where the size of each generation is a function of the previous one. However, real life problems cannot usually be expressed with such a proper mathematical model. Unpredictable effects are included in the model as random variables $\{\xi_n\}$ resulting in a stochastic difference equation.

$$x_{n+1} = F(n, x_n) + G(x, x_n)\xi_{n+1}, n \in N \tag{3}$$

The solution of (3) is the discrete time stochastic process of $\{\xi_n\}$ compatible with natural filters. Although stochastic difference equations also occur as discretization of stochastic differential equations, analysis of asymptotic properties may be more difficult. Here, both the modeling developed in discrete time and the numerical method analysis for stochastic differential equations has effects [13]. Stochastic difference equations appear in numerical analysis because they are the final product of discretization of stochastic differential equations

$$dX_n = f(X(n))dn + g(X(n))dW_n \tag{4}$$

where W_n is a standard Brownian motion. The (4) equation can be discretized by the one-step Euler-Maruyama numerical order. This stochastic difference equation

$$x_{n+1} = x_n + hf(x_n) + \sqrt{h} g(x_n)\xi_{n+1}, n = 0,1, \dots \tag{5}$$

where $\{\xi_n\}$ is the standard normal sequence of random variables, while h is the network dimension [3-9].

3. DISCRETE TIME PROBABILITY DISTRIBUTIONS

In this section, definitions related to some probability concepts used are given.

3.1. Discrete Uniform Distribution

Definition. Let k be a positive bit integer. A random variable X with probability function

$$P(x,k) = \begin{cases} 1/k, & x = 1,2,3,\dots,k \\ 0, & other \end{cases}$$

is called a discrete uniform chance variable [1-2].

Theorem. If X has a discrete uniform distribution, then

a. $E(X) = \frac{k+1}{2}$,
b. $V(X) = \frac{k^2-1}{12}$,
c. $M_x(t) = \frac{1}{k}\sum_{x=1}^{k} e^{tx}$

3.2. Bernoulli Distribution

Definition. If there are only two results for an X random variable, X is called a Bernoulli random variable [15]. Bernoulli variables are obtained with the probability function

$$f(x,p) = p^x(1-p)^{1-x} \;\; x = 0,1$$

Theorem. If X has a Bernoulli distribution,

a. $E(X) = p$,
b. $V(X) = p(1-p)$,
c. $M_x(t) = e^t p + (1-p)$.

3.3. Binomial Distribution

Definition. Let the total number of those who succeeded in n independent Bernoulli trials be the random variable X. For a single experiment, the probability of success is denoted by p, and the probability of failure is $(1-p)$. The binomial random variable X has the following probability function

$$f(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x}; x = 0,1,2,\dots,n.$$

Calculation of consecutive binomial probabilities,

$$f(x+1;n,p) = \frac{(n-x)p}{(x+1)(1-p)} f(x;n,p); x = 0,1,\dots,n-1.$$

Theorem. If X has a binomial distribution,

a. $E(X) = np$,
b. $V(X) = np(1-p)$,
c. $M_x(t) = [e^t p + (1-p)]^n$.

3.4. Negative Binomial (Pascal) Distribution

Definition. Let X be the random variable of the number of trials required to achieve success $K \geq 1$, with the probability of success p in each experiment for independent Bernoulli trials. In this case, X is called a negative binomial random variable and its probability function [1-2]:

$$f(x) = \binom{x-1}{K-1} p^K (1-p)^{x-K}; x = K, K+1, \dots$$

Theorem. If X has a negative binomial distribution,

a. $E(X) = \frac{k}{p}$,
b. $V(X) = \frac{k(1-p)}{p^2}$,
c. $M_x(t) = \frac{p^k e^{tk}}{(1-e^t q)^k}$

3.5. Geometric Distribution

Definition. The number of experiments done to obtain the first desired result (success or unsuccessful) in a Bernoulli experiment repeated n times in succession is called a geometric random variable X. The distribution of this variable is called the geometric distribution and the probability function of the geometric random variable X, with probability of unsuccessfulness $q = 1 - p$ and probability of success p in a single experiment [1-2];

$$f(x) = P(X = x) = q^{x-1}p; x = 1,2,3, \dots$$

Theorem. If X has a geometric distribution,

a. $E(X) = \frac{1}{p}$,
b. $V(X) = \frac{(1-p)}{p^2}$,
c. $M_x(t) = pe^t \frac{1}{1-[e^t(1-p)]}$.

3.6. Hypergeometric Distribution

Definition. Let a be the number of elements of a given Type A in a mass consisting of a finite number of N elements. Let X be the number of elements of its type in a sample of n units that are randomly drawn without replacing them again. X is a random hypergeometric variable and the hypergeometric probability mass function is given as [1-2];

$$f(x; N, M, n) = \frac{\binom{M}{X}\binom{N-M}{n-x}}{\binom{N}{n}}; x = 0,1, \dots, n.$$

Theorem. If X has a hypergeometric distribution,

a. $E(X) = \frac{nM}{N}$,
b. $V(X) = \frac{M(M-1)n(n-1)}{N(N-1)} + M\frac{n}{N}$,
c. $\frac{N-n}{N-1} n \frac{M}{N}(1 - \frac{M}{N})$

3.7. Poisson Distribution

Definition. $f(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}; x = 0,1,2, \dots, \lambda > 0$. The Taylor expansion of the function e^y and the probability function gives ($e^y = \sum_{i=0}^{\infty} \frac{y^i}{i!}$):

$$\sum_{x=0}^{\infty} f(X = x; \lambda) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1.$$

Theorem. If X has a Poisson distribution,

a. $E(X) = \lambda$,
b. $V(X) = \lambda$,
c. $M_x = e^{\lambda(e^t-1)}$.

4. NUMERICAL EXAMPLES

In this section, the population model is introduced. After giving information about this model, random models will be established and examined.

4.1. Discrete Time Probability Distribution

4.1.1. Uniform Distribution

In the random logistic difference equation defined as

$$X(n+1) = X(n) + r * X(n) * \left(1 - \frac{X(n)}{K}\right),$$

if r is a random variable with a parameterized uniform distribution, K=1000 and $N = 50$, then the probability characteristics obtained from 10^5 simulations are given below.

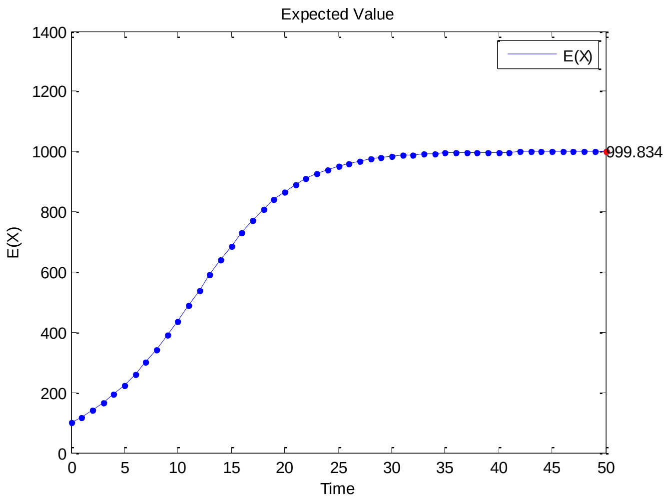


Figure 1. Expected value of random model

In the logistics model process ($t \in [0,50]$), it is observed that the variability increases in the beginning and then remains stable. The extreme values are shown in the Table (Table 1 and Fig 1).

Table 1. Expected value, extreme values and times in random model

Variable	Minimum	Time	Maximum	Time
$E(x(t))$	100	0	999.834	50

It appears that the expected population reached its highest level at the time of $t = 50$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, $E\,(x(50)) = 999.834$ was obtained for the expected value at the end of the process ($t = 50$).

