Sigma J Eng & Nat Sci 38 (2), 2020, 1031-1050

Sigma Journal of Engineering and Natural Sciences
Sigma Mühendislik ve Fen Bilimleri Dergisi

Research Article

A THREE STEPS ITERATIVE PROCESS FOR APPROXIMATING THE FIXED POINTS OF MULTIVALUED GENERALIZED α-NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX HYPERBOLIC SPACES

İbrahim KARAHAN*[1], Lateef Olakunle JOLAOSO[2]

[1]*Department of Mathematics , Erzurum Technical University, ERZURUM;* ORCID: 0000-0001-6191-7515
[2]*School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, SOUTH AFRICA;* ORCID: 0000-0002-4838-7465

Received: 30.10.2019 Revised: 02.05.2020 Accepted: 12.05.2020

ABSTRACT

In this paper, we prove some fixed point properties and demiclosedness principle for multivalued generalized α-nonexpansive mappings in uniformly convex hyperbolic spaces. We also proposed a three steps iterative scheme for approximating the common fixed points of generalized α-nonexpansive mapping and prove some strong and Δ-convergence theorems for such operator in the setting of uniformly convex hyperbolic space. We provide a numerical example to show that the three steps scheme proposed in this paper performs better than the modified SP-iterative scheme. The results obtained in this paper extend and generalized the corresponding results in uniformly convex Banach spaces, CAT(0) space and many other results in this direction.
Keywords: Generalized nonexpansive, three steps iteration, multivalued mappings, hyperbolic spaces, fixed point theorems.
2000 Mathematics Subject Classification: 47A06, 47H09, 47H10, 49M05.

1. INTRODUCTION

The study of fixed point theory for nonexpansive mappings have found numerous applications in differential equations, integral equations, signal processing, convex optimization and control theory. The existence of fixed points for single-valued nonexpansive mappings was first studied by Browder [4] in 1965 in a real Hilbert space. This was further extended to a uniformly convex Banach space by Browder in [5] and Göhde in [11], and to a reflexive Banach space by Goebel and Kirk [10, 18]. The study of fixed points of multivalued nonexpansive mappings was initiated by Markin [24] and Nadler [26] using the concept of Hausdorff metric.

Let X be a metric space and K be a nonempty subset of X. The set K is called proximal if for each $x \in X$, there exists an element $y \in K$ such that

$$d(x,y) = d(x,K) := \inf\{d(x,z) : z \in K\}.$$

It is well known that each weakly compact subset of a Banach space and each closed convex subset of a uniformly convex Banach space are proximal. Let $CB(X)$ be the collection of all

* Corresponding Author: e-mail: ibrahimkarahan@erzurum.edu.tr, tel: (442) 444 53 88 / 2094

nonempty closed and bounded subsets and $P(X)$ be the collection of all nonempty proximal bounded and closed subsets of X. The Hausdorff distance on $CB(X)$ is defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, \quad \forall\, A,B \in CB(X).$$

Note that for all $a \in A$, $d(a,B) \leq H(A,B)$.

A multivalued mapping $T: X \to CB(X)$ is said to be nonexpansive if

$$H(Tx,Ty) \leq d(x,y) \quad \forall\, x,y \in X.$$

A point $x \in X$ is called a fixed point of T if $x \in Tx$. We denote the set of fixed points of T by $F(T)$ and $P_T(x) = \{y \in Tx: d(x,y) = d(x,Tx)\}$. If $F(T) \neq \emptyset$, then T is said to be quasi nonexpansive if

$$H(Tx,p) \leq d(x,p) \quad \forall\, x \in X, p \in F(T).$$

The theory of multivalued mapping is more difficult than the corresponding theory of single-valued. Shimizo and Takahashi [35] showed the existence of fixed points for a multivalued nonexpansive mapping in a convex metric space. Since then, many authors have studied the approximation of fixed points of multivalued nonexpansive mappings using different iterative schemes (see, [1, 25, 15, 17, 29, 32, 36, 37, 38]).

Beside the nonlinear mappings involved in the study of fixed point theory, the role played by the ambient spaces involved in a fixed point problem is also very important. It is well known that the Banach spaces with convex structures have been studied to a great extent in this regard. This is mainly because the Banach spaces are vector space and so it is easier to introduce a convex structure in them. However, the metric space does not naturally enjoy such structure. Hence, there is need to introduce convex structure in the metric space. The notion of convex metric spaces was first coined and introduced by Takahashi [41] for studying fixed point theory for nonexpansive mappings in convex metric space. Several other attempts have been made to introduce different convex structure on metric spaces. The hyperbolic space is an example of a metric space with convex structure. In fact, different convex structures have been introduce to the hyperbolic space which results in different definitions of hyperbolic space (see [8, 20, 31]).

We note that although, the class of hyperbolic spaces introduced by Kohlenbach [20] is slightly restrictive than the class of hyperbolic spaces defined in [8], it is however, more general than the class of hyperbolic spaces introduced in [31]. More so, the Banach space and CAT(0) spaces are examples of hyperbolic spaces introduced in [20]. More examples of this class of hyperbolic space includes Hadamard manifolds, Hilbert ball with hyperbolic metric, Cartesian products of Hilbert balls and ℝ-trees. For more examples and details on hyperbolic spaces, the reader can see (for instance) [8, 9, 20, 31].

Throughout this paper, we consider the hyperbolic space which is defined by Kohlenbach [20] as follows:

Definition 1.1 *A hyperbolic space (X,d,W) is a metric space (X,d) together with a convexity mapping $W: X^2 \times [0,1] \to X$ satisfying*

(i) $d(u,W(x,y,\alpha)) \leq \alpha d(u,x) + (1-\alpha)d(u,y)$;
(ii) $d(W(x,y,\alpha),W(x,y,\beta)) = |\alpha-\beta| d(x,y)$;
(iii) $W(x,y,\alpha) = W(y,x,1-\alpha)$;
(iv) $d(W(x,z,\alpha),W(y,w,\alpha) \leq (1-\alpha)d(x,y) + \alpha d(z,w)$,

for all $x,y,z,w \in X$ and $\alpha,\beta \in [0,1]$. In the sequel, we shall use the term hyperbolic space instead of Kohlenbach hyperbolic space for the sake of simplicity.

The normal Mann iteration scheme [23] have played a very helpful role in approximating the fixed point of a nonexpansive mapping in a Banach space. In 1974, Ishikawa [14] introduced a new iterative process which performs better than the Mann iteration for approximating the fixed points of nonexpansive mapping as follows:

$$\begin{cases} x_0 \in K, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \quad n \geq 0, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, and K is a nonempty closed and convex subset of a real Hilbert space. Sastry and Babu [32] further developed an analogue of the Ishikawa iteration for multivalued nonexpansive mappings in Hilbert space as follows:

$$\begin{cases} x_0 \in K, \\ y_n = (1-\beta_n)x_n + \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n z_n', \quad n \geq 0, \end{cases}$$

where $\alpha, \beta \in 0,1]$, $z_n \in Tx_n$ such that $||z_n - p|| = d(p, Tx_n)$ and $z_n' \in Ty_n$ such that $||z_n' - p|| = d(p, Ty_n)$, K is a nonempty closed and convex subset of a Hilbert space.

Phuengrattana and Suntai [30] also introduced the following algorithm called SP-iteration as a generalization of the Mann, Ishikawa and Noor [27] iterations for approximating the fixed points of a nonexpansive mapping $T: K \to K$ in a uniformly convex Banach space:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n Tv_n, \\ y_n = (1-\beta_n)w_n + \beta_n Tw_n, \\ w_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \end{cases} \tag{1.2}$$

$\{\alpha_n\}, \{\beta_n\}, \{\gamma\}$ are real sequences in $[0,1]$. They also showed that the SP-iteration converges faster than the Mann, Ishikawa and Noor iterations for the class of continuous and non-decreasing function.

The following iteration process is a translation of the SP-iteration scheme from Banach space to hyperbolic space (see [30]): For a given $x_0 \in K$, $\{x_n\}$ is defined by

$$\begin{cases} z_n = W(x_n, Tx_n, \gamma_n), \\ y_n = W(z_n, Tz_n, \beta_n), \\ x_{n+1} = W(y_n, Ty_n, \alpha_n), \end{cases}$$

where K is a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity, T is a self mapping and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$.

