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Research Article

ON STABILITY OF SOME INTEGRAL EQUATIONS IN 2-BANACH SPACES

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ABSTRACT

The objective of this article is to investigate the Ulam-Hyres stability and Ulam-Hyres-Rassias stability for some general integral equations $f(x) = \int_E F(x, f(x))dx, \quad x \in E$, where E is a nonempty set of a Banach space. The main tool used in the analysis is a recent fixed point theory. In this way, we obtain results in 2-Banach Spaces.
Keywords: Ulam stability, integral equation, fixed point theory, Banach Spaces.
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1. INTRODUCTION

It is well-known that the interesting talk given by S. M. Ulam in 1940 at the University of Wisconsin kindled the spark of the theory of stability of functional equations (see e.g. [1, 2, 3, 4, 5, 6, 7] for more details). One interesting open problem in that famous talk can be stated as follows:

Let G be a group and (G^*, d) a metric group. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if $g: G \to G^*$ satisfies

$$d(g(xy), g(x)g(y)) < \delta$$

for all $x, y \in G$, then a homomorphism $f: G \to G^*$ exists such that

$$d(g(x), f(x)) < \varepsilon$$

for all $x, y \in G$?

For the last 70 years, that stability issue has been a very popular subject of investigations and we refer the reader to [8, 9] for further information and references.

Many mathematicians have interacted with the interesting open question given by Ulam in his famous talk. For instance, an affirmative answer to the equation of Ulam was given by D. H. Hyers in 1941 (see [10]) in case of Banach spaces. This answer, in this case, says that the Cauchy functional equation is stable in the sense of Hyers-Ulam. In 1950, T. Aoki (see [11]) is known as

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the second author who treat this problem for additive mappings (see also [12]). In 1978, Th. M. Rassias [13] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. The new type of stability introduced by Rassias in [13] is nowadays called the Hyers-Ulam-Rassias stability. The result obtained by Th. M. Rassias reads as follows (see [13]):

Theorem 1 *Consider E_1, E_2 to be two Banach spaces, and let $f: E_1 \to E_2$ be a mapping such that the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E_2 is continuous for each fixed $x \in E_1$. Assume that there exists $\theta \geq 0$ and $p \in [0,1)$ such that*

$$\| f(x+y) - f(x) - f(y) \| \leq \theta(\| x \|^p + \| y \|^p),\ x, y \in E_1 \backslash \{0\}. (1)$$

Then there exists a unique solution $T: E_1 \to E_2$ of the Cauchy equation with

$$\| f(x) - T(x) \| \leq \frac{2\theta \|x\|^p}{|2-2^p|},\ x \in E_1 \backslash \{0\}. (2)$$

It should be remarked that the results of D. H. Hyers and Th. M. Rassias have been generalized in several directions to other settings. For instance, several authors have studied the stability for differential equations (see [14], [15], [16],[17], [18], [19], [20], [21]). In [22] and [23], M. Akkouchi and E. Elqorachi have studied the stability of the Cauchy and Wilson equations and the generalized Cauchy and Wilson equations by using tools from harmonic analysis.

The fixed point method is the second most popular method in proving the stability of functional equations. Baker in 1991 (see [24]) was the first who used fixed point approach in the investigations of stability of functional equations. In fact, Baker applied a version of Banach’s fixed point theorem to obtain the Hyers-Ulam stability of functional equations. The stability of many integral equations have been studied by many authors. For instance, in [25] established the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability for a general class of nonlinear Volterra integral equations in Banach spaces using some alternative fixed point approach.

It should be remarked that Diaz and Margolis in [26] proved a theorem of the alternative for any contraction mapping on a generalized complete metric space. The theorem of Diaz and Margolis have been used by many authors see e.g. [25]. The interesting survey by K. Ciepliński in [27] presented some applications of various fixed-point theorems to the theory of the Hyers-Ulam stability of functional equations. J. BrzdĘk in [28] proved a fixed-point theorem for (not necessarily) linear operators and used it to obtain Hyers-Ulam stability results for a class of functional equations. J. BrzdĘk and K. Ciepliński proved in [29] A fixed-point result of the same type in complete non-Archimedean metric spaces as well as in complete metric spaces.