Similarly, variations of variances for the model (1) are also seen below (Figure 2).

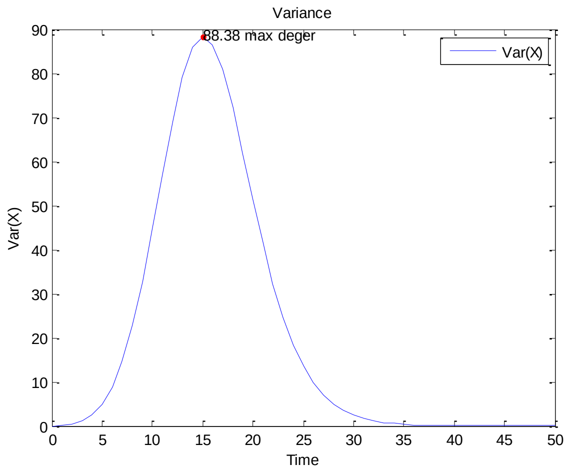


Figure 2. Variance of random model

In the logistics model process ($t \in [0,50]$), it is observed that variability increases first and decreases later. The extreme values are shown in the Table (Table 2).

Table 2. Extreme values and times of variance in random model

Variable	Minimum	Time	Maximum	Time
$Var(x(t))$	0	0	88.38	15

It is observed that the population has reached its highest level of deviation from the average at the time of $t = 15$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, at the end of the process, $Var\ (x(15)) = 88.38$ was obtained for variance, ($t = 15$).

Similar to variance, variations in the standard deviation for the (1) model are also seen below (Figure 3).

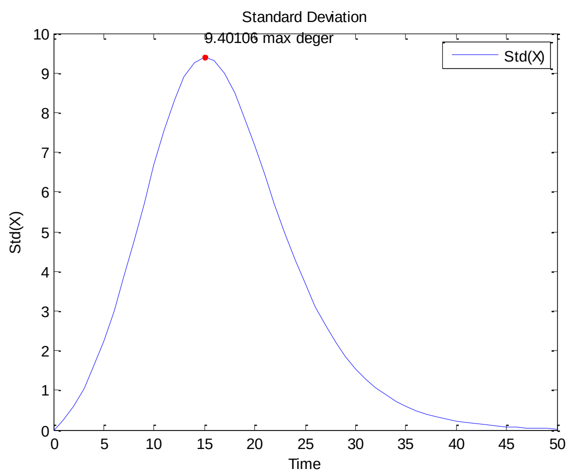


Figure 3. Standard deviation of the random model

As the standard deviation is the square root of the variance, the result is that these two numerical characteristics are expected to exhibit similar behavior. The extreme values for standard deviations are shown below (Table 3).

Table 3. Extreme values and times of standard deviation in random model

Variable	Minimum	Time	Maximum	Time
$Std(x(t))$	0	0	9.40106	15

It is observed that the population has reached its highest level of deviation from the average at the time of $t = 15$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, $Std\ (x(15)) = 9.40106$ was obtained for variance $(t = 15)$ at the end of the process.

Using the results obtained for the standard deviations and expected values, the variation coefficients for the variables $x\ (t)$ in the random model (1) were also calculated as follows (Figure 4).

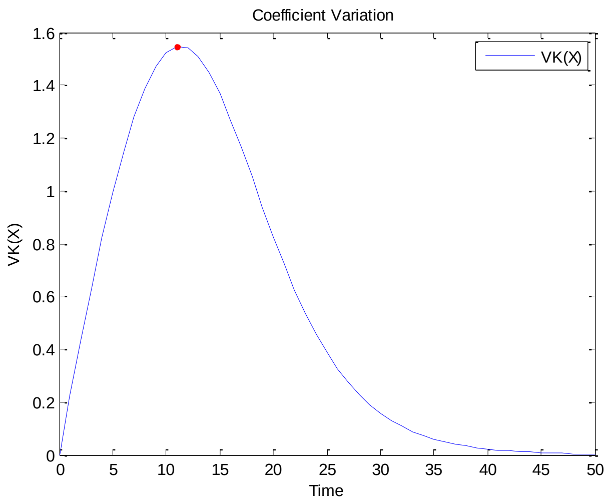


Figure 4. Variation coefficient of the random model

Coefficient of Variation (CV) is calculated by definition as $100 \times std\ (x\ (t))\ /\ E\ (x\ (t))$ and random r parameters for the installation of model (1) are defined to have %5 coefficient of variation. However, as a result of examining the model, it is seen that the coefficient of variation of $x\ (t)$ variables increased to higher rates. The extreme values of the variation coefficients are given in the table below (Table 4).

Table 4. Extreme values and times of the coefficient of variation in the random model

Variable	Minimum	Time	Maximum	Time
$CV(x(t))$	0	0	1.54683	11

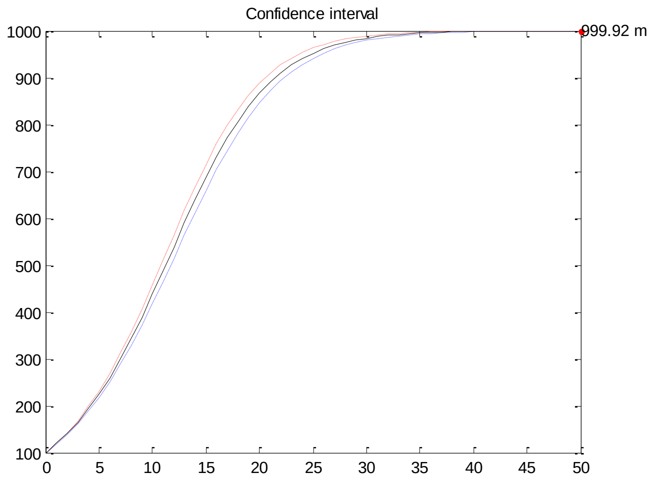
Despite the %5 coefficient of variation in the parameters, it is observed that the variation rate of $x\ (t)$ is constantly increasing and reaches %1.5 at $t = 11$. Therefore, it can be interpreted that the variability in random results increases as logistic model and then decreases.

The results obtained for the expected values of the model (1) are given below (Figure 5).

The confidence intervals given in the figure are calculated as $CI = (E\,(x\,(t)) - 3.std\,(x\,(t)), E\,(x\,(t)) + 3.std\,(x\,(t)))$ and three gives the range of variation within the standard deviation. This range includes about %99 of the random variable's values for uniform distribution. Therefore, the extreme values obtained for the expected values in these ranges are given below (Table 5).

Table 5. Extreme values and times of confidence in the random model

Variable	Minimum	Time	Maximum	Time
$E(x(t))$	100	0	999.92	50



At the end of the process, three standard deviation intervals for $x(t)$ variables are obtained as follows: $E\,(x(50)) \in (971.63082{,}1028.03718)$.

Model (1) states that the expectation for this value is $E\,(x\,(50)) = 999.92$, that is, approximately $\%9.9992$, and the expected approximate population is in the range of $\%99$ probability (971.63082,1028.03718) at time $t = 50$.

4.1.2. Binomial Distribution

In the random logistic difference equation defined as

$$X(n+1) = X(n) + r * X(n) * \left(1 - \frac{X(n)}{K}\right),$$

if r is a random variable with a parameterized binomial distribution ($n = 6$) and $N = 50$, then the probability characteristics obtained from 10^5 simulations are given below.