Recently, Gunduz and Karahan [12] modified the SP-iteration for approximating the common fixed points of three multivalued nonexpansive mappings in hyperbolic space:

$$\begin{cases} x_{n+1} = W(u_n, y_n, \alpha_n), \\ y_n = W(v_n, z_n, \beta_n), \\ z_n = W(w_n, x_n, \gamma_n), \end{cases} \tag{1.3}$$

where $u_n \in P_T(y_n), v_n \in P_S(z_n)$, $w_n \in P_R(x_n)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences in $(0,1)$. It is well known that an iterative process that approximates the fixed points of nonlinear mappings using fewer number of iteration is preferable to iterative process with more number of iterations. The three steps iteration was shown by Glowonski and Le Tallec [7] to yield better numerical results than the one or two steps iterations. Glowonski and Le Tallec [7] employed three steps iterative process to approximate the solutions of the elastoviscoplasticity problem in liquid crystal theory and eigenvalues computations. Haubruge et al. [13] further applied the Glowinski and Le Tallec [7] iteration scheme to obtain a new splitting type algorithms for solving variational inequalities, separable convex programming and minimizing the sum of convex functions. They also showed that three steps iteration process lead to highly parallel iterations under certain conditions. All these show the importance of studying three steps iteration process for approximating solutions of real life problems.

In this paper, we introduce a new three steps iteration for approximating the common fixed point of three multivalued mappings in hyperbolic spaces. Our algorithm is defined as follows:

Let K be a nonempty convex subset of a hyperbolic space X. Let $R, S, T: K \to P(K)$ be three multivalued mappings. Choose $x_0 \in K$ and define $\{x_n\}$ as follow:

$$\begin{cases} x_{n+1} = W\left(u_n, W\left(x_n, v_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), \\ y_n = W\left(v_n, W\left(w_n, x_n, \frac{c_n}{1-b_n}\right), b_n\right), \\ z_n = W(x_n, w_n, a_n), \end{cases} \tag{1.4}$$

where $u_n \in P_R(y_n)$, $v_n \in P_S(z_n)$, $w_n \in P_T(x_n)$ and $a_n, b_n, c_n, \alpha_n, \beta_n \in (0,1)$ such that $0 < \alpha_n + \beta_n < 1$ and $0 < b_n + c_n < 1$.

It is easy to prove that algorithm (1.4) is well defined. Using algorithm (1.4), we study the approximation of common fixed points of three generalized α-nonexpansive mappings. The generalized α-nonexpansive mapping was recently introduced by Pant and Shukla [28] for single-valued mapping in Banach space. Pant and Shukla [28] showed that the class of generalized α-nonexpansive mappings is more general than the class of nonexpansive mappings, Suzuki's generalized nonexpansive mappings and α-nonexpansive mappings. It is worth mentioning that as far as we know, no work has been done on generalized α-nonexpansive mapping in hyperbolic space. Hence, it is necessary to extend the results on generalized α-nonexpansive mapping from uniformly convex Banach spaces to hyperbolic space.

In this article, we introduce the notion of multivalued generalized α-nonexpansive mapping in hyperbolic spaces. We also give some properties of the fixed points and demiclosedness principle of such mapping. Further, using algorithm (1.4), we prove some strong and Δ- convergences for approximating the common fixed points of the class of such maps. Hence, our results in this paper improve and unify the corresponding results of Pant and Shukla [28], Gunduz and Karahan [12], Suanoom et al [39], Mebawondu et al. [25], Khan et al. [15, 16, 17] and many other results in this direction.

2. PRELIMINARIES

In this section, we give some preliminaries, definitions and results which will be used in the sequel.

A hyperbolic space (X, d, W) is said to be uniformly convex if for any $\sigma > 0$ and $\epsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that for all $x, y, z \in X$

$$d(W(x, y, \frac{1}{2}), z) \leq (1-\delta)\sigma,$$

provided that $d(x,z) \leq \sigma$, $d(y,z) \leq \sigma$ and $d(x,y) \geq \epsilon\sigma$. The mapping $\eta: (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(\sigma, \epsilon)$ for given $\sigma > 0$ and $\epsilon \in (0,2]$ is called modulus of uniform convexity. We call η monotone if it decreases with σ (for a fixed ϵ). Also, a subset K of a hyperbolic space X is convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0,1]$.

Definition 2.1 *[39] Let K be a nonempty subset of a metric space X and $\{x_n\}$ be any bounded sequence in K. For $x \in X$, define a continuous functional $r(\cdot, \{x_n\}): X \to [0, \infty)$ by*

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(K, \{x_n\})$ of $\{x_n\}$ with respect to K is given by

$$r(K, \{x_n\}) = \inf\{r(x, \{x_n\}): x \in X\}.$$

A point $x \in K$ is said to be an asymptotic center of the sequence $\{x_n\}$ with respect to a subset $K \subseteq X$ if

$$r(x,\{x_n\}) = \inf\{r(y,\{x_n\}): y \in K\}.$$

The set of all asymptotic center of $\{x_n\}$ is denoted by $A(K,\{x_n\})$. If the asymptotic radius and the asymptotic center are taken with respect to X, then we simply denote them by $r(\{x_n\})$ and $A(\{x_n\})$ respectively.

Definition 2.2 *[19] A sequence $\{x_n\}$ in X is said to be Δ-convergence to $x \in X$ if x is the unique asymptotic center of $\{x_{n_k}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.*

It is well known that Δ-convergence coincides with weak convergence in Banach spaces with Opial's property (see [21]). We denote the strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$.

The following lemmas will be used in the sequel.

Lemma 2.3 *[22] Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniformly convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*

Lemma 2.4 *[6] Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{x\}$. Suppose that $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$ with $A(\{x_{n_k}\}) = \{x_1\}$ and $\{d(x_n,x_1)\}$ converges. Then $x = x_1$.*

Lemma 2.5 *[16] Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequence in X such that $limsup_{n\to\infty} d(x_n,x) \le c$, $limsup_{n\to\infty} d(y_n,x) \le c$ and $lim_{n\to\infty} d(W(x_n,y_n,\alpha_n),x) = c$ for some $c \ge 0$, then $lim_{n\to\infty} d(x_n,y_n) = 0$.*

Definition 2.6 *Let K be a nonempty subset of a hyperbolic space X and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a Fejér monotone sequence with respect to K if for all $x \in K$ and $n \in \mathbb{N}$*

$$d(x_{n+1},x) \le d(x_n,x).$$

Lemma 2.7 *[3] Let K be a nonempty closed subset of a complete metric space X and $\{x_n\}$ be a Féjer monotone sequence with respect to K. Then $\{x_n\}$ converges to some $x^* \in K$ if and only if $lim_{n\to\infty} d(x_n,K) = 0$.*

Lemma 2.8 *[3] Let $\{x_n\}$ be a sequence in X and K be a nonempty subset of X. Suppose $T: K \to K$ is any nonlinear mapping and the sequence $\{x_n\}$ is Fejér monotone with respect to K, then we have the following: (i) $\{x_n\}$ is bounded, (ii) The sequence $\{d(x_n,x^*)\}$ is decreasing and converges for all $x^* \in F(T)$, (iii) $lim_{n\to\infty} d(x_n,F(T))$ exists.*

Definition 2.9 *A nonlinear mapping $T: K \to K$ is said to be*

(i) Suzuki's generalized nonexpansive (or satisfied condition C) [40] if for all $x,y \in K$

$$\frac{1}{2}d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) \le d(x,y), \tag{2.1}$$

(ii) α-nonexpansive mapping [2] if for all $x,y \in K$, $\alpha \in (0,1)$ and

$$d(Tx,Ty)^2 \le \alpha d(Tx,y)^2 + \alpha d(Ty,x)^2 + (1-2\alpha)d(x,y)^2, \tag{2.2}$$

(iii) generalized α-nonexpansive mapping [28] if for all $x,y \in K$, $\alpha \in (0,1)$ and

$$\frac{1}{2}d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) \le \alpha d(Tx,y) + \alpha d(Ty,x) + (1-2\alpha)d(x,y).$$

Remark 2.10 *It is worth mentioning that when $\alpha = 0$, the class of generalized α-nonexpansive mapping reduces to the class of mapping satisfying condition C. Pant and Shukla [28] gave the following example of mapping which is generalized α-nonexpansive but not α-nonexpansive nor satisfies condition C.*

Example 2.11 *[28] Let* $X = \{(0,0),(2,0),(0,4),(4,0),(4,5),(5,4)\}$ *be a subset of* $\mathbb{R}^2$. *Define a norm* $\|\cdot\|$ *on* X *by* $\|(x_1,x_2)\| = |x_1| + |x_2|$. *Then* $(X, \|\cdot\|)$ *is a Banach space. Define a mapping* $T: X \to X$ *by*

$$T: \begin{pmatrix} (0,0),(2,0),(0,4),(4,0),(4,5),(5,4) \\ (0,0),(0,0),(0,0),(2,0),(4,0),(0,4) \end{pmatrix}, \tag{2.3}$$

For $\alpha = \frac{1}{5}$,

$$||Tx - Ty|| \leq \alpha||Tx - y|| + \alpha||Ty - x|| + (1 - 2\alpha)||x - y||$$

if $(x, y) \neq ((4,5),(5,4))$. In the case $x = (4,5)$ and $y = (5,4)$, we have

$$\frac{1}{2}||x - Tx|| = \frac{1}{2}||y - Ty|| = \frac{5}{2} > 2 = ||x - y||.$$

Therefore T is generalized α-nonexpansive mapping.

