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Research Article

ON LIE GROUP ANALYSIS OF BOUNDARY VALUE PROBLEM WITH CAPUTO FRACTIONAL DERIVATIVE

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ABSTRACT

Lie symmetry analysis of initial and boundary value problem for partial differential equations with Caputo fractional derivative is investigated. Also given generalized definition and theorem for symmetry method for partial differential equation with Caputo fractional derivative. The group symmetries and examples on reduction of fractional partial differential equations with initial and boundary conditions to nonlinear ordinary differential equations with initial condition are present.
Keywords: Lie group method, Caputo type fractional derivative, boundary value problem.

1. INTRODUCTION

The investigated by Sophus Lie (1842–1899) method to find Lie point symmetry method has been widely used and described in many books and research articles (see [1–3], and references therein). Lie group method is a powerful and direct approach to construct exact solutions of nonlinear partial differential equation (PDE), by analyzing the symmetries of the nonlinear PDE. In addition, based on Lie symmetry method, many other types of exact solutions of PDE can be obtained, such as traveling wave solutions, soliton solutions, power series solutions, and so on [4–8]. After Sophus Lie, the symmetry method was investigated by Ovsyannikov, Olver and many other big minds studied it by contributing to the theory of symmetries. Bluman investigated the symmetry analysis of initial and boundary value problems (IBVPs) for PDE, by given the properties of boundary conditions and boundary surfaces under symmetries (see [9, 10]). And we in our work by using Lie symmetry method and its analysis search symmetries and obtain solutions for IBVPs for PDE, containing Caputo fractional derivative.

So, in this work we focus on IBVP for fractional partial differential equation (FPDE)

$$\begin{cases} cD_t^{\alpha}u(t,x) = (A(u)u_x)_{xx}, \\ u(t,0) = f(t), \qquad t \in \mathbb{R}^+, \\ u(0,x) = g(x), \qquad x \in \mathbb{R}^+, \end{cases}$$

where $u(t,x)$ is a function on $\mathbb{R}^+ \times \mathbb{R}^+$, $0 < \alpha < 1$ with fractional derivative in the sense of Caputo [11] in a form

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$$cD_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & if \ \ n-1 < \alpha < n, \ \ n \in \mathbb{N}, \\ \frac{d^n}{dt^n} f(t), & if \ \ \alpha = n \,. \end{cases}$$

The FPDE with Caputo derivative is more useful in searching the solution of boundary value problems, th an Riemann–Liouville derivative, which has a form

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, & if \ \ n-1 < \alpha < n, \ \ n \in \mathbb{N}, \\ \frac{d^n}{dt^n} f(t), & if \ \ \alpha = n \,. \end{cases}$$

Also we have next relationship between these fractional derivatives

$$D_t^{\alpha} f(t) = cD_t^{\alpha} f(t) + \sum_{i=0}^{n-1} \frac{t^{i-\alpha}}{\Gamma(i-\alpha+1)} f^i(0^+).$$

For zero initial conditions, the two derivatives are the same. This property allows us switch between the two derivatives according to our necessity. While $\alpha > n$ the Caputo derivative becomes a conventional n-th derivative of the function $f(t)$. Thus, we can assume that Caputo derivative is more handy since the initial value for fractional differential equation with Caputo derivative is the same as the initial value for integer PDE [12]. So, we consider the boundary value problems with Caputo fractional derivative.

2. SYMMETRY ANALYSIS FOR CAPUTO TIME–FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

Let

$$F(t, x, u, cD_t^{\alpha} u, \partial_x u, \partial_x^2 u, \dots, \partial_x^s u) = 0, \ \ 0 < \alpha \le 1, \tag{1}$$

be a FPDE with two independent variables $x \in \mathbb{R}$ and $t > 0$, where $cD_t^{\alpha} u$ is Caputo time-fractional derivative of u and $\partial_x^i u = \frac{d^i u}{dx^t}$, $i = 1, \dots, s$.