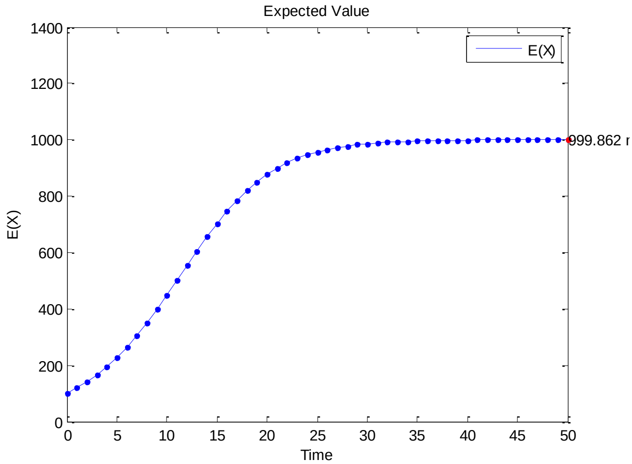


Figure 6. Expected value of random model

In the logistics model process ($t \in [0,50]$), it is observed that the variability increases first and then remains stable. The extreme values are shown in the Table (Table 6 and Fig 6).

Table 6. Expected value, extreme values and times in random model

Variable	Minimum	Time	Maximum	Time
$E(x(t))$	100	0	999.862	50

It appears that the expected population reached its highest level at the time of $t = 50$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, $E\ (x(50)) = 999.862$ was obtained for the expected value at the end of the process ($t = 50$).

Similarly, variations of variances for the model (1) are also seen below (Figure 7).

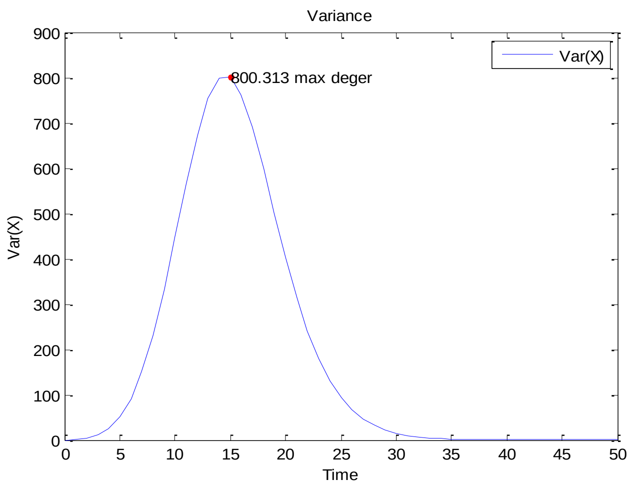


Figure 7. Variance of random model

In the logistics model process ($t \in [0,50]$), it is observed that variability increases first and decreases later. The extreme values are shown in the Table (Table 7).

Table 7. Extreme values and times of variance in random model

Variable	Minimum	Time	Maximum	Time
$Var(x(t))$	0	0	800.313	15

It is observed that the population has reached its highest level of deviation from the average at the time of $t = 15$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, at the end of the process, $Var\ (x(15)) = 800.313$ was obtained for variance ($t = 15$).

Similar to variance, variations in the standard deviation for the model (1) are also seen below (Figure 8).

As the standard deviation is the square root of the variance, the result is that these two numerical characteristics are expected to exhibit similar behavior. The extreme values for standard deviations are shown below (Table 8).

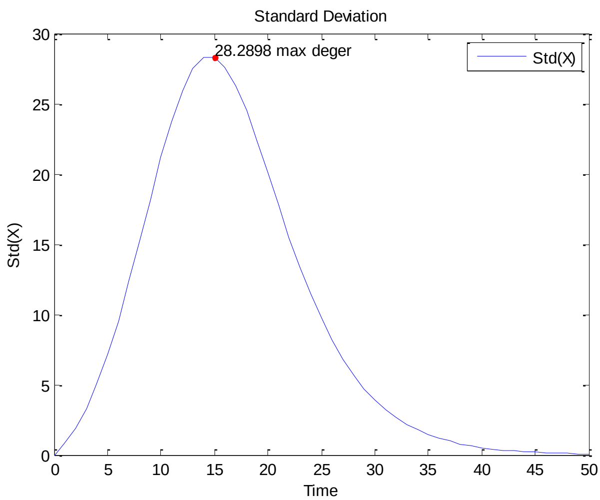


Figure 8. Standard deviation of the random model

Table 8. Extreme values and times of standard deviation in random model

Variable	Minimum	Time	Maximum	Time
$Std(x(t))$	0	0	28.2898	15

It is observed that the population has reached its highest level of deviation from the average at the time of $t = 15$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, $Std\ (x(15)) = 28.2898$ was obtained for variance ($t = 15$) at the end of the process.

Using the results obtained for the standard deviations and expected values, the variation coefficients for the variables $x\,(t)$ in the random model (1) were also calculated as follows (Figure 9).

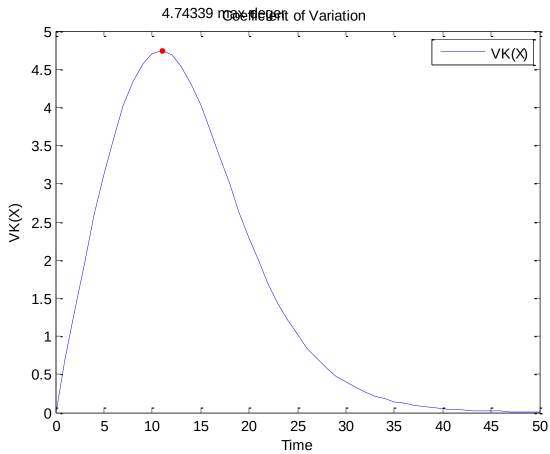


Figure 9. Variation coefficient of the random model

Coefficient of Variation (CV) is calculated by definition as $100 \times std\,(x\,(t))\,/\,E\,(x\,(t))$ and random r parameters for the installation of model (1) are defined to have %5 coefficient of variation. However, as a result of examining the model, it is seen that the coefficient of variation of $x\,(t)$ variables increased to higher rates. The extreme values of the variation coefficients are given in the table below (Table 9).

Table 9. Extreme values and times of the coefficient of variation in the random model

Variable	Minimum	Time	Maximum	Time
$CV(x(t))$	0	0	4.74339	11

Despite the %5 coefficient of variation in the parameters, it is observed that the variation rate of $x\,(t)$ is constantly increasing and reaches %0.0474 at $t\ =\ 11$. Therefore, it can be interpreted that the variability in random results increases as logistic and then decreases.

The results obtained for the expected values of the model (1) are given below (Figure 10).

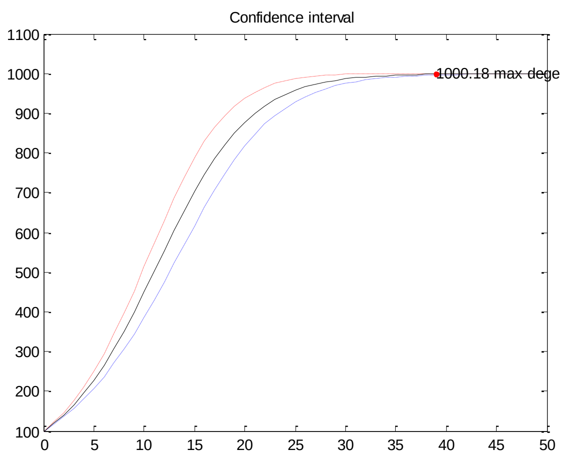


Figure 10. Confidence interval of the expected values of the random model

The confidence intervals given in the figure are calculated as $CI = (E\ (x\ (t)) - 3.std\ (x\ (t)), E\ (x\ (t)) + 3.std\ (x\ (t)))$ and three gives the range of variation within the standard deviation. This range includes about %99 of the random variable's values for binom distribution. Therefore, the extreme values obtained for the expected values in these ranges are given below (Table 10).