However, for $x = (4,5)$ and $y = (5,4)$

$$\begin{aligned} ||Tx - Ty||^2 &= 64 > 42\alpha + 4 \\ &= 25\alpha + 25\alpha + (1 - 2\alpha) \cdot 4 \\ &= \alpha||Tx - y||^2 + \alpha||Ty - x||^2 + (1 - 2\alpha)||x - y||^2. \end{aligned}$$

This shows that T is not an α-nonexpansive mapping for any $\alpha < 1$. Further, for $x = (4,0)$ and $y = (5,4)$

$$\frac{1}{2}||x - Tx|| = 1 < 5 = ||x - y||,$$

but

$$||Tx - Ty|| = 6 > 5 = ||x - y||.$$

So, T is not a Suzuki's generalized nonexpansive mapping.

Now, we give the definition of generalized alpha-nonexpansive mappings for the multivalued mappings in hyperbolic spaces.

Definition 2.12 *A multivalued mapping* $T: K \to CB(K)$ *is said to be generalized* α*-nonexpansive mapping if for all* $x, y \in K$, *and* $\alpha \in (0,1)$

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow$$
$$H(Tx, Ty) \leq \alpha d(x, Ty) + \alpha d(y, Tx) + (1 - 2\alpha)d(x, y). \tag{2.4}$$

Remark 2.13 *Let* $T: K \to CB(K)$ *be a multivalued mapping. If* T *is a nonexpansive mapping, then it is clear that* T *is Suzuki's generalized nonexpansive and thus,* T *is generalized* α*-nonexpansive.*

The following example shows that the converse of Remark 2.13 does not hold.

Example 2.14 *Let* $T: [0,3] \to CB([0,3])$ *be a mapping defined by*

$$Tx = \begin{cases} [1, \frac{3}{2}], & \text{if } x = 3, \\ \{0\}, & \text{if } x \neq 3. \end{cases}$$

If $x = 3$ and $y = 2$, then

$$\frac{1}{2}d(3, T3) = \frac{3}{4} < 1 = |3 - 2|,$$

and

$$H(T3, T2) = \frac{3}{2} > 1 = |3 - 2|.$$

Then T is neither nonexpansive nor Suzuki's generalized nonexpansive. However T is generalized α-nonexpansive for $\alpha = \frac{1}{3}$.

The following Lemma is useful for our results. It gives some properties of the P_T mapping in metric (hence hyperbolic) spaces (see [26, 36]).

Lemma 2.15 *Let $T: K \to P(K)$ be a multivalued mapping and $P_T x = \{u \in Tx: d(x,u) = d(x,Tx)\}$. Then the following are equivalent:*

(i) $F(T) = F(P_T)$,
(ii) $P_T p = \{p\}$ for each $p \in F(T)$,
(iii) For each $x \in K$, $P_T x$ is a closed subset of Tx and so it is compact,
(iv) $d(x,Tx) = d(x,P_T x)$ for each $x \in K$,
(v) P_T is a multivalued mapping K to $P(K)$.

3. MAIN RESULTS

In this section, we present our main results in this paper. First we give some properties of the fixed points set of generalized α-nonexpansive mappings in a convex hyperbolic space.

3.1. Some fixed point properties of generalized α-nonexpansive mapping

Lemma 3.1 *Let K be a nonempty closed and convex subset of a hyperbolic space X with monotone modulus of convexity η and $T: K \to CB(X)$ be a generalized α-nonexpansive mapping such that $F(T) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$. Then $F(T)$ is closed and convex.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ which converges to some $z \in K$. We will show that $z \in F(T)$. Since $Tp = \{p\}$ for each $p \in F(T)$, we have

$$\tfrac{1}{2} d(x_n, Tx_n) = 0 < d(x_n, z),$$

and

$$\begin{aligned} d(x_n, Tz) \leq H(Tx_n, Tz) &\leq \alpha d(z, x_n) + \alpha d(x_n, Tz) + (1-2\alpha) d(z, x_n) \\ &= \alpha d(x_n, Tz) + (1-\alpha) d(z, x_n). \end{aligned}$$

Hence

$$d(x_n, Tz) \leq d(z, x_n) \tag{3.1}$$

Taking limit as $n \to \infty$, we have that

$$\lim_{n\to\infty} d(x_n, Tz) \leq \lim_{n\to\infty} d(x_n, z) = 0.$$

Hence, by the uniqueness of the limit, we have that $z \in Tz$.

Next, we show that $F(T)$ is convex. Let $x, y \in F(T)$ and $z \in K$, then by the definition of generalized α-nonexpansive mapping, we have

$$d(x, Tz) \leq d(x, z) \tag{3.2}$$

and

$$d(y, Tz) \leq d(y, z). \tag{3.3}$$

For $z = W(x, y, \lambda)$, from (3.2) and (3.3), we have

$$\begin{aligned} d(x,y) &\leq d(x,Tz) + d(Tz,y) \\ &\leq d(x,z) + d(z,y) \\ &= d(x, W(x,y,\lambda)) + d(W(x,y,\lambda), y) \\ &\leq \lambda d(x,x) + (1-\lambda) d(x,y) + \lambda d(x,y) + (1-\lambda) d(y,y) \\ &= d(x,y). \end{aligned}$$

Thus $d(x,Tz) = d(x,z)$ and $d(Tz,y) = d(z,y)$ because, if $d(x,Tz) < d(x,z)$ or $d(Tz,y) < d(z,y)$, then we will have a contradiction that $d(x,y) < d(x,y)$. So $z \in Tz$. Thus, $W(x,y,\lambda) \in F(T)$. This implies that $F(T)$ is convex.

Lemma 3.2 *Let K be a nonempty closed and convex subset of a hyperbolic space X and $T: K \to P(K)$ be a generalized α-nonexpansive mapping such that $F(T) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$. Then T is quasi nonexpansive.*

Proof. Let $p \in F(T)$ and $x \in K$. Note that

$$\frac{1}{2}d(p,Tp) = 0 \leq d(x,p).$$

Then

$$\begin{aligned} d(p,Tx) &\leq H(Tp,Tx) \\ &\leq \alpha d(p,Tx) + \alpha d(x,Tp) + (1-2\alpha)d(x,p). \end{aligned}$$

This implies that

$$(1-\alpha)d(Tx,p) \leq (1-\alpha)d(x,p). \tag{3.4}$$

Since $(1-\alpha) > 0$, then we have $H(Tx,Tp) \leq d(x,p)$ for all $x \in K$ and $p \in F(T)$. Therefore, T is quasi nonexpansive.

We now establish the demiclosedness principle for multivalued generalized α-nonexpansive mapping.

Lemma 3.3 *Let X be a complete uniformly convex hyperbolic space with monotone modules of uniform convexity η. Let K be a nonempty closed and convex subset of X and $T: K \to P(K)$ be a generalized α-nonexpansive mapping such that $F(T) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$. If $\{x_n\}$ is a bounded sequence in K such that Δ-$lim_{n\to\infty} x_n = x^*$ and $lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $x^* \in F(T)$.*

Proof. Since $\{x_n\}$ is a bounded sequence in X, we have from Lemma 2.3 that $\{x_n\}$ has a unique asymptotic center in K. Also, since Δ-$\lim_{n\to\infty} x_n = x^*$, we have that $A(\{x_n\}) = \{x^*\}$. Now note that

$$\begin{aligned} d(x_n,Tx^*) &\leq d(x_n,Tx_n) + H(Tx_n,Tx^*) \\ &\leq d(x_n,Tx_n) + \alpha d(x_n,Tx^*) + \alpha d(x^*,Tx_n) + (1-2\alpha)d(x_n,x^*) \\ &\leq d(x_n,Tx_n) + \alpha d(x_n,Tx^*) + \alpha d(x^*,x_n) + \alpha d(x_n,Tx_n) + (1-2\alpha)d(x_n,x^*). \end{aligned}$$

This implies that

$$d(x_n,Tx^*) \leq \frac{1+\alpha}{1-\alpha} d(x_n,Tx_n) + d(x_n,x^*).$$

Taking $\limsup_{n\to\infty}$ from both sides, we have

$r(Tx^*,\{x_n\}) = \limsup_{n\to\infty} d(x_n,Tx^*) \leq \frac{1+\alpha}{1-\alpha} \limsup_{n\to\infty} d(x_n,Tx_n) + \limsup_{n\to\infty} d(x_n,x^*) \leq \limsup_{n\to\infty} d(x_n,x^*) = r(x^*,\{x_n\})$. By the uniqueness of the asymptotic center of $\{x_n\}$, we have $x^* \in Tx^*$ and hence, $x^* \in F(T)$.