Lie symmetry transformation acting on a space of two independent variables (t, x) and dependent variable u is determined as

$$\begin{aligned} \bar{t} &= t + \varepsilon \tau(t, x, u) + O(\varepsilon^2), \\ \bar{x} &= x + \varepsilon \xi(t, x, u) + O(\varepsilon^2), \\ \bar{u} &= u + \varepsilon \eta(t, x, u) + O(\varepsilon^2), \end{aligned} \tag{2}$$

where $\varepsilon > 0$ is a group parameter and τ, ξ, η , are the infinitesimals of the transformation. According to above transformation an infinitesimal generator has a following form

$$X = \xi(t, x, u) \frac{\partial}{\partial x} + \tau(t, x, u) \frac{\partial}{\partial t} + \eta(t, x, u) \frac{\partial}{\partial u}. \tag{3}$$

Definition 2.1. *$u = \theta(t, x)$ is an invariant solution of the equation (1) obtaining from the invariance of the equation (1) under the symmetry (2) with infinitesimal generator (3) if only if (see [3])*

- *$u = \theta(t, x)$ is an invariance surface of X,*
- *$u = \theta(t, x)$ solves the equation (1).*

In other words, Lie point symmetries for (1) are given by the vector field (3) with Lie transformation (2) if only if an invariant solution $u = \theta(t, x)$ satisfies:

- $X\big(u - \theta(t, x)\big) = 0$ for $u = \theta(t, x)$, which gives us $\xi\big(t, x, \theta(t, x)\big) \frac{\partial}{\partial x} + \tau\big(t, x, \theta(t, x)\big) \frac{\partial}{\partial t} = \eta\big(t, x, \theta(t, x)\big)$,
- $F\big(t, x, \theta(t, x), cD_t^{\alpha} \theta(t, x), \partial_x \theta(t, x), \partial_x^2 \theta(t, x), \dots, \partial_x^s \theta(t, x)\big) = 0$, for $u = \theta(t, x)$.

According to the infinitesimal transformation (2) a fractional prolongation $pr^{(\alpha,n)}X$ of the equation (1)

$$pr^{(\alpha,s)}X(E)\big|_{E=0} = 0, \;\; where\ E = F(t,x,u,cD_t^\alpha u, \partial_x u, \partial_x^2 u, \dots, \partial_x^s u),$$

here

$$\begin{aligned}
\eta_\alpha^t &= cD_t^\alpha(\eta) + \xi cD_t^\alpha(u_x) - cD_t^\alpha(\xi u_x) + \tau cD_t^\alpha(u_t) - cD_t^\alpha(\tau u_t),\\
\eta_1^x &= D_x\eta - u_x D_x\xi - u_t D_x\tau,\\
\eta_2^x &= D_x\eta_1^x - u_{xx}D_x\xi - u_{xt}D_x\tau,\\
&\vdots\\
\eta_3^x &= D_x\eta_{s-1}^x - \partial_x^s u D_x\xi - \partial_x^{s-1}u_t D_x\tau,
\end{aligned} \tag{4}$$

with total derivative D_i in a form

$$D_i = \partial_i + u_i\partial_u + u_{it}\partial_{u_t} + u_{ji}\partial_{u_t} + u_{ii}\partial_{u_i} + u_{jj}\partial_{u_j} + \cdots.$$

The expressions for η_i^x, $i = 1, \dots, s$ in (4) can be easily obtained (see [3]), here we focus our attention on η_α^t [3,13].

The fractional integral can be defined as

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau.$$

So, we can see the below representations of Caputo and Riemann–Liouville fractional derivatives (see [11,14])

$$cD_t^\alpha f(t) = I_t^{n-\alpha}D^n f(t), \tag{5}$$

$$D_t^\alpha f(t) = D^n I_t^{n-\alpha} f(t). \tag{6}$$

According to the generalized Leibnitz rule in [15]

$$D_t^\alpha(f(t), g(t)) = \sum_{n=0}^{\infty}\binom{\alpha}{n} I_t^{n-\alpha}f(t)D_t^n g(t), \quad \binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)},$$

we have

$$\xi cD_t^\alpha(u_x) - cD_t^\alpha(\xi u_x) = -\sum_{n=1}^{\infty}\binom{\alpha}{n} I_t^{n-\alpha}(u_x)D_t^n(\xi) + \sum_{i=0}^{n-1}\frac{(\xi u_x)^{(i)}(0)}{\Gamma(i-\alpha+1)}t^{i-\alpha},$$

and

$$\tau cD_t^\alpha(u_t) - cD_t^\alpha(\tau u_t) = -\sum_{n=1}^{\infty}\binom{\alpha}{n} I_t^{n-\alpha}(u_t)D_t^n(\tau) + \sum_{i=0}^{n-1}\frac{(\tau u_t)^{(i)}(0)}{\Gamma(i-\alpha+1)}t^{i-\alpha}.$$