In this article, we focus on the investigations of the Ulam-Hyers and Ulam-Hyers-Rassias stability of some general integral equations. The main tool used in the analysis is some fixed point theory.

2. PRELIMINARIES

In what follows, $\mathbb{R}$ will be used to denote the set of reals, $\mathbb{R}_+$ the set of positive reals, $\mathbb{C}$ to denote the set of complex numbers, $\mathbb{F}$ $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and C^D to denote the family of functions from the set D into the set C. Let X be a (real or complex) normed space over the (real or complex) field $\mathbb{K}$ and let $I = [a, b]$ be a closed and bounded interval. Let $F: I \times X \to X$ be a continuous mapping. The objective of this article is to investigate the stability of the following integral equation

$$y(x) = \int_E F(x, y(x))dx, \qquad x \in E \tag{3}$$

in $2-$Banach spaces, where $y: I \to X$ is some unknown function. The set of solutions will be the Banach space $C(I, X)$ of continuous functions from I to X. For this purpose, we need first to recall (see, for instance, [30]) some definitions.

Definition 2 *By a linear 2-normed space we mean a pair* $(X, \|\cdot,\cdot\|)$ *such that* X *is an at least two-dimensional real linear space and*

$$\|\cdot,\cdot\|: X \times X \to \mathbb{R}$$

is a function satisfying the following conditions:

$$\| x_1, x_2 \| = 0 \text{ if and only if } x_1 \text{ and } x_2 \text{ are linearly dependent;}$$
$$\| x_1, x_2 \| = \| x_2, x_1 \| \text{ for } x_1, x_2 \in X$$
$$\| x_1, x_2 + x_3 \| \leq \| x_1, x_2 \| + \| x_1, x_3 \| \text{ for } x_i \in X, i = 1,2,3$$
$$\| \beta x_1, x_2 \| = |\beta| \, \| x_1, x_2 \| \text{ for } \beta \in \mathbb{R} \text{ and } x_1, x_2 \in X$$

Definition 3 *A sequence* $(x_n)_{n\in\mathbb{N}}$ *of elements of a linear 2-normed space* X *is called a Cauchy sequence if there are linearly independent* $y, z \in X$ *such that*

$$\lim_{n,m\to\infty} \| x_n - x_m, z \| = 0 = \| x_n - x_m, y \|,$$

whereas $(x_n)_{n\in\mathbb{N}}$ is said to be convergent if there exists an $x \in X$ (called a limit of this sequence and denoted by $\lim_{n\to\infty} X_n$) with

$$\lim_{n,m\to\infty} \| x_n - x, y \| = 0, y \in X.$$

A linear 2-normed space in which every Cauchy sequence is convergent is called a *2-Banach space*.

Let us also mention that in linear 2-normed spaces, every convergent sequence has exactly one limit and the standard properties of the limit of a sum and a scalar product are valid. Next, it is easily seen that we have the following property.

Lemma 4 *If* X *is a linear 2-normed space,* $x, y, z \in X$, y, z *are linearly independent, and*

$$\| x, y \| = 0 = \| x, z \|,$$

then $x = 0$.

Let us yet recall an important lemma from [31].

Lemma 5 *If* X *is a linear 2-normed space and* $(x_n)_{n\in\mathbb{N}}$ *is a convergent sequence of elements of* X, *then*

$$\lim_{n\to\infty} \| x_n, z \| = \| \lim_{n\to\infty} x_n, z \|, z \in X.$$

It is easy to check that (in view of the Cauchy-Schwarz inequality), if $\langle\cdot,\cdot\rangle$ is a real inner product in a real linear space X, of dimension greater than 1, and

$$\| x_1, x_2 \| := \sqrt{\| x_1 \|^2 \| x_2 \|^2 - \langle x_1, x_2 \rangle^2}, \qquad x_1, x_2 \in X$$

then conditions (N1)-(N4) are satisfied.