3.2. Strong and Δ- convergence theorems for generalized α-onexpansive mapping.

We state and prove the following lemmas which will be needed in the proof of our main theorems.

Lemma 3.4 *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X. Let $R, S, T: K \to P(K)$ be three multivalued mappings such that P_R, P_S and P_T are generalized α-nonexpansive mappings and $F := F(R) \cap F(S) \cap F(T) \neq \emptyset$. For arbitrary*

$x_0 \in K$, let $\{x_n\}$ be the sequence define by algorithm (1.4). Then $\{x_n\}$ is bounded and the limit $lim_{n\to\infty} d(x_n, x^*)$ exists for each $x^* \in F$.

Proof. Let $x^* \in F$, using Lemma 3.2, we have

$$\begin{aligned}
d(y_n, x^*) &= d\left(W(v_n, W\left(w_n, x_n, \frac{c_n}{1-b_n}\right), b_n), x^*\right) \\
&\le b_n d(v_n, x^*) + (1-b_n) d\left(W\left(w_n, x_n, \tfrac{c_n}{1-b_n}\right), x^*\right) \\
&\le b_n d(v_n, P_S(x^*)) + (1-b_n)\left[\tfrac{c_n}{1-b_n} d(w_n, x^*) + \left(1-\tfrac{c_n}{1-b_n}\right) d(x_n, x^*)\right] \\
&\le b_n H(P_S(z_n), P_S(x^*)) + c_n d(w_n, x^*) + (1-b_n-c_n) d(x_n, x^*) \\
&\le b_n d(z_n, x^*) + c_n d(w_n, P_T(x^*)) + (1-b_n-c_n) d(x_n, x^*) \\
&\le b_n d(W(x_n, w_n, a_n), x^*) + c_n H(P_T(x_n), P_T(x^*)) + (1-b_n-c_n) d(x_n, x^*) \\
&\le b_n[a_n d(x_n, x^*) + (1-a_n) d(w_n, x^*)] + c_n d(x_n, x^*) + (1-b_n-c_n) d(x_n, x^*) \\
&\le b_n[a_n d(x_n, x^*) + (1-a_n) d(w_n, P_T(x^*))] + (1-b_n) d(x_n, x^*) \\
&\le b_n[a_n d(x_n, x^*) + (1-a_n) H(P_T(x_n), P_T(x^*))] + (1-b_n) d(x_n, x^*) \\
&\le b_n[a_n d(x_n, x^*) + (1-a_n) d(x_n, x^*)] + (1-b_n) d(x_n, x^*) \\
&= d(x_n, x^*). \qquad (3.5)
\end{aligned}$$

Also

$$\begin{aligned}
d(x_{n+1}, x^*) &= d\left(W\left(u_n, W\left(x_n, v_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), x^*\right) \\
&\le \alpha_n d(u_n, x^*) + (1-\alpha_n) d\left(W\left(x_n, v_n, \tfrac{\beta_n}{1-\alpha_n}\right), x^*\right) \qquad (3.6) \\
&\le \alpha_n d(u_n, P_R(x^*)) + (1-\alpha_n)\left[\tfrac{\beta_n}{1-\alpha_n} d(x_n, x^*) + \left(1-\tfrac{\beta_n}{1-\alpha_n}\right) d(v_n, x^*)\right] \\
&\le \alpha_n H(P_R(y_n), P_R(x^*)) + \beta_n d(x_n, x^*) + (1-\alpha_n-\beta_n) d(v_n, x^*) \\
&\le \alpha_n d(y_n, x^*) + \beta_n d(x_n, x^*) + (1-\alpha_n-\beta_n) d(v_n, P_S x^*) \\
&\le \alpha_n d(x_n, x^*) + \beta_n d(x_n, x^*) + (1-\alpha_n-\beta_n) H(P_S(z_n), P_S(x^*)) \\
&\le (\alpha_n+\beta_n) d(x_n, x^*) + (1-\alpha_n-\beta_n) d(z_n, x^*) \\
&= (\alpha_n+\beta_n) d(x_n, x^*) + (1-\alpha_n-\beta_n) d(W(x_n, w_n, a_n), x^*) \\
&\le (\alpha_n+\beta_n) d(x_n, x^*) + (1-\alpha_n-\beta_n)[a_n d(x_n, x^*) + (1-a_n) d(w_n, x^*)] \\
&\le (\alpha_n+\beta_n) d(x_n, x^*) + (1-\alpha_n-\beta_n)[a_n d(x_n, x^*) + (1-a_n) d(w_n, P_T x^*)] \\
&\le (\alpha_n+\beta_n) d(x_n, x^*) + (1-\alpha_n-\beta_n)[a_n d(x_n, x^*) + (1-a_n) H(P_T x_n, P_T x^*)] \\
&\le (\alpha_n+\beta_n) d(x_n, x^*) + (1-\alpha_n-\beta_n)[a_n d(x_n, x^*) + (1-a_n) d(x_n, x^*)] \\
&= d(x_n, x^*).
\end{aligned}$$

This shows that the sequence $\{x_n\}$ is a Fejér monotone sequence, and hence it is bounded. Consequently, $\lim_{n\to\infty} d(x_n, x^*)$ exists.

Lemma 3.5 *Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X. Let $R, S, T: K \to P(K)$ be three multivalued mappings such that P_R, P_S and P_T are generalized α-nonexpansive mappings and $F := F(R) \cap F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.4), then we have*

$$\lim_{n\to\infty} d(x_n, P_T x_n) = \lim_{n\to\infty} d(x_n, P_R x_n) = \lim_{n\to\infty} d(x_n, P_S x_n) = 0.$$

Proof. By Lemma 3.4, $\lim_{n\to\infty} d(x_n, x^*)$ exists for each $x^* \in F$. Assume that $\lim_{n\to\infty} d(x_n, x^*) = c$ for some $c \ge 0$. If $c = 0$, the results is trivial. So we suppose that $c > 0$. Using Lemma 3.2, we have

$$\begin{aligned}
d(w_n, x^*) &\le d(w_n, P_T x^*) \\
&\le H(P_T x_n, P_T x^*) \\
&\le d(x_n, x^*).
\end{aligned}$$

Hence

$$\limsup_{n\to\infty} d(w_n, x^*) \le c.$$

Also we get

$$\begin{aligned}
d(v_n, x^*) &\le d(v_n, P_S x^*)\\
&\le H(P_S z_n, P_S x^*)\\
&\le d(z_n, x^*)\\
&= d(W(x_n, w_n, a_n), x^*)\\
&\le a_n d(x_n, x^*) + (1 - a_n) d(w_n, x^*)\\
&\le a_n d(x_n, x^*) + (1 - a_n) H(P_T x_n, P_T x^*)\\
&\le a_n d(x_n, x^*) + (1 - a_n) d(x_n, x^*)\\
&\le d(x_n, x^*). \qquad (3.7)
\end{aligned}$$

Then, we deduce that

$$\limsup_{n\to\infty} d(v_n, x^*) \le c. \qquad (3.8)$$

More so from (3.5), we get

$$\begin{aligned}
d(u_n, x^*) &= d(u_n, Rx^*)\\
&\le H(Ry_n, Rx^*)\\
&\le d(y_n, x^*)\\
&\le d(x_n, x^*),
\end{aligned}$$

hence

$$\limsup_{n\to\infty} d(u_n, p) \le c.$$

On the other hand, it follows from (3.6) that

$$\begin{aligned}
d\left(W\left(x_n, v_n, \tfrac{\beta_n}{1-\alpha_n}\right), x^*\right) &\le \tfrac{\beta_n}{1-\alpha_n} d(x_n, x^*) + \left(1 - \tfrac{\beta_n}{1-\alpha_n}\right) d(v_n, x^*)\\
&\le \tfrac{\beta_n}{1-\alpha_n} d(x_n, x^*) + \left(1 - \tfrac{\beta_n}{1-\alpha_n}\right) d(v_n, Sx^*)\\
&\le \tfrac{\beta_n}{1-\alpha_n} d(x_n, x^*) + \left(1 - \tfrac{\beta_n}{1-\alpha_n}\right) H(Sz_n, Sx^*)\\
&\le \tfrac{\beta_n}{1-\alpha_n} d(x_n, x^*) + \left(1 - \tfrac{\beta_n}{1-\alpha_n}\right) d(z_n, x^*)\\
&\le \tfrac{\beta_n}{1-\alpha_n} d(x_n, x^*) + \left(1 - \tfrac{\beta_n}{1-\alpha_n}\right) d(x_n, x^*)\\
&= d(x_n, x^*).
\end{aligned}$$