Thus, we get the expression

$$\begin{aligned}
\eta_\alpha^t = D_t^\alpha(\eta) + \eta_0 - \sum_{n=1}^{\infty}\binom{\alpha}{n} I_t^{n-\alpha}(u_x)D_t^n(\xi) + \sum_{i=0}^{n-1}\frac{(\xi u_x)^{(i)}(0)}{\Gamma(i-\alpha+1)}t^{i-\alpha}\\
-\sum_{n=1}^{\infty}\binom{\alpha}{n+1} I_t^{n-\alpha}(u)D_t^{n+1}(\tau) - \alpha D_t(\tau)D_t^\alpha(u_t).
\end{aligned}$$

Here by using a generalized form of the chain rule (see [16])

$$\frac{d^m f(g(t))}{dt^m} = \sum_{k=0}^{m}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}\big(-g(t)\big)^r\frac{d^m}{dt^m}\big(g(t)^{k-r}\big)\frac{d^k}{dg^k}f(g),$$

which for fractional m takes a form

$$\frac{d^\alpha f(g(t))}{dt^\alpha}=\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{l}\binom{l}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{l-\alpha}}{\Gamma(l-\alpha+1)}\left(-g(t)\right)^r\frac{d^m}{dt^m}\left(g(t)^{k-r}\right)\frac{d^{l-m+k}}{dt^{l-m}dg^k}f(g),$$

and so we have

$$\eta_0=\sum_{j=0}^{n-1}\frac{(\eta)^{(i)}(0)}{\Gamma(j-\alpha+1)}t^{n-\alpha}$$
$$=\sum_{j=0}^{n-1}\frac{t^{n-\alpha}}{\Gamma(j-\alpha+1)}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{l}\binom{l}{m}\binom{k}{r}\frac{1}{k!}\left(\frac{t^{l-\alpha}}{\Gamma(l-\alpha+1)}(-u)^r\frac{d^j}{dt^j}(u^{k-r})\frac{d^{l-j+k}\eta}{dt^{l-j}du^k}\right)(0).$$

And so, the infinitesimal η_α^t takes a form

$$\eta_\alpha^t=\frac{c\partial^\alpha\eta}{\partial t^\alpha}+\left(\eta_u-\alpha(\tau_t+\tau_u u_t)\right)\frac{c\partial^\alpha u}{\partial t^\alpha}-u\frac{c\partial^\alpha\eta_u}{\partial t^\alpha}+\mu-\sum_{n=1}^{\infty}\binom{\alpha}{n}I_t^{n-\alpha}(u_x)D_t^n(\xi) \tag{7}$$

where

$$\mu=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}(-u)^r}{\Gamma(n-\alpha+1)}\frac{d^m}{dt^m}(u^{k-r})\frac{d^{n-m+k}\eta}{dt^{n-m}du^k}. \tag{8}$$

Here we can proof the next lemma.

Lemma 2.1. *If (1) is invariant under infinitesimal transformation (2) with infinitesimal generator (3) and the equation (1) has no the second and higher order derivative of u with respect to t, then* $\eta=A(t,x)u+B(t,x)$ *with A(t,x) and B(t,x) arbitrary functions.*

Proof. By expanding the $cD_t^\alpha(\eta)$ we get the expression (8). As the equation has not any variations of second and high order derivative u with respect to t, then for $k=2$ we have

$$\mu=\frac{1}{2!}\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}\eta_{uu}u_{tt},$$

and $\eta_{uu}=0$. Here by integration we get $\eta=A(t,x)u+B(t,x)$.
Q.E.D.

3. LIE SYMMETRY ANALYSIS FOR CAPUTO TIME-FRACTIONAL INITIAL AND BOUNDARY VALUE PROBLEMS

Here we study Lie symmetry analysis for Caputo fractional initial and boundary value problems. It is known that the PDE can describe real processes according nature and society if there are given initial and boundary conditions for the PDE. In case of FPDE the advantage of the Caputo fractional derivative is that it enables standard initial and boundary conditions for differential equations. And although Lie symmetry analysis is one of the most widely-applicable methods of finding exact solutions of differential equations, but it was not widely used for solving boundary value problems. The reason is the initial and boundary conditions usually are not invariant under any obtained Lie symmetries [9,10]. Thus we can say that an invariant solution for PDE resulting by applying Lie symmetry method solves a given boundary value problem, when the symmetry transformation leaves invariant all boundary conditions and the domain of the boundary value problem [9].