Table 10. Extreme values and times of confidence in the random model

Variable	Minimum	Time	Maximum	Time
$E(x(t))$	100	0	1000.18	39

At the end of the process, three standard deviation intervals for $x(t)$ variables are obtained as follows: $E\ (x(39)) \in (914.9926{,}1084.7314)$.

Model (1) states that the expectation for this value is $E\ (x\ (39))\ = 1000.18$, that is, approximately %10.0018, and the expected approximate population is in the range of %99 probability $(914.9926{,}1084.7314)$ at time $t\ =\ 39$.

4.1.3. Geometric Distribution

In the random logistic difference equation defined as

$$X(n+1) = X(n) + r * X(n) * \left(1 - \frac{X(n)}{K}\right),$$

if r is a random variable with a parameterized geometric distribution and $N = 50$, then the probability characteristics obtained from 10^5 simulations are given below.

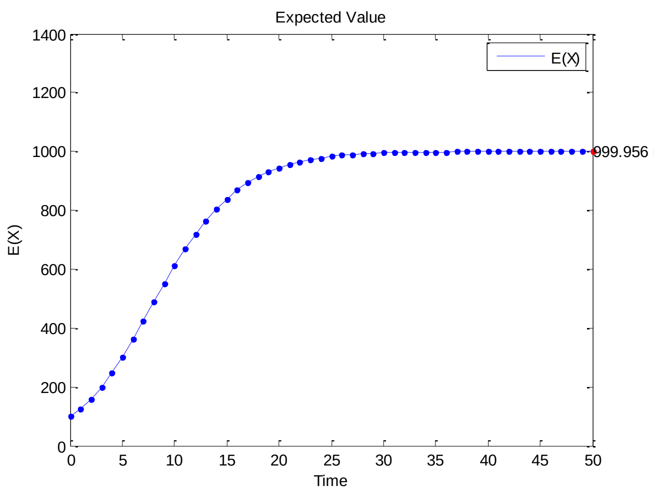


Figure 11. Expected value of random model

In the logistics model process ($t \in [0{,}50]$), it is observed that the variability increases first and then remains stable. The extreme values are shown in the Table (Table 11 and Fig 11).

Table 11. Expected value, extreme values and times in random model

Variable	Minimum	Time	Maximum	Time
$E(x(t))$	100	0	999.956	50

It appears that the expected population reached its highest level at the time of $t = 50$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, $E\ (x(50)) = 999.956$ was obtained for the expected value at the end of the process$(t = 50)$.

Similarly, variations of variances for the model (1) are also seen below (Figure 12).

In the logistics model process $(t \in [0,50])$, it is observed that variability increases first and decreases later. The extreme values are shown in the Table (Table 12).

Table 12. Extreme values and times of variance in random model

Variable	Minimum	Time	Maximum	Time
$Var(x(t))$	0	0	25561.9	10

It is observed that the population has reached its highest level of deviation from the average at the time of $t\ =\ 10$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, at the end of the process, $Var\ (x(10)) = 25561.9$ was obtained for variance $(t = 10)$.

Similar to variance, variations in the standard deviation for the model (1) are also seen below (Figure 13).

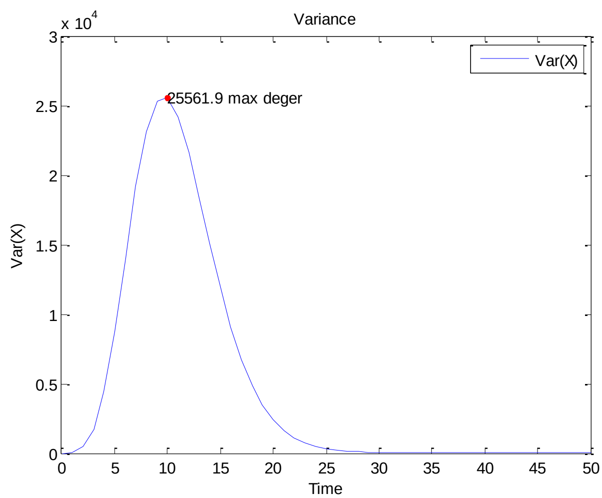


Figure 12. Variance of random model

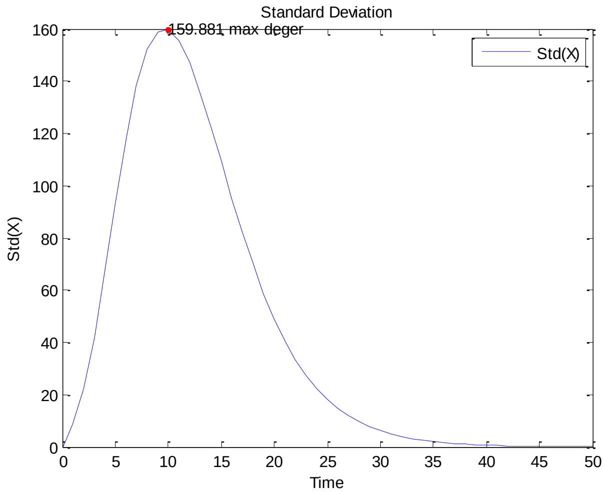


Figure 13. Standard deviation of the random model

As the standard deviation is the square root of the variance, the result is that these two numerical characteristics are expected to exhibit similar behavior. The extreme values for standard deviations are shown below (table 13).

Table 13. Extreme values and times of standard deviation in random model

Variable	Minimum	Time	Maximum	Time
$Std(x(t))$	0	0	159.881	10

It is observed that the population has reached its highest level of deviation from the average at the time of $t = 10$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, $Std\ (x(10)) = 159.881$ was obtained for variance $(t = 10)$ at the end of the process.

Using the results obtained for the standard deviations and expected values, the variation coefficients for the variables $x\ (t)$ in the random model (1) were also calculated as follows (Figure 19).

Coefficient of Variation (CV) is calculated by definition as $100 \times std\ (x\ (t))\ /\ E\ (x\ (t))$ and random r parameters for the installation of model (1) are defined to have %5 coefficient of variation. However, as a result of examining the model, it is seen that the coefficient of variation of $x\ (t)$ variables increased to higher rates. The extreme values of the variation coefficients are given in the table below (Table 14).

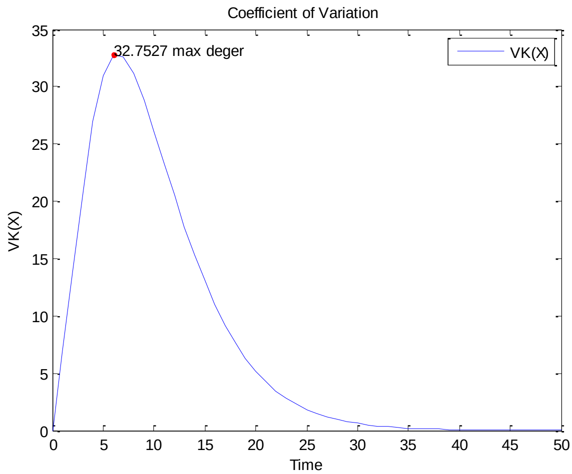


Figure 14. Variation coefficient of the random model

Table 14. Extreme values and times of the coefficient of variation in the random model

Variable	Minimum	Time	Maximum	Time
$CV(x(t))$	0	0	32.7527	6

Despite the %5 coefficient of variation in the parameters, it is observed that the variation rate of $x\,(t)$ is constantly increasing and reaches %0.32 at $t = 6$. Therefore, it can be interpreted that the variability in random results increases as logistic model and then decreases.