This implies that

$$\limsup_{n\to\infty} d\left(W\left(x_n, v_n, \tfrac{\beta_n}{1-\alpha_n}\right), x^*\right) \le c. \qquad (3.9)$$

Also we can rewrite (3.6) as

$$\begin{aligned}
(1-\alpha_n) d(x_{n+1}, x^*) &\le \alpha_n d(u_n, x^*) + (1-\alpha_n) d\left(W\left(x_n, v_n, \tfrac{\beta_n}{1-\alpha_n}\right), x^*\right) - \alpha_n d(x_{n+1}, x^*)\\
&\le (1-\alpha_n) d\left(W\left(x_n, v_n, \tfrac{\beta_n}{1-\alpha_n}\right), x^*\right) + \alpha_n [d(x_n, x^*) - d(x_{n+1}, x^*)].
\end{aligned}$$

Thus

$$d(x_{n+1}, x^*) \le d\left(W\left(x_n, v_n, \tfrac{\beta_n}{1-\alpha_n}\right), x^*\right) + \tfrac{\alpha_n}{1-\alpha_n} [d(x_n, x^*) - d(x_{n+1}, x^*)],$$

and hence

$$c \le \liminf_{n\to\infty} d\left(W\left(x_n, v_n, \tfrac{\beta_n}{1-\alpha_n}\right), x^*\right). \qquad (3.10)$$

Therefore from (3.9) and (3.10), we have

$$\lim_{n\to\infty} d\left(W\left(x_n, v_n, \frac{\beta_n}{1-\alpha_n}\right), x^*\right) = c. \tag{3.11}$$

Since $\limsup_{n\to\infty} d(v_n, x^*) \le c$, using Lemma 2.5, we have

$$\lim_{n\to\infty} d(x_n, v_n) = 0. \tag{3.12}$$

Similarly, we can show that

$$\lim_{n\to\infty} d(x_n, w_n) = 0, \tag{3.13}$$

that is

$$\lim_{n\to\infty} d(x_n, P_T x_n) = 0.$$

Clearly

$$d(z_n, x_n) \le \alpha_n d(x_n, x_n) + (1-\alpha_n) d(w_n, x_n),$$

then from (3.13), we have

$$\lim_{n\to\infty} d(z_n, x_n) = 0. \tag{3.14}$$

Therefore

$$\lim_{n\to\infty} d(v_n, z_n) \le \lim_{n\to\infty} [d(v_n, x_n) + d(x_n, z_n)] = 0.$$

This implies that

$$\lim_{n\to\infty} d(z_n, P_S z_n) = 0, \tag{3.15}$$

and

$$\lim_{n\to\infty} d(x_n, P_S z_n) \le \lim_{n\to\infty} [d(x_n, z_n) + d(z_n, P_S z_n)] = 0. \tag{3.16}$$

More so

$$\begin{aligned} d(x_n, P_S x_n) &\le d(x_n, z_n) + d(z_n, P_S x_n) \\ &\le d(x_n, z_n) + d(z_n, P_S z_n) + H(P_S z_n, P_S x_n) \\ &\le d(x_n, z_n) + d(z_n, P_S z_n) + \alpha d(z_n, P_S x_n) + \alpha d(x_n, P_S z_n) + (1-2\alpha) d(x_n, z_n) \\ &\le d(x_n, z_n) + d(z_n, P_S z_n) + \alpha d(z_n, x_n) + \alpha d(x_n, P_S x_n) \\ &\quad + \alpha d(x_n, P_S z_n) + (1-2\alpha) d(x_n, z_n). \end{aligned}$$

This implies that

$$d(x_n, P_S x_n) \le \frac{(2-\alpha)}{1-\alpha} d(x_n, z_n) + \frac{1}{1-\alpha} d(z_n, P_S z_n) + \frac{\alpha}{1-\alpha} d(x_n, P_S z_n). \tag{3.17}$$

Since $\alpha \in (0,1)$, it follows from (3.14), (3.15), (3.16) and (3.17) that

$$\lim_{n\to\infty} d(x_n, P_S x_n) = 0.$$

Also observe that $\limsup_{n\to\infty} d(y_n, x^*) \le c$. From (3.6), we have

$$\begin{aligned} c = \lim_{n\to\infty} d(x_{n+1}, x^*) &\le \lim_{n\to\infty} \left[\alpha_n d(u_n, x^*) + (1-\alpha_n) d\left(W\left(x_n, v_n, \frac{\beta_n}{1-\alpha_n}\right), x^*\right)\right] \\ &\le \lim_{n\to\infty} \left[\alpha_n d(y_n, x^*) + (1-\alpha_n) d\left(W\left(x_n, v_n, \frac{\beta_n}{1-\alpha_n}\right), x^*\right)\right]. \end{aligned}$$

It follows from (3.11) that $c \le \liminf_{n\to\infty} d(y_n, x^*)$, hence $\lim_{n\to\infty} d(y_n, x^*) = c$.

Now, let $z'_n = W\left(x_n, w_n \frac{c_n}{1-b_n}\right)$, then, $y_n = W(v_n, z'_n, b_n)$. Clearly

$$d(z'^*_n) \le d(x_n, x^*),$$

then $\limsup_{n\to\infty} d(z'^*_n) \le c$. Since $\limsup_{n\to\infty} d(v_n, x^*) \le c$, using Lemma 2.5, we have that

$$\lim_{n\to\infty} d(v_n, z'_n) = 0.$$

Therefore

$$\lim_{n\to\infty} d(y_n, v_n) \leq \lim_{n\to\infty} [b_n d(v_n, v_n) + (1 - b_n) d(z_n', v_n)] = 0,$$

and

$$\lim_{n\to\infty} d(y_n, x_n) \leq \lim_{n\to\infty} [d(y_n, v_n) + d(v_n, x_n)] = 0. \tag{3.18}$$

Also, let $y_n' = W(y_n, v_n, \frac{\beta_n}{1-\alpha_n})$, then $x_{n+1} = W(u_n, y_n', \alpha_n)$. Since $\lim_{n\to\infty} d(x_{n+1}, p) = c$, $\limsup_{n\to\infty} d(u_n, p) \leq c$ and from (3.9) $\limsup_{n\to\infty} d(y_n', p) \leq c$, using Lemma 2.5, we have

$$\lim_{n\to\infty} d(u_n, y_n') = 0. \tag{3.19}$$

Also, since $0 < \alpha_n + \beta_n < 1$, we have

$$\begin{aligned} d(y_n', x_n) &= d\left(W\left(x_n, v_n, \frac{\beta_n}{1-\alpha_n}\right), x_n\right) \\ &\leq \frac{\beta_n}{1-\alpha_n} d(x_n, x_n) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(v_n, x_n). \end{aligned}$$

Hence, from (3.12), we have

$$\lim_{n\to\infty} d(y_n', x_n) = 0. \tag{3.20}$$

Therefore, using (3.19) and (3.20), we get

$$\lim_{n\to\infty} d(u_n, x_n) \leq \lim_{n\to\infty} [d(u_n, y_n') + d(y_n', x_n)] = 0.$$

and from (3.18), we have

$$\lim_{n\to\infty} d(u_n, y_n) \leq \lim_{n\to\infty} [d(u_n, x_n) + d(x_n, y_n)] = 0.$$

Therefore we obtain

$$\lim_{n\to\infty} d(y_n, P_R y_n) = 0, \tag{3.21}$$

and from (3.18), we get

$$\lim_{n\to\infty} d(x_n, P_R y_n) \leq \lim_{n\to\infty} [d(x_n, y_n) + d(y_n, P_R y_n)] = 0. \tag{3.22}$$

Also

$$\begin{aligned} d(x_n, P_R x_n) &\leq d(x_n, y_n) + d(y_n, P_R x_n) \\ &\leq d(x_n, y_n) + d(y_n, P_R y_n) + H(P_R y_n, P_R x_n) \\ &\leq d(x_n, y_n) + d(y_n, P_R y_n) + \alpha d(x_n, P_R y_n) + \alpha d(y_n, P_R x_n) + (1 - 2\alpha) d(x_n, y_n) \\ &\leq d(x_n, y_n) + d(y_n, P_R y_n) + \alpha d(x_n, P_R y_n) + \alpha d(y_n, x_n) \\ &\quad + \alpha d(x_n, P_R x_n) + (1 - 2\alpha) d(x_n, y_n). \end{aligned}$$