Definition 6 *We say that the integral equation (3) has the Hyers-Ulam stability in* $2-$*Banach spaces, if for all* $\epsilon > 0$ *and all function* $f: I \to X$ *satisfying the inequality*

$$\| f(x_1) - \int_E F(x_1, f(x_1)) dx_1, y \| \leq \epsilon, \ \forall x_1 \in E, y \in Y_0 \tag{4}$$

there exists a solution $g: I \to X$ of the integral equation

$$g(x_1) = \int_E G(x_1, g(x_1)) dx_1, \ \forall x_1 \in E, \tag{5}$$

such that

$$\| f(x_1) - g(x_1), y \| \leq \delta(\epsilon), \qquad \forall x_1 \in I, y \in Y_0$$

where $\delta(\epsilon)$ is an expression of ϵ only. If the above statement is also true when we replace ϵ and $\delta(\epsilon)$ by $\phi(x)$ and $\Phi(x)$, where $\phi, \Phi: I \to [0, \infty)$ are functions not depending on f and g explicitly, then we say that the corresponding integral equation has the Hyers-Ulam-Rassias stability (or the generalized Hyers-Ulam stability).

Here we recall the main tool in our study, which is the fixed point theorem introduced by J. BrzdĘk, K. Ciepliński in [32].

2.1. Fixed Point Theorem

Let us assume first the following:

E is a nonempty set, $(Y, \|\cdot,\cdot\|)$ is a $2-$Banach space, Y_0 is a subset of Y containing two linearly independent vectors, $j \in \mathbb{N}$,

$$f_i: E \to E, \quad g_i: Y_0 \to Y_0,$$

and

$$L_i: E \times Y_0 \to \mathbb{R}_+ \text{ for } i = 1, \cdots, j.$$

$\mathrm{T}: Y^E \to Y^E$ is an operator satisfying the inequality

$$\| \mathrm{T}\xi(x) - \mathrm{T}\mu(x), y \| \leq \sum_{i=1}^{j} L_i(x,y) \| \xi(f_i(x)) - \mu(f_i(x)), g_i(y) \|,$$

where $\xi, \mu \in Y^E, x \in E,$ and $y \in Y_0$.

$\Lambda: \mathbb{R}^{E \times Y_0} \to \mathbb{R}^{E \times Y_0}$ is an operator defined by

$$\Lambda\delta(x,y) := \sum_{i=1}^{j} L_i(x,y)\delta(f_i(x), g_i(y)), \qquad \delta \in \mathbb{R}^{E \times Y_0}, x \in E, y \in Y_0.$$

The following theorem is the main tool in this article.

Theorem 7 (Theorem 1 in [32]) *Let assumptions (A1)-(A3) hold and functions*

$$\varepsilon: E \times Y_0 \to \mathbb{R}_+ \text{ and } \varphi: E \to Y$$

fulfill the following two conditions:

$$\| \mathrm{T}\varphi(x) - \varphi(x), y \| \leq \varepsilon(x,y), \qquad x \in E, y \in Y_0,$$

$$\varepsilon^*(x,y) := \sum_{i=1}^{\infty} (\Lambda^i \varepsilon)(x,y) < \infty, \qquad x \in E, y \in Y_0.$$

Then, there exists a unique fixed point ψ of T for which

$$\| \varphi(x) - \psi(x), y \| \leq \varepsilon^*(x,y), \qquad x \in E, y \in Y_0.$$

Moreover,

$$\psi(x) = \lim_{l \to \infty} (\mathrm{T}^l \varphi)(x), \qquad x \in E.$$

It should be remarked that V. Radu [33] and L. Cădariu and V. Radu [34] have used some fixed point theorem to study the stability for the Cauchy functional equation and the Jensen functional equation; and they present proofs for Hyers-Ulam-Rassias stability. By their work, they unified the results of Hyers, Rassias and Gajda [35]. We point out that the stability of these equations have been studied by S.-M. Jung [36], W. Jian [37] and other authors. Subsequently, certain authors have adopted fixed point methods to study the stability of some functional equations. In a recent paper, S.-M. Jung in [18] has used the fixed point approach to prove the stability of ceratin differential equations of first order. The aim of this paper is to use Theorem 7 above to establish the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of some general integral equations in $2-$Banach spaces.