The results obtained for the expected values of the model (1) are given below (Figure 15).

The confidence intervals given in the figure are calculated as $CI = (E\,(x\,(t)) - 3.std\,(x\,(t)), E\,(x\,(t)) + 3.std\,(x\,(t)))$ and three gives the range of variation within the standard deviation. This range includes about %99 of the random variable's values for geometric distribution. Therefore, the extreme values obtained for the expected values in these ranges are given below (Table 15).

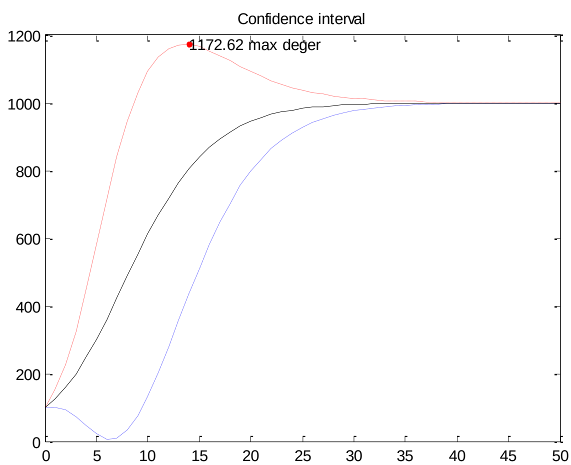


Figure 15. Confidence interval of the expected values of the random model

Table 15. Extreme values and times of confidence in the random model

Variable	Minimum	Time	Maximum	Time
$E(x(t))$	100	0	1172.62	14

At the end of the process, three standard deviation intervals for $x(t)$ variables are obtained as follows: $E\ (x(14)) \in (520.313{,}1479.599)$.

Model (1) states that the expectation for this value is $E\ (x\ (14))\ = 1172.62$ that is, approximately $\%11.7262$, and the expected approximate population is in the range of $\%99$ probability $(520.313{,}1479.599)$ at time $t\ = 14$.

4.1.4. Hypergeometric Distribution

In the random logistic difference equation defined as

$$X(n+1) = X(n) + r * X(n) * \left(1 - \frac{X(n)}{K}\right),$$

if r is a random variable with a parameterized hypergeometric distribution and $N = 50$, then the probability characteristics obtained from 10^5 simulations are given below.

In the logistics model process ($t \in [0{,}50]$), it is observed that the variability increases first and then remains stable. The end values are shown in the Table (Table 16 and Fig 16).

Table 16. Expected value, end values and times in random model

Variable	Minimum	Time	Maximum	Time
$E(x(t))$	100	0	999.913	50

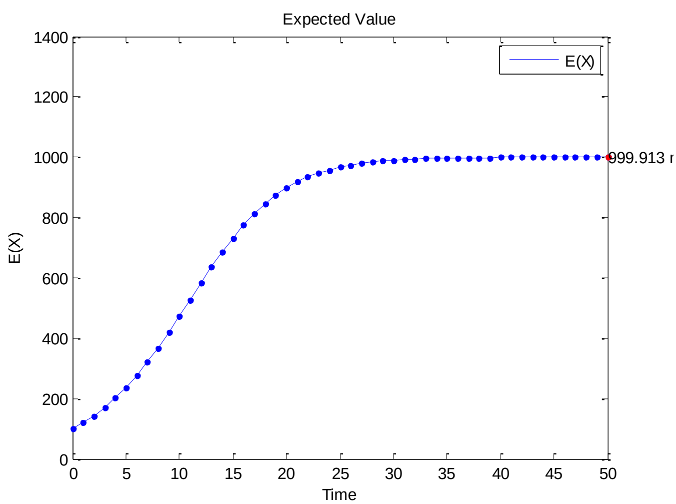


Figure 16. Expected value of random model

It appears that the expected population reached its highest level at the time of $t = 50$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, $E\ (x(50)) = 999.913$ was obtained for the expected value at the end of the process$(t = 50)$.

Similarly, variations of variances for the model (1) are also seen below (Figure 17).

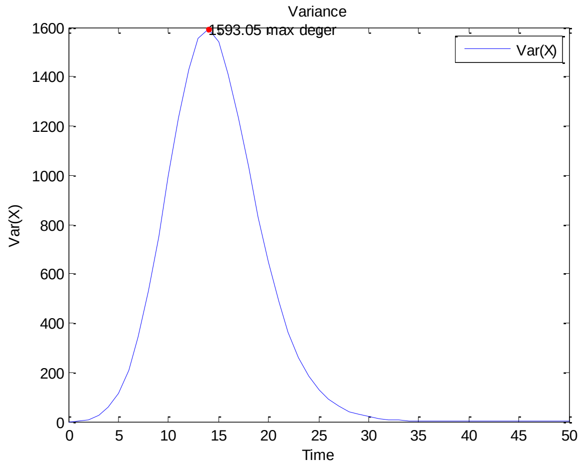


Figure 17. variance of random model

In the logistics model process ($t \in [0,50]$), it is observed that variability increases first and decreases later. The extreme values are shown in the Table (Table 17).

Table 17. Extreme values and times of variance in random model

Variable	Minimum	Time	Maximum	Time
$Var(x(t))$	0	0	1593.05	14

It is observed that the population has reached its highest level of deviation from the average at the time of $t = 14$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, at the end of the process, $Var\ (x(14)) = 1593.05$ was obtained for variance ($t = 14$).Similar to variance, variations in the standard deviation for the model (1) are also seen below (Figure 18).

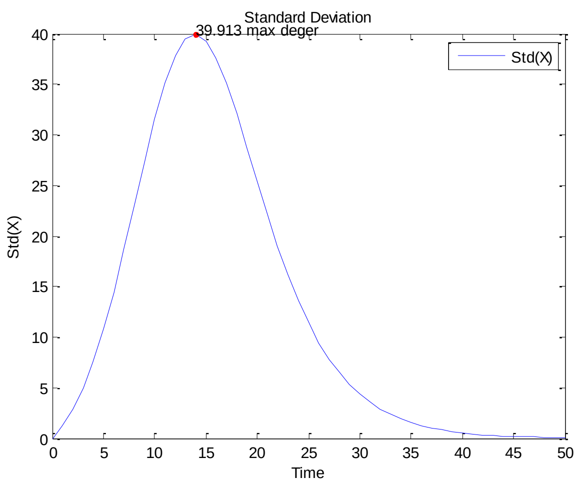


Figure 18. Standard deviation of the random model

As the standard deviation is the square root of the variance, the result is that these two numerical characteristics are expected to exhibit similar behavior. The end values for standard deviations are shown below (table 18).

Table 18. Extreme values and times of standard deviation in random model

Variable	Minimum	Time	Maximum	Time
$Std(x(t))$	0	0	39.913	14

It is observed that the population has reached its highest level of deviation from the average at the time of $t = 14$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, $Std\ (x(14)) = 39.913$ was obtained for variance ($t = 14$) at the end of the process.

Using the results obtained for the standard deviations and expected values, the variation coefficients for the variables $x\ (t)$ in the random model (1) were also calculated as follows (Figure 19).

Coefficient of Variation (CV) is calculated by definition as $100 \times std\ (x\ (t))\ /\ E\ (x\ (t))$ and random r parameters for the installation of model (1) are defined to have %5 coefficient of variation. However, as a result of examining the model, it is seen that the coefficient of variation of $x\ (t)$ variables increased to higher rates. The extreme values of the variation coefficients are given in the table below (Table 19).