In [9] Bluman gives a definition of Lie symmetry invariance for initial and boundary by means of that there are some classes initial and boundary problems which can be solved. Bluman studies the PDE with finite values of x and t boundary and initial conditions. Later Cherniha extends the definitions of Bluman for the PDE with infinite values of x and t boundary and initial conditions [17,18]. In this work we investigate the symmetry analysis of initial and boundary value problems for fractional nonlinear PDE with Caputo time-fractional derivative.

Let Lie symmetry infinitesimal generator X

$$X = \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{9}$$

is admitted by the boundary value problem defined on a domain Ω:

$$u_t = f\left(t,x,u,\frac{\partial u}{\partial x},\dots,\frac{d^k u}{dx^k}\right), \quad (t,x) \in \Omega \subset \mathbb{R}^2, \tag{10}$$

$$d_a(t,x) = 0: B_a\left(t,x,u,\frac{\partial u}{\partial x},\dots,\frac{d^{k-1} u}{dx^{k-1}}\right) = 0, \quad a = 1,\dots,p. \tag{11}$$

Here $B_a\left(t,x,u,\frac{\partial u}{\partial x},\dots,\frac{d^{k-1} u}{dx^{k-1}}\right) = 0$ boundary condition on $d_a(t,x) = 0$. Suppose that the above boundary value problem has a unique solution.

Definition 3.1. *[9] The symmetry X which has the form (9) is allowed by the boundary value problem (10)-(11) if:*

- $X^{(k)}\left(u_t - f\left(t,x,u,\frac{\partial u}{\partial x},\dots,\frac{d^k u}{dx^k}\right)\right) = 0$ *for* $u_t = f\left(t,x,u,\frac{\partial u}{\partial x},\dots,\frac{d^k u}{dx^k}\right)$;
- $Xd_a(t,x) = 0$ *for* $d_a(t,x) = 0$, *a=1,…,p;*
- $X^{(k-1)}B_a\left(t,x,u,\frac{\partial u}{\partial x},\dots,\frac{d^{k-1} u}{dx^{k-1}}\right) = 0$ *for* $B_a\left(t,x,u,\frac{\partial u}{\partial x},\dots,\frac{d^{k-1} u}{dx^{k-1}}\right) = 0$ *on* $d_a(t,x) = 0$.

Further we extend the Blusman's definition for FPDE with Caputo derivative. Let us consider the boundary value problem for FPDE defined on a domain Ω

$$cD_t^{\alpha} u = g\left(t,x,u,\frac{\partial u}{\partial x},\dots,\frac{d^k u}{dx^k}\right), \quad (t,x) \in \Omega \subset \mathbb{R}^2, \tag{12}$$

$$cd_a(t,x) = 0: cB_a\left(t,x,u,\frac{\partial u}{\partial x},\dots,\frac{d^{k-1} u}{dx^{k-1}}\right) = 0, \quad a = 1,\dots,p. \tag{13}$$

Here $cB_a\left(t,x,u,\frac{\partial u}{\partial x},\dots,\frac{d^{k-1} u}{dx^{k-1}}\right) = 0$ is a boundary condition on $cd_a(t,x) = 0$. Suppose the boundary value problem (12)-(13) has a unique solution. Then we can give next definition.

Definition 3.2. *The symmetry X which has the form (9) is allowed by the boundary value problem (12)-(13) if:*

- $X^{(k)}\left(cD_t^{\alpha} u - g\left(t,x,u,\frac{\partial u}{\partial x},\dots,\frac{d^k u}{dx^k}\right)\right) = 0$ *for* $u_t = f\left(t,x,u,\frac{\partial u}{\partial x},\dots,\frac{d^k u}{dx^k}\right)$;
- $Xd_a(t,x) = 0$ *for* $cd_a(t,x) = 0$, *a=1,…,p;*
- $X^{(k-1)}cB_a\left(t,x,u,\frac{\partial u}{\partial x},\dots,\frac{d^{k-1} u}{dx^{k-1}}\right) = 0$ *for* $cB_a\left(t,x,u,\frac{\partial u}{\partial x},\dots,\frac{d^{k-1} u}{dx^{k-1}}\right) = 0$ *on* $cd_a(t,x) =$

0.