3. ULAM-HYERS STABILITY RESULTS IN 2-BANACH SPACES

In this section, we investigate the Ulam-Hyers stability of (3) in 2-Banach Spaces. For this purpose, we introduce the following theorem.

Theorem 8 *Suppose that the function* $F(x,y) \in \mathbb{R}_+^{E\times Y_0}$ *satisfies*

$$\|F(x,y_1) - F(x,y_2),\ y\| \le \sum_{i=1}^{j} K_i(x,y)\|y_1(f_i(x)) - y_2(f_i(x)),\ g_i(y)\| \tag{6}$$

for all $x \in E$ and $y_1, y_2 \in Y^E$. If the inequality

$$\left\|y(x) - \int_E F(x,y(s))d\sigma,\ y\right\| \le \varepsilon(x,y) \tag{7}$$

holds for all $x \in E$, all $y \in Y^E$ and for bounded function $\varepsilon \in \mathbb{R}_+^{E\times Y_0}$ satisfying the condition

$$\varepsilon(f_i(x), g_i(y)) \int_E K_i(x,y) < 1. \tag{8}$$

Then there exists a unique solution $y_0 \in Y^E$ of (3) satisfying

$$\|y(x) - y_0(x),\ y\| \le \frac{1}{1-\sum_{i=1}^{j} \varepsilon(f_i(x),g_i(y)) \int_E K_i(x,y)} \tag{9}$$

for all $x \in E$.

Proof.

Define the operator $T: Y^E \to Y^E$ by

$$Ty(x) := \int_E F(x,y(s))d\sigma,$$

it is clear that the fixed points of T solves (3).

Then, for any $\varphi_1, \varphi_2 \in Y^E$, we have

$$\begin{aligned}\|T\varphi_1(x) - T\varphi_2(x),\ y\| &= \int_E \|F(x,\varphi_1(x)) - F(x,\varphi_2(x)),\ y\|d\sigma \\ &\le \int_E \sum_{i=1}^{j} K_i(x,y)\|y_1(f_i(x)) - y_2(f_i(x)),\ g_i(y)\|d\sigma \\ &= \sum_{i=1}^{j} \left(\int_E K_i(x,y)d\sigma\right)\|y_1(f_i(x)) - y_2(f_i(x)),\ g_i(y)\|\end{aligned}$$

for all $x \in E$. Therefore, the condition (A2) is satisfied with $L_i(x,y) := \int_E K_i(x,y)d\sigma$. Also, since (8) holds, we have

$$\begin{aligned}\varepsilon^*(x,y) &:= \sum_{i=1}^{\infty} (\Lambda^i\varepsilon)(x,y) \\ &= \sum_{n=1}^{\infty} \left[\sum_{i=1}^{j} \varepsilon(f_i(x), g_i(y)) \int_E K_i(x,y)\right]^n \\ &= \frac{1}{1-\sum_{i=1}^{j} \varepsilon(f_i(x),g_i(y)) \int_E K_i(x,y)}.\end{aligned}$$

Now, according to Theorem 7, there exists a unique solution of (3) such that

$$\|y(x) - y_0(x),\ y\| \le \varepsilon^*(x,y) = \frac{1}{1-\sum_{i=1}^{j} \varepsilon(f_i(x),g_i(y)) \int_E K_i(x,y)}$$

for all $x \in E$.