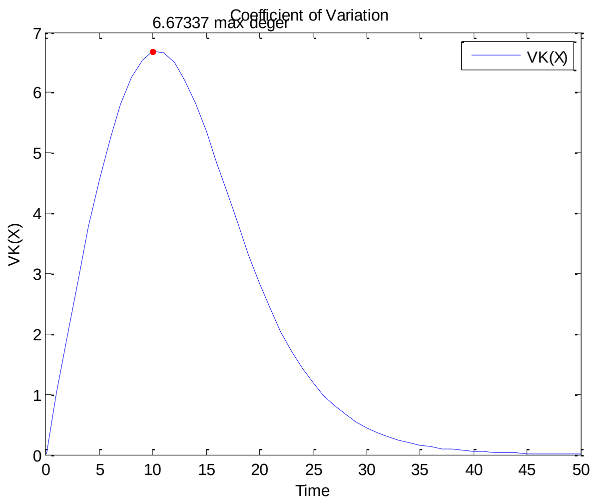


Figure 19. Variation coefficient of the random model

Table 19. Extreme values and times of the coefficient of variation in the random model

Variable	Minimum	Time	Maximum	Time
$CV(x(t))$	0	0	6.67337	10

Despite the %5 coefficient of variation in the parameters, it is observed that the variation rate of $x\ (t)$ is constantly increasing and reaches %0.0667 at $t\ =\ 10$. Therefore, it can be interpreted that the variability in random results increases as logistic model and then decreases.

The results obtained for the expected values of the model (1) are given below (Figure 20).

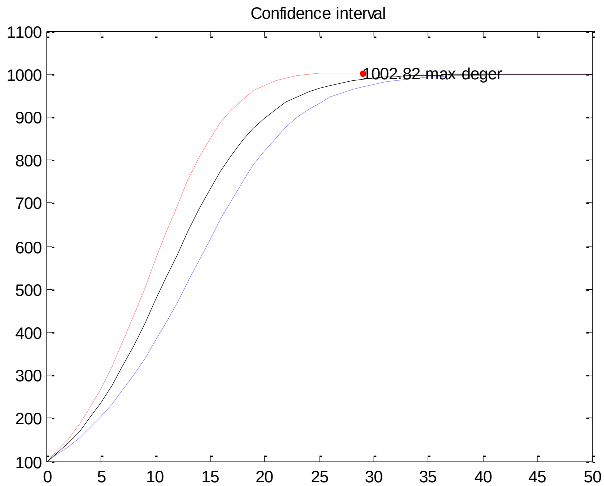


Figure 20. Confidence interval of the expected values of the random model

The confidence intervals given in the figure are calculated as $CI = (E\,(x\,(t)) - 3.std\,(x\,(t)), E\,(x\,(t)) + 3.std\,(x\,(t)))$ and three gives the range of variation within the standard deviation. This range includes about %99 of the random variable's values for uniform distribution. Therefore, the extreme values obtained for the expected values in these ranges are given below (Table 20).

Table 20. Extreme values and times of confidence in the random model

Variable	Minimum	Time	Maximum	Time
$E(x(t))$	100	0	1002.82	29

At the end of the process, three standard deviation intervals for $x(t)$ variables are obtained as follows: $E\,(x(29)) \in (880.174{,}1119.652)$.

Model (1) states that the expectation for this value is $E\,(x\,(29)) = 1002.82$ that is, approximately $\%10.0282, and$ the expected approximate population is in the range of $\%99$ probability $(880.174{,}1119.652)$ at time $t\ = 29$.

4.1.5. Poisson distribution

In the random logistic difference equation defined as

$$X(n+1) = X(n) + r * X(n) * \left(1 - \frac{X(n)}{K}\right),$$

if r is a random variable with a parameterized Poisson distribution and $N = 50$, then the probability characteristics obtained from 10^5 simulations are given below.

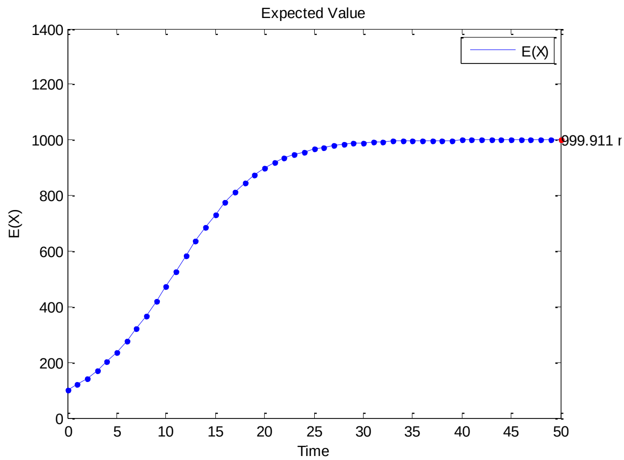


Figure 21. Expected value of random model

In the logistics model process ($t \in [0,50]$), it is observed that the variability increases first and then remains stable. The extreme values are shown in the Table (Table 21 and Fig 21).

Table 21. Expected value, end values and times in random model

Variable	Minimum	Time	Maximum	Time
$E(x(t))$	100	0	999.911	50

It appears that the expected population reached its highest level at the time of $t = 50$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, $E\ (x(50)) = 999.911$ was obtained for the expected value at the end of the process($t = 50$).

Similarly, variations of variances for the model (1) are also seen below (Figure 22).

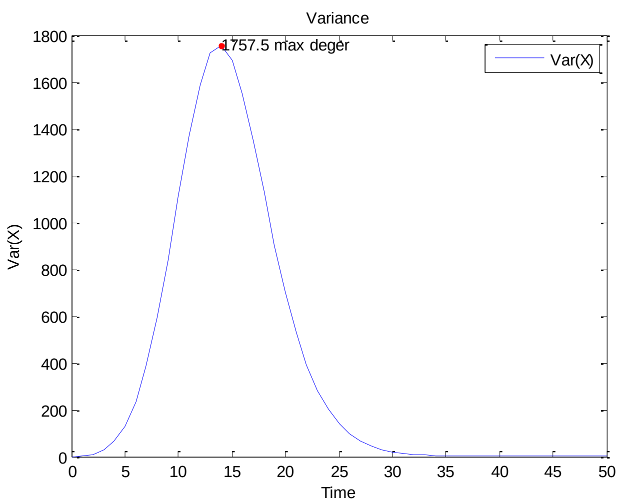


Figure 22. Variance of random model

In the logistics model process ($t \in [0,50]$), it is observed that variability increases first and decreases later. The extreme values are shown in the Table (Table 22).

Table 22. Extreme values and times of variance in random model

Variable	Minimum	Time	Maximum	Time
$Var(x(t))$	0	0	1757.5	14

It is observed that the population has reached its highest level of deviation from the average at the time of $t = 14$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, at the end of the process, $Var\ (x(14)) = 1757.5$ was obtained for variance $(t = 14)$.

Similar to variance, variations in the standard deviation for the model (1) are also seen below (Figure 23).

As the standard deviation is the square root of the variance, the result is that these two numerical characteristics are expected to exhibit similar behavior. The end values for standard deviations are shown below (Table 23).

Table 23. Extreme values and times of standard deviation in random model

Variable	Minimum	Time	Maximum	Time
$Std(x(t))$	0	0	41.9225	14

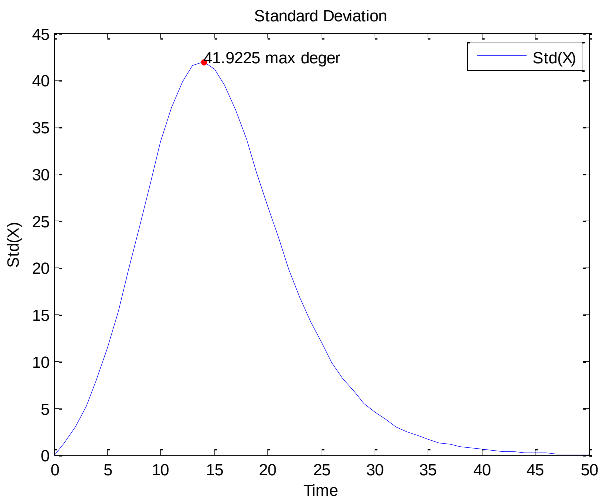


Figure 23. standard deviation of the random model

It is observed that the population has reached its highest level of deviation from the average at the time of $t = 14$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, $Std\ (x(14)) = 41.9225$ was obtained for variance $(t = 14)$ at the end of the process.