Hence

$$d(x_n, P_R x_n) \leq \frac{2-\alpha}{1-\alpha} d(x_n, y_n) + \frac{1}{1-\alpha} d(y_n, P_R y_n) + \frac{\alpha}{1-\alpha} d(x_n, P_R y_n). \tag{3.23}$$

Since $\alpha \in (0,1)$, it follows from (3.18), (3.21), (3.22) and (3.23) that

$$\lim_{n\to\infty} d(x_n, P_R x_n) = 0.$$

Theorem 3.6 *Let K be a nonempty closed and convex subset of a complete hyperbolic space X with a monotone modulus of uniform convexity η. Let $R, S, T: K \to P(K)$ be three multivalued mappings such that P_R, P_S and P_T are generalized α-nonexpansive mappings and $F := F(R) \cap F(S) \cap F(T) \neq \emptyset$. If $\{x_n\}$ is the sequence defined by (1.4), then $\{x_n\}$ Δ-converges to a point in F.*

Proof. Let $p \in F$. By Lemma 3.4, $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, p)$ exists. Thus $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{p\}$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $A(\{x_{n_k}\}) = \{q\}$. From Lemma 3.5, we get $\lim_{k\to\infty} d(x_{n_k}, Tx_{n_k}) = 0$. We claim that $q \in F(T)$.
To prove this, we take another sequence $\{v_m\}$ in $T(q)$. Then

$$\begin{aligned} r(v_m, \{x_{n_k}\}) &= \limsup_{k\to\infty} d(v_m, x_{n_k}) \\ &\leq \lim_{k\to\infty} [d(v_m, Tx_{n_k}) + d(Tx_{n_k}, x_{n_k})] \\ &\leq \lim_{k\to\infty} [H(Tq, Tx_{n_k}) + d(Tx_{n_k}, x_{n_k})] \\ &\leq \lim_{k\to\infty} [d(q, x_{n_k}) + d(Tx_{n_k}, x_{n_k})] \\ &\leq \limsup_{k\to\infty} d(q, x_{n_k}) \\ &= r(p, \{x_{n_k}\}). \end{aligned}$$

This implies that $|r(v_m, \{x_{n_k}\}) - r(q, \{x_{n_k}\})| \to 0$ for $k \to \infty$. By Lemma 2.4, we get $\lim_{m\to\infty} v_m = q$. Hence $T(q)$ is either closed or bounded. Consequently, $\lim_{m\to\infty} v_m = q \in F(T)$. Similarly, we can show that $q \in F(S)$ and $q \in F(R)$. Hence $q \in F$. From the uniqueness of the asymptotic center, we have

$$\begin{aligned} \limsup_{k\to\infty} d(x_{n_k}, q) &< \limsup_{k\to\infty} d(x_{n_k}, p) \\ &\leq \limsup_{n\to\infty} d(x_n, p) \\ &< \limsup_{n\to\infty} d(x_n, q) \\ &= \limsup_{k\to\infty} d(x_{n_k}, q). \end{aligned}$$

This is a contradiction, and hence, $p = q$. Thus $A(\{x_{n_k}\}) = \{q\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ-converges to a common fixed point in F.

Theorem 3.7 *Let K be a nonempty closed and convex subset of a complete hyperbolic space X with a monotone modulus of uniform convexity η. Let $R, S, T: K \to P(K)$ be three multivalued mappings such that P_R, P_S and P_T are generalized α-nonexpansive mappings and $F := F(R) \cap F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined in (1.4), then $\{x_n\}$ converges strongly to a common fixed point $p \in F$ if and only if $liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. Suppose that $\{x_n\}$ converges to a fixed point $p \in F$. Then $\lim_{n\to\infty} d(x_n, p) = 0$ and since $0 \leq d(x_n, F) \leq d(x_n, p)$, it follows that $\liminf_{n\to\infty} d(x_n, F) = 0$. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. From Lemma 3.4 we have that

$$d(x_{n+1}, p) \leq d(x_n, F),$$

which implies that

$$d(x_{n+1}, F) \leq d(x_n, F).$$

This means $\lim_{n\to\infty} d(x_n, F)$ exists. Therefore by the hypothesis of our theorem, $\liminf_{n\to\infty} d(x_n, F) = 0$. Thus, we have $\lim_{n\to\infty} d(x_n, F) = 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence in K. Let $m, n \in \mathbb{N}$ and suppose $m > n$. Then, it follows that $d(x_m, p) \leq d(x_n, p)$ for all $p \in F$. Hence, we get

$$d(x_m, x_n) \leq d(x_m, p) + d(x_n, p) \leq 2d(x_n, p).$$

Taking inf on the set F, we have $d(x_m, x_n) \leq d(x_n, F)$. On letting $m, n \to \infty$ in the inequality $d(x_m, x_n) \leq d(x_n, F)$, we have that it converges to a point $q \in K$. Next, we show that $q \in F$. Clearly $d(x_n, F(T)) = \inf_{x^* \in F(T)} d(x_n, x^*)$. So for each $\epsilon > 0$, there exists $p_n^{(\epsilon)} \in F(T)$ such that

$$d(x_n, p_n^{(\epsilon)}) < d(x_n, F(T)) + \frac{\epsilon}{3}.$$

This implies that $\lim_{n\to\infty} d(x_n, p_n^{(\epsilon)}) \leq \frac{\epsilon}{3}$. From $d(p_n^{(\epsilon)}, q) \leq d(x_n, p_n^{(\epsilon)}) + d(x_n, q)$, it follows that

$$\limsup_{n\to\infty} d(p_n^{(\epsilon)}, q) \leq \frac{\epsilon}{3}.$$

Hence, we obtain

$$\begin{aligned} d(T(q), q) &\leq d(q, p_n^{(\epsilon)}) + d(p_n^{(\epsilon)}, T(q)) \\ &\leq d(q, p_n^{(\epsilon)}) + H(T(p_n^{(\epsilon)}), T(q)) \\ &\leq 2d(p_n^{(\epsilon)}, q) \end{aligned}$$

which shows that $d(T(q), q) < \epsilon$. So $d(T(q), q) = 0$ since ϵ is arbitrary chosen. Similarly, we can show that $d(S(q), q) = 0$ and $d(R(q), q) = 0$. Since F is closed, then $q \in F$. This complete the proof.

We now give the definition of condition (I) of Senter Dotson [33] for three mappings and also the definition of semi-compactness.

Definition 3.8 *The multivalued mappings $S, R, T: K \to P(K)$, where K is a subset of X are said to satisfy condition (I) if there exists a nondcreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that*

$$\frac{1}{3}[d(x, Sx) + d(x, Tx) + d(x, Rx)] \geq f(d(x, F)) \quad \text{forall} \quad x \in K.$$

Definition 3.9 *A mapping $T: K \to P(K)$ is called semi-compact if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a convergent subsequence.*

We now state the following application of our above Theorem 3.7.

Theorem 3.10 *Let K be a nonempty, closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let R, S, T, P_R, P_S, P_T and F be as defined in Lemma 3.5. Suppose P_S, P_T and P_R satisfy condition (I), then the iterative process defined in (1.4) converges strongly to $p \in F$.*

Proof. For all $p \in F$, $\lim_{n\to\infty} d(x_n, p)$ exists. Let us put $\lim_{n\to\infty} d(x_n, p) = c$ for some $c \geq 0$. If $c = 0$, then the result follows directly. So suppose that $c > 0$. Now $d(x_{n+1}, p) \leq d(x_n, p)$ gives that

$$\inf_{p\in F(T)} d(x_{n+1}, p) \leq \inf_{p\in F(T)} d(x_n, p),$$

which means that $d(x_{n+1}, F) \leq d(x_n, F)$. Hence $\lim_{n\to\infty} d(x_n, F)$ exists. By using condition (I) and Lemma 3.5, we get

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \frac{1}{3}[d(x_n, P_S x_n) + d(x_n, P_T x_n) + d(x_n, P_R x_n)] = 0.$$

Thus

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

By the properties of f, we get that $\lim_{n\to\infty} d(x_n, F) = 0$. Using Theorem 3.7, we obtain the desired result.

The following can be obtain as corollaries of our result.