4. ULAM-HYERS-RASSIAS STABILITY RESULTS IN 2-BANACH SPACES

In this section, we are interested in the investigation of the Ulam-Hyers-Rassias stability of (3) in 2-Banach Spaces. For this purpose, we introduce the following theorem.

Theorem 9 *Suppose that the function* $F(x,y) \in \mathbb{R}_+^{E\times Y_0}$ *satisfies*

$$\|F(x,y_1) - F(x,y_2),\ y\| \le \sum_{i=1}^{j} K_i(x,y)\|y_1(f_i(x)) - y_2(f_i(x)),\ g_i(y)\| \tag{10}$$

for all $x \in E$ and $y_1, y_2 \in Y^E$. If the inequality

$$\left\|y(x) - \int_E F(x, y(s))d\sigma,\ y\right\| \leq \varepsilon(x, y)\Phi(x, y) \tag{11}$$

holds for all $x \in E$, all $y \in Y^E$ and for bounded function $\varepsilon \in \mathbb{R}_+^{E \times Y_0}$ satisfying the condition

$$\varepsilon(f_i(x), g_i(y)) \int_E K_i(x, y) < 1. \tag{12}$$

Then there exists a unique solution $y_0 \in Y^E$ of (3) satisfying

$$\|y(x) - y_0(x),\ y\| \leq \frac{\Phi(x,y)}{1 - \sum_{i=1}^{j} \varepsilon(f_i(x), g_i(y)) \int_E K_i(x,y)} \tag{13}$$

for all $x \in E$.

Proof.

Define the operator $T: Y^E \to Y^E$ by

$$Ty(x) := \int_E F(x, y(s))d\sigma,$$

it is clear that the fixed point of T solves (3).

Then, for any $\varphi_1, \varphi_2 \in Y^E$, we have

$$\begin{aligned} \|T\varphi_1(x) - T\varphi_2(x),\ y\| &= \int_E \|F(x, \varphi_1(x)) - F(x, \varphi_2(x)),\ y\| d\sigma \\ &\leq \int_E \sum_{i=1}^{j} K_i(x, y) \|y_1(f_i(x)) - y_2(f_i(x)),\ g_i(y)\| d\sigma \\ &= \sum_{i=1}^{j} \left(\int_E K_i(x, y)d\sigma\right) \|y_1(f_i(x)) - y_2(f_i(x)),\ g_i(y)\| \end{aligned}$$

for all $x \in E$. Therefore, the condition (A2) is satisfied with $L_i(x, y) := \int_E K_i(x, y)d\sigma$.

Also, since (12) holds, we have

$$\begin{aligned} \varepsilon^*(x, y) &:= \sum_{i=1}^{\infty} (\Lambda^i \varepsilon)(x, y) \\ &= \sum_{n=1}^{\infty} \left[\sum_{i=1}^{j} \varepsilon(f_i(x), g_i(y)) \int_E K_i(x, y)\right]^n \\ &= \frac{\Phi(x,y)}{1 - \sum_{i=1}^{j} \varepsilon(f_i(x), g_i(y)) \int_E K_i(x,y)}. \end{aligned}$$

Now, according to Theorem 7, there exists a unique solution of (3) such that

$$\|y(x) - y_0(x),\ y\| \leq \varepsilon^*(x, y) = \frac{\Phi(x,y)}{1 - \sum_{i=1}^{j} \varepsilon(f_i(x), g_i(y)) \int_E K_i(x,y)}$$

for all $x \in E$.

5. CONCLUSION

Fixed point methods have been used to investigate the stability of functional equations. Basically, the method based on Banach fixed point theory and the method introduced by Diaz and Margolis have been extensively used in the literature. In this article, we managed to investigate the Ulam-Hyers and Ulam-Hyers-Rassias stability of some general integral equation. In the investigation we used a recent fixed point theory. The obtained results are in $2-$ Banach spaces. In this way, we employ to the first time a recent different fixed point approach to investigate the stability of some general integral equations. Potential future work could be to investigate the stability for more general nonlinear equations.

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