Using the results obtained for the standard deviations and expected values, the variation coefficients for the variables $x\,(t)$ in the random model (1) were also calculated as follows (Figure 24).

Coefficient of Variation (CV) is calculated by definition as $100 \times std\ (x\ (t))\ /\ E\ (x\ (t))$ and random r parameters for the installation of model (1) are defined to have %5 coefficient of variation. However, as a result of examining the model, it is seen that the coefficient of variation of $x\ (t)$ variables increased to higher rates. The extreme values of the variation coefficients are given in the table below (Table 24).

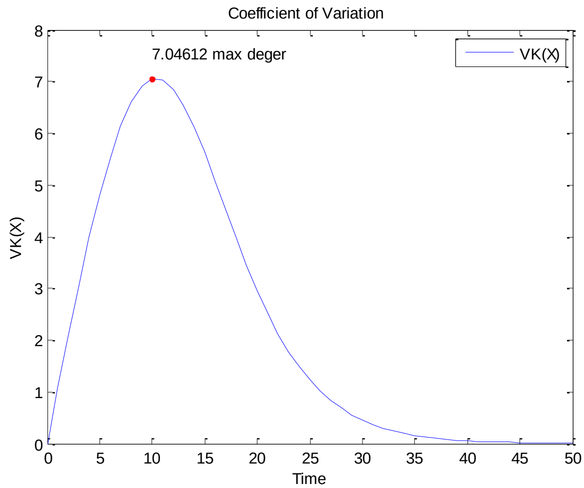


Figure 24. Variation coefficient of the random model

Table 24. Extreme values and times of the coefficient of variation in the random model

Variable	Minimum	Time	Maximum	Time
$CV(x(t))$	0	0	7.04612	10

Despite the %5 coefficient of variation in the parameters, it is observed that the variation rate of $x\ (t)$ is constantly increasing and reaches %0.07 at $t\ =\ 10$. Therefore, it can be interpreted that the variability in random results increases as logistic model and then decreases.

The results obtained for the expected values of the model are (1) given below (Figure 25).

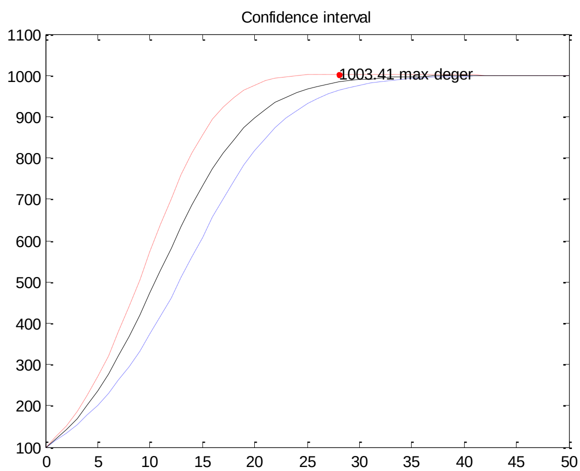


Figure 25. Confidence interval of the expected values of the random model

The confidence intervals given in the figure are calculated as $CI = (E\,(x\,(t)) - 3.std\,(x\,(t)), E\,(x\,(t)) + 3.std\,(x\,(t)))$ and three gives the range of variation within the standard deviation. This range includes about %99 of the random variable's values for poisson distribution. Therefore, the extreme values obtained for the expected values in these ranges are given below (Table 25).

Table 25. Extreme values and times of confidence in the random model

Variable	Minimum	Time	Maximum	Time
$E(x(t))$	100	0	1003.41	28

At the end of the process, three standard deviation intervals for $x(t)$ variables are obtained as follows: $E\,(x(28)) \in (874.1435, 1125.6785)$.

Model (1) states that the expectation for this value is $E\,(x\,(28)) = 1003.41$ that is, approximately $\%10.0341, and$ the expected approximate population is in the range of %99 probability $(874.1435, 1125.6785)$ at time $t = 28$.

4.2. White Noise

In the random logistic difference equation defined as

$$X(n+1) = X(n) + r * X(n) * \left(1 - \frac{X(n)}{K}\right) + noise * (rand - 0.5) * 2;$$

if r is a random variable with a parameterized uniform distribution ($noise = 1$) and $N = 50$, then the probability characteristics obtained from 10^5 simulations are given below.

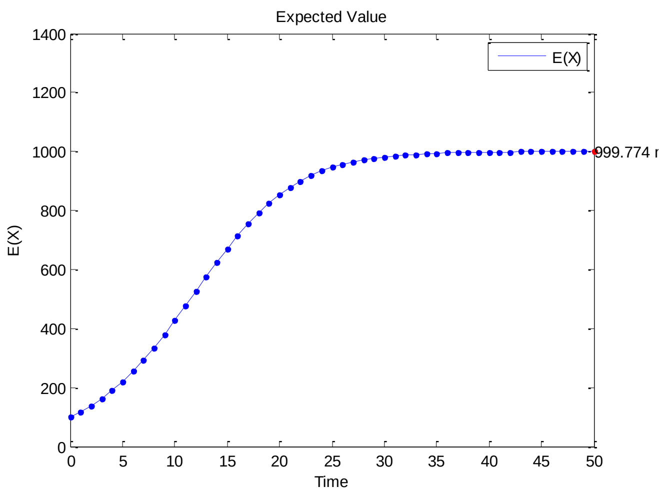


Figure 26. Expected value of random model

In the logistics model process ($t \in [0,50]$), it is observed that the variability increases first and then remains stable. The extreme values are shown in the Table (Table 26 and Fig 26).

Table 26. Expected value, end values and times in random model

Variable	Minimum	Time	Maximum	Time
$E(x(t))$	100	0	999.774	50

It appears that the expected population reached its highest level at the time of $t = 50$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, $E\ (x(50)) = 999.774$ was obtained for the expected value at the end of the process $(t = 50)$.

Similarly, variations of variances for the model (1) are also seen below (Figure 27).

In the logistics model process $(t \in [0,50])$, it is observed that variability increases first and decreases later. The extreme values are shown in the Table (Table 27).

Table 27. Extreme values and times of variance in random model

Variable	Minimum	Time	Maximum	Time
$Var(x(t))$	0	0	10.0728	13

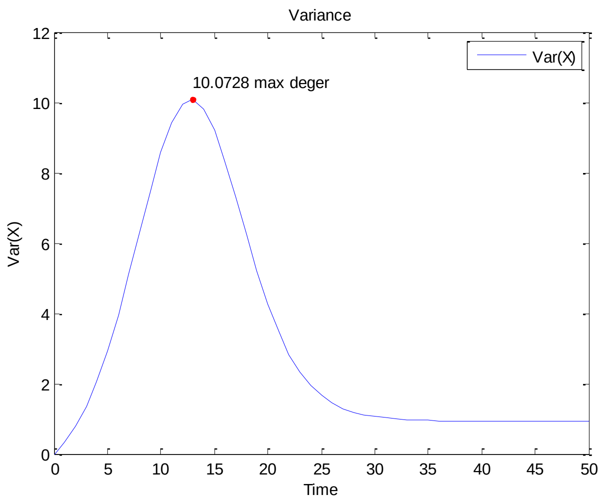


Figure 27. Variance of random model

It is observed that the population has reached its highest level of deviation from the average at the time of $t = 13$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, at the end of the process, $Var\ (x(13)) = 10.0728$ was obtained for variance $(t = 13)$.