Corollary 3.11 *Let K be a nonempty closed and convex subset of a complete hyperbolic space X with a monotone modulus of uniform convexity η. Let $R, S, T: K \to P(K)$ be three multivalued mappings such that P_R, P_S and P_T be multivalued mappings satisfying condition (C) and $F := F(R) \cap F(S) \cap F(T) \neq \emptyset$. If $\{x_n\}$ is the sequence defined by (1.4), then $\{x_n\}$ Δ-converges to a point in F.*

Corollary 3.12 *Let K be a nonempty closed and convex subset of a complete hyperbolic space X with a monotone modulus of uniform convexity η. Let $R,S,T: K \to P(K)$ be three multivalued mappings such that P_R, P_S and P_T are α-nonexpansive mappings and $F := F(R) \cap F(S) \cap F(T) \neq \emptyset$. If $\{x_n\}$ is the sequence defined by (1.4), then $\{x_n\}$ Δ-converges to a point in F.*

Corollary 3.13 *Let K be a nonempty closed and convex subset of a complete hyperbolic space X with a monotone modulus of uniform convexity η. Let $T: K \to P(K)$ be a multivalued mapping such that P_T is generalized α-nonexpansive mappings and $F(T) \neq \emptyset$. Let $\{x_n\}$ be defined as*

$$\begin{cases} x_{n+1} = W\left(u_n, W\left(x_n, v_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), \\ y_n = W\left(v_n, W\left(w_n, x_n, \frac{c_n}{1-b_n}\right), b_n\right), \\ z_n = W(x_n, w_n, a_n), \end{cases} \tag{3.24}$$

where $u_n \in P_T(y_n)$, $v_n \in P_T(z_n)$, $w_n \in P_T(x_n)$ and $a_n, b_n, c_n, \alpha_n, \beta_n \in (0,1)$ such that $0 < \alpha_n + \beta_n < 1$ and $0 < b_n + c_n < 1$. then $\{x_n\}$ Δ-converges to a point in $F(T)$.

4. NUMERICAL EXAMPLE

In this section, we present a numerical example to show that our proposed algorithm (1.4) converges faster than Ishikawa Iteration and SP iteration.

Let $(X,d) = \mathbb{R}$ with $d(x,y) = |x-y|$ and $K = [0,3]$. Denote by

$$W(x,y,\alpha) := \alpha x + (1-\alpha)y, \quad \forall\ x,y \in X \quad \text{and} \quad \alpha \in [0,1],$$

then (X,d,W) is a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity and K is a nonempty closed and convex subset of X. Let $R,S,T: K \to CB(K)$ be defined by

$$Tx = \begin{cases} [0, \frac{x}{6}], & \text{if} \quad x \neq 3, \\ \{1\}, & \text{if} \quad x = 3, \end{cases}$$

$$Sx = \begin{cases} [1, \frac{3}{2}], & \text{if} \quad x = 3, \\ \{0\}, & \text{if} \quad x \neq 3, \end{cases}$$

and

$$Rx = \begin{cases} [0, \frac{x}{2}] & \text{if} \quad x \neq 2, \\ \{1\}, & \text{if} \quad x = 2. \end{cases}$$

It is easy to prove that R,S,T are generalized α-nonexpansive for $\alpha \in (0,1)$ and the convex value $0 \in K$ which is the unique fixed point in K and $F(T) \cap F(S) \cap F(R) = \{0\}$. Let $\{a_n\}$ be a constant sequence such that $a_n = \frac{1}{2}$ and $b_n = \frac{3n+1}{4n+5}, c_n = \frac{1}{4n+5}, \alpha_n = \frac{1}{2n}$ and $\beta_n = \frac{1}{4n}$ for all $n \geq 0$. Then algorithm (1.4) becomes:

$$\begin{cases} z_n = \frac{1}{2}(x_n + w_n), \\ y_n = \frac{3n+1}{4n+5} v_n + \left(1 - \frac{3n+1}{4n+5}\right)\left(\frac{w_n}{n+4} + \frac{n+3}{n+4} x_n\right), \\ x_{n+1} = \frac{1}{2n} u_n + \frac{1}{2}\left(1 - \frac{1}{2n}\right)\left(\frac{x_n}{2n-1} + \frac{4n-3}{2n-1} v_n\right). \end{cases}$$

We make different choices of x_0 with stopping criterion $\frac{||x_{n+1}-x_n||}{||x_2-x_1||} < 10^{-4}$. Using Mathlab version 2016(b), we plot the graph of x_{n+1} against the number of iteration for algorithm 1.4 and modified SP-iteration (1.3) using the following initial values. Case 1: Choose $x_0 = 0.5$, Case 2: Choose $x_0 = 1$, Case 3: Choose $x_0 = 2.25$. Case 4: Choose $x_0 = 3$.

See Figure 1, Figure 2, Figure 3 and Figure 4 for the graphs. We deduce from this example that algorithm 1.4 performs better than the modified SP-iteration (1.3) in terms of number of iterations and cpu time taken for computation.

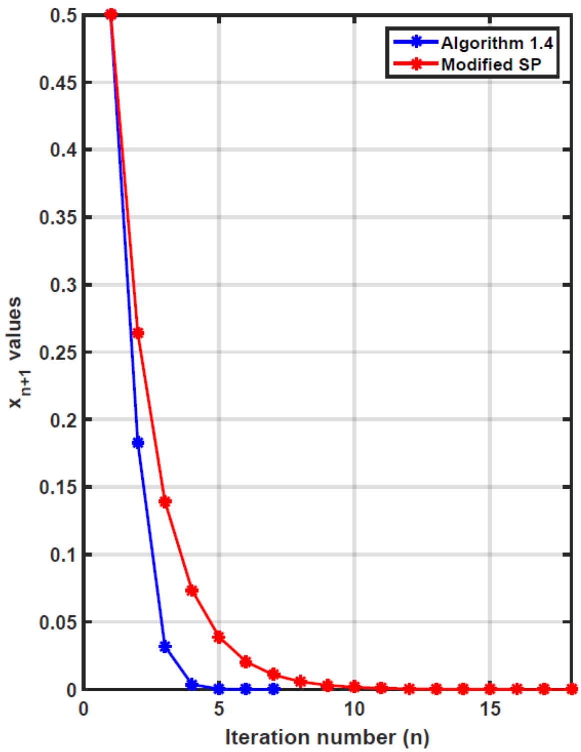

Figure 1. Case 1, $x_1 = 0.5$ (cpu time: Algorithm (1.4): 0.0014 sec, Modified SP: 0.0028sec).

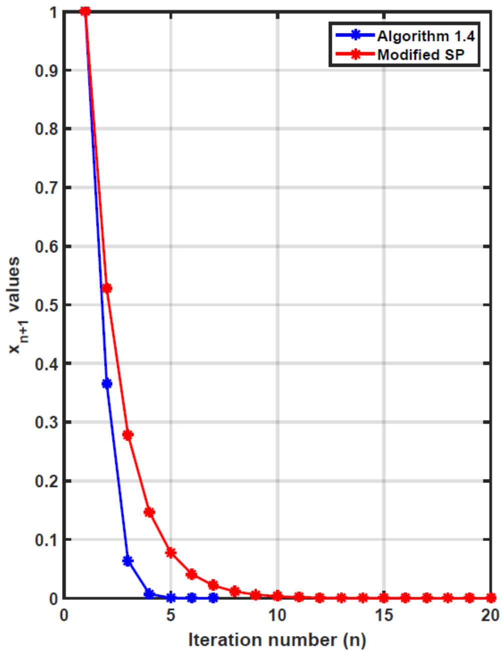

Figure 2. Case 2, $x_1 = 1$ (cpu time: Algorithm (1.4): 0.0035 sec, Modified SP: 0.0101sec).

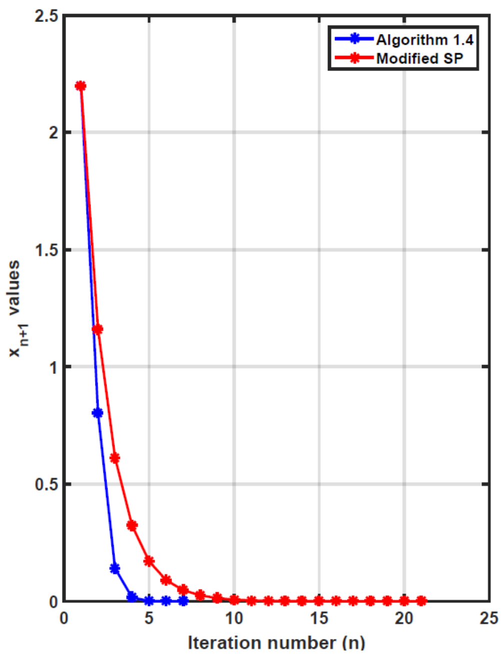

Figure 3. Case 3, $x_1 = 2.25$ (cpu time: Algorithm (1.4): 0.0015 sec, Modified SP: 0.0132sec).