Similar to variance, variations in the standard deviation for the (1) model are also seen below (Figure 28).

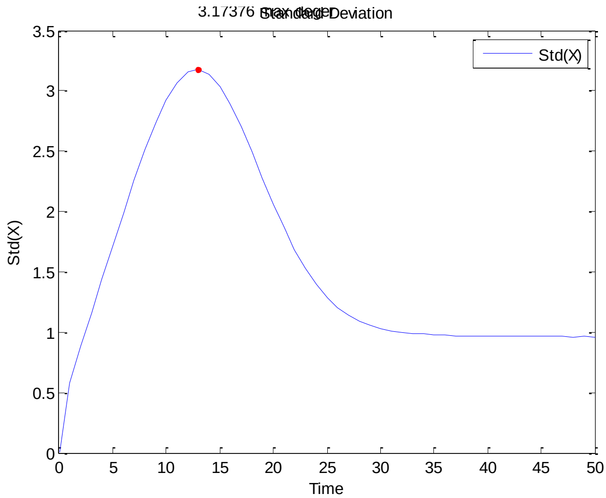


Figure 28. Standard deviation of the random model

As the standard deviation is the square root of the variance, the result is that these two numerical characteristics are expected to exhibit similar behavior. The extreme values for standard deviations are shown below (Table 28).

Table 28. Extreme values and times of standard deviation in random model

Variable	Minimum	Time	Maximum	Time
$Std(x(t))$	0	0	3.17376	13

It is observed that the population has reached its highest level of deviation from the average at the time of $t = 13$. Therefore, the results obtained from the deterministic model are more likely to be observed differently in an experiment that takes place randomly at these moments. In addition, $Std\ (x(13)) = 3.17376$ was obtained for variance ($t = 13$) at the end of the process.

Using the results obtained for the standard deviations and expected values, the variation coefficients for the variables $x\ (t)$ in the random model (1) were also calculated as follows (Figure 29).

Coefficient of Variation (CV) is calculated by definition as $100 \times std\ (x\ (t))\ /\ E\ (x\ (t))$ and random r parameters for the installation of model (1) are defined to have %5 coefficient of variation. However, as a result of examining the model, it is seen that the coefficient of variation of $x\ (t)$ variables increased to higher rates. The extreme values of the variation coefficients are given in the table below (Table 29).

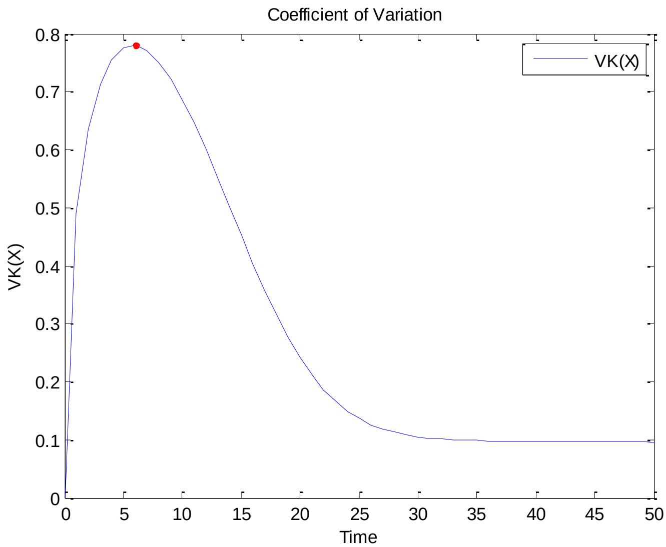


Figure 29. Variation coefficient of the random model

Table 29. Extreme values and times of the coefficient of variation in the random model

Variable	Minimum	Time	Maximum	Time
$CV(x(t))$	0	0	0.780106	6

Despite the %5 coefficient of variation in the parameters, it is observed that the variation rate of $x\ (t)$ is constantly increasing and reaches %0.0078 at $t\ =\ 6$. Therefore, it can be interpreted that the variability in random results increases as logistic model and then decreases.

The results obtained for the expected values of the model (1) are given below (Figure 30).

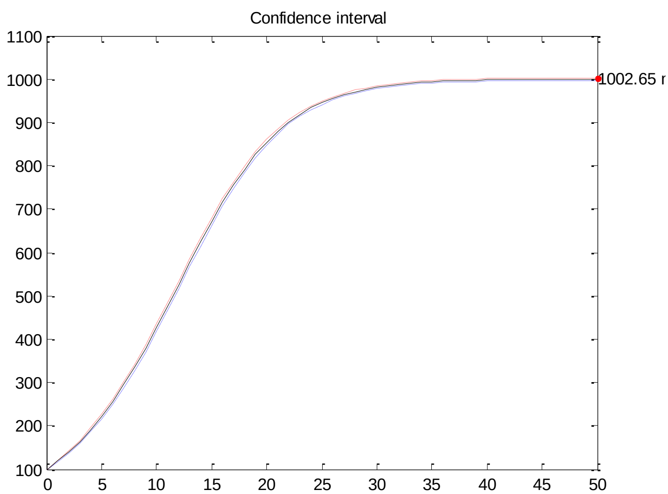


Figure 30. Confidence interval of the expected values of the random model

The confidence intervals given in the figure are calculated as $CI = (E\,(x\,(t)) - 3.std\,(x\,(t)), E\,(x\,(t)) + 3.std\,(x\,(t)))$ and three gives the range of variation within the standard deviation. This range includes about %99 of the random variable's values for normal distribution. Therefore, the extreme values obtained for the expected values in these ranges are given below (Table 30).

Table 30. Extreme values and times of confidence in the random model

Variable	Minimum	Time	Maximum	Time
$E(x(t))$	100	0	1002.65	50

At the end of the process, three standard deviation intervals for $x(t)$ variables are obtained as follows: $E\,(x(50)) \in (990.45272{,}1009.49528)$.

Model (1) states that the expectation for this value is $E\big(x(50)\big) = 1002.67$ that is, approximately %10.0265, and the expected approximate population is in the range of %99 probability $(990.45272{,}1009.49528)$ at time $t = 50$.

5. CONCLUSION

In this study, the model was randomized by adding random effects to the Logistics difference model. A random model has been established for Logistic difference model with random parameters with uniform distribution. With this difference model, expected value, variance, standard deviation, coefficient of variation and confidence intervals for population changes are calculated. The calculations were obtained using simulations of the random model.

The binomial distribution, which has parameters of the deterministic model, has been randomly rendered and the random model has been simulated. Using the obtained results, the expected value, variance, standard deviation, coefficient of variation and confidence intervals for the model were calculated.

Similarly, the parameters of the model were made random with geometric distribution and the random model was simulated. Using the obtained results, the expected value, variance, standard deviation, coefficient of variation and confidence intervals for the model were calculated.

In addition, the parameters of the deterministic difference model within the hypergeomteric and poisson distributions were randomized with these distributions, and the expected value, variance, standard deviation, coefficient of variation and confidence intervals of the model were calculated using the results of the model obtained.

The variation coefficients for the five distributions are compared and defined to have a variation coefficient of %5 for the parameter $X(t)$ for each distribution. Although a %5 deviation rate was used for random parameters, the simulation results showed variability in the proportion of the population. Analysis of random logistic difference model is given with the help of graphics and tables.

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