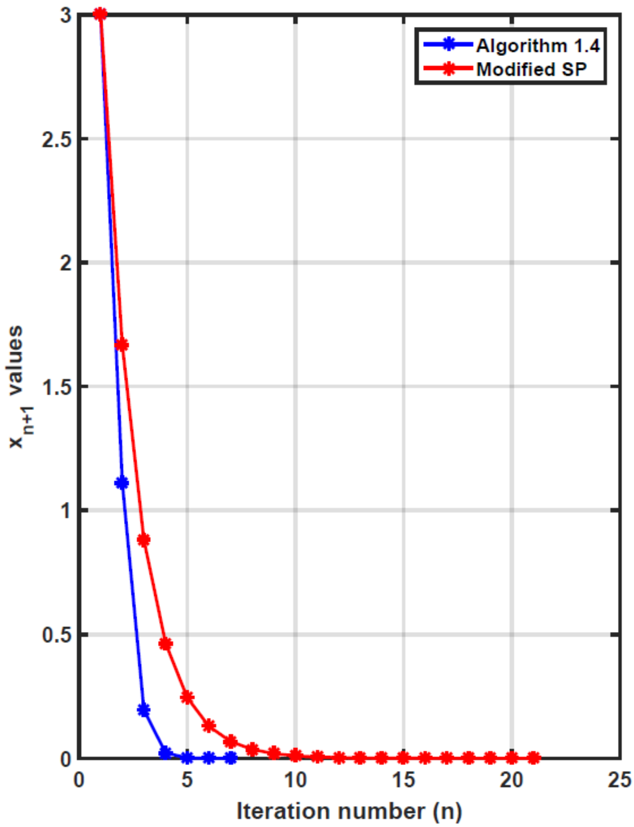

Figure 4. Case 4, $x_1 = 3$ (cpu time: Algorithm (1.4): 0.0019 sec, Modified SP: 0.0111sec).

REFERENCES

[1] Abbas M., Khan S. H., Khan A. R., Agarwal R. P., (2011) Common fixed points of two multivalued nonexpansive mappings by one-step iterative scheme, *Appl. Math. Lett.*, 24, 97-102.

[2] Aoyama K. and Kohsaka F., (2011) Fixed point theorem for α-nonexpansive mappings in Banach spaces, *Nonlinear Anal.*, 74, 4387-4391.

[3] Bauschke H. H. and Combettes P. L., (2011)Convex analysis and monotone operator theory in Hilbert spaces, *ser. CMS Books in Mathematics*, Berlin, Germany.

[4] Browder F. E., (1965) Fixed-point theorems for noncompact mappings in Hilbert space, *Proc. Natl. Acad. Sci.* 53, 1272-1276.

[5] Browder F. E., (1965) Nonexpansive nonlinear operators in Banach spaces, *Proc. Natl. Acad. Sci.*, 54, 1041-1044.

[6] Chang S. S., Wang G., Wang L., Tang Y. K. and Ma Z. L., (2014) Δ-convergence theorems for multi-valued nonexpansive mappings in hyperbolic spaces, *Appl. Math. Comp.*, 249, 535-540.

[7] Glowinski R. and Le Tallec P., (1989) Augmented Lagrangian and operator-splitting methods in non linear mechanics, 9, SIAM.

[8] Goebel K. and Kirk W. A., (1983) Iteration processes for nonexpansive mappings, In Topological Methods in Nonlinear Functional Analysis, S. P. Singh, S. Thomeier, and B.Watson, Eds., vol. 21 of Contemporary Mathematics, 115-123, American Mathematical Society, Providence, RI, USA.

[9] Goebel K. and Reich S., (1984) Uniform convexity, Hyperbolic Geometry and Nonexpansive mappings, *Marcel Dekket*, New York.

[10] Goebel K. and Kirk W. A., (1990) Topics in Metric Fixed Point Theory, *Cambridge University Press*, Cambridge, England.

[11] Göhde D., (1965) Zum Prinzip def Kontraktiven Abbilding, *Math. Nachr.*, 30, 251-258.

[12] Gunduz B. and Karahan I., (2018) Convergence of SP iterative scheme for three multivalued mappings in hyperbolic space, *J. Comput. Analy. Appl.*, 24, 815-827.

[13] [aubruge S., Nguyen V. H. and Strodiot J., (1998) Convergence analysis and applications of the glowinski-le tallec splitting method for finding a zero of the sum of two maximal monotone operators, *J. Optim. Theory Appl.*, 97 (3), 645-673.

[14] Ishikawa S., (1974) Fixed points by a new iteration method, *Proc. Amer. Math. Soc.* 44, 147- 150.

[15] [han S H., Abbas M., Rhoades B. E., (2010) A new one-step iterative scheme for approximating common fixed points of two multivalued nonexpansive mappings, *Rend del Circ Mat*, 59, 149-157.

[16] Khan A. R., Fukhar-ud-din H. and Khan M. A. A., (2012) An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces, *Fixed Point Theory Appl.* 2012, 54.

[17] Khan A. R., Khamsi M. A. and Fukhar-ud-din H., (2011) Strong convergence of a general iteration scheme in CAT(0) spaces, *Nonlinear Anal*, 74, 783-791.

[18] Kirk W. A., (1965) A fixed point theorem for mappings which do not increase distances, *Am. Math. Monthly*, 72, 1004-1006.

[19] Kirk W. A. and Panyanak B., (2008) A concept of convergence in geodesic spaces, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 68, 12, 3689-3696.

[20] Kohlenbach U., (2005) Some logical metathorems with applications in functional analysis, *Trans. Amer. Math. Soc.*, 357 (1), 89-128.

[21] Kuczumow T., (1978) An almost convergence and its applications, *Ann. Univ. Mariae Curie-SkÅ ,odowska, Sect. A*, 32, 79-88.

[22] Leustean L., (2010) Nonexpansive iteration in uniformly convex W-hyperbolic spaces, In A. Leizarowitz, B.S. Mordukhovich, I. Shafrir, A. Zaslavski, Nonlinear Analysis and Optimization I. Nonlinear analysis Contemporary Mathematics. Providence. RI Ramat Gan American Mathematical Soc. Bar Ilan University, 513, 193-210.

[23] Mann W.R., (1953) Mean value methods in iteration, *Proc. Amer. Math. Soc.*, 4 , 506-510.

[24] Markin J. T., (1973) Continuous dependence of fixed point sets, *Proc. Amer. Math. Soc.*, 38, 545-547.

[25] Mebawondu A. A., Jolaoso L. O. and Abass H. A., (2017) On Some Fixed Points Properties and Convergence Theorems for a Banach Operator in Hyperbolic Spaces, *Inter J Nonlinear Analy Appl*, 8(2), 293-306.

[26] Nadler S. B. , Jr., (1969) Multivalued contraction mappings, *Pacific J. Math.*, 30, 475-488.

[27] Noor M. A., (2000) New approximation schemes for general variational inequalities, *J. Math. Anal. Appl.* 251, 217-229.

[28] Pant R. and Shukla R., (2017) Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces, *Numer. Funct. Anal. Optim.*, 38(2), 248-266.

[29] Panyanak B., (2007) Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces, *Comp Math Appl.*, 54, 872-877.

[30] Phuengrattana W. and Suantai S., (2011) On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval, *J. Comput. Appl. Math.*, 235, 3006-3014.
[31] Reich S. and Shafrir I., (1990) Nonexpansive iterations in hyperbolic spaces, *Nonlinear Anal.* 15, 537-558.
[32] Sastry K. P. R. and Babu G. V. R., (2005) Convergence of Ishikawa iterates for a multivalued mapping with a fixed point, *Czechoslovak Math J.*, 55, 817-826.
[33] Senter H.F. and Dotson W.G., (1974) Approximating fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc*., 44(2), 375-380.
[34] Shahzad N. and Zegeye H., (2009) On Mann and Ishikawa iteration schemes for multivalued maps in Banach space, *Nonlinear Anal*. 71, 838-844.
[35] Shimizu T. and Takahashi W., (1996) Fixed points of multivalued mappings in certain convex metric spaces, *Topol. Methods Nonlinear Anal* 8,197-203.
[36] Song Y. and Cho Y. J., (2011) Some notes on Ishikawa iteration for multivalued mappings, *Bull. Korean Math Soc.*, 48, 575-584.
[37] Song Y., Wang H., (2008) Erratum to Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces, *Comp Math Appl* **55**, , 2999-3002.
[38] Song Y. and Wang H., (2009) Convergence of iterative algorithms for multivalued mappings in Banach spaces, *Nonlinear Anal.*, 70, 1547-1556.
[39] Suanoom C. and Klin-eam C., (2016) Remark on fundamentally nonexpansive mappings in hyperbolic spaces, *Bull. Austral J. Nonlinear Sci. Appl.*, 9, 1952-1956.
[40] Suzuki T., (2008) Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, *J. Math. Anal. Appl.*, 340, 1088-1095.
[41] Takahashi W. A., (1970) A convexity in metric space and nonexpansive mappings, *I. Kodai Math. Sem. Rep.*, 22, 142-149.