Theorem 3.1. *The solution* $u = \upsilon(t,x)$ *for (12) is invariant if only if for infinitesimal generator X the curve (mapping)* $\upsilon(t,x)$ *admits:*

$$\eta\big(t,x,\upsilon(t,x)\big) - \xi\big(t,x,\upsilon(t,x)\big) - \tau\big(t,x,\upsilon(t,x)\big) cD_t^{1-\alpha} g\left(t,x,\upsilon(t,x),\frac{\partial \upsilon(t,x)}{\partial x},\dots,\frac{d^k \upsilon(t,x)}{dx^k}\right) = 0. \tag{14}$$

Proof. As the solution surface $u = \upsilon(t,x)$ is invariant for (12) if only if $X\big(u - \upsilon(t,x)\big) = 0$ which gives

$$0 = X\big(u - \upsilon(t,x)\big) = \left(\xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \eta(t,x,u)\frac{\partial}{\partial u}\right)\big(u - \upsilon(t,x)\big)$$
$$= \eta(t,x,u)\frac{\partial}{\partial u}u - \xi(t,x,u)\frac{\partial}{\partial x}\upsilon(t,x)$$
$$-\tau(t,x,u)\frac{\partial}{\partial t}\upsilon(t,x) = \eta(t,x,u) - \xi(t,x,u)\frac{\partial}{\partial x}\upsilon(t,x) - \tau(t,x,u)\frac{\partial}{\partial t}\upsilon(t,x).$$

Further, as $u = \upsilon(t,x)$, then $u_t = \frac{\partial}{\partial t}\upsilon(t,x)$. From the property of Caputo derivative (5) $u_t = cD_t^{1-\alpha} g\left(t, x, u, \frac{\partial u}{\partial x}, \dots, \frac{d^k u}{dx^k}\right)$, i.e

$$\eta(t,x,u) - \xi(t,x,u)\frac{\partial}{\partial x}\upsilon(t,x) - \tau(t,x,u) cD_t^{1-\alpha} g\left(t, x, \upsilon(t,x), \frac{\partial \upsilon(t,x)}{\partial x}, \dots, \frac{d^k \upsilon(t,x)}{dx^k}\right).$$

Q.E.D.

4. SYMMETRY ANALYSIS FOR HARRY-DYM CAPUTO TIME–FRACTIONAL INITIAL AND BOUNDARY VALUE PROBLEMS

Here we will study the symmetries of initial and boundary value problems with Caputo fractional derivatives.

Let $0 < \alpha \leq 1$

$$cD_t^\alpha u(t,x) = (A(u)u_x)_{xx}, \tag{15}$$

with initial and boundary conditions

$$\begin{cases} u(t,0) = f(t), & for\ \ t > 0, \\ u(0,x) = g(x), & for\ \ x > 0, \end{cases} \tag{16}$$

is IBVP for Caputo time-fractional diffusion equation.

By applying Lie symmetry method to equation (15) we obtain next cases

Case 1. For $A(u) = u^s$, $s \neq 1$ and , $s \neq 0$ we get

$$\xi = c_1 x + c_2,$$
$$\tau = \frac{3c_1}{\alpha(s+1)} t + c_3,$$
$$\eta = 3c_1 u,$$

which gives us an infinitesimal generator in form

$$X = (c_1 x + c_2)\frac{\partial}{\partial x} + \left(\frac{3c_1}{\alpha(s+1)} t + c_3\right)\frac{\partial}{\partial t} + (3c_1 u)\frac{\partial}{\partial u}, \tag{17}$$

here each c_i, $i = 1,2,3$ represents a symmetry for equation (15) with $A(u) = u^s$. The definition of invariance of the initial and boundary problems allows us make the following argument. The invariance of $t = 0$ leads to $c_3 = 0$ and the invariance of $x = 0$ leads to $c_2 = 0$. And

$$\frac{3c_1}{\alpha(s+1)} t \frac{\partial}{\partial t} f(t) = 3c_1 f(t),$$

the solution to this equation is $f(t) = t^{\alpha(s+1)} k_1$, here k_1 is an arbitrary constant.

And for $t = 0$ we get

$$c_1 x \frac{\partial}{\partial x} g(x) = 3c_1 g(x),$$

or $g(x) = x^3 k_2$ here k_2 is an arbitrary constant.

That means the initial and boundary problem for equation (15) with $A(u) = u^s$ is invariant according to infinitesimal generator (17) if the boundary and initial conditions have above form and there is an infinitesimal operator

$$X_1 = x\frac{\partial}{\partial x} + \frac{3c_1}{\alpha(s+1)} t \frac{\partial}{\partial t} + 3u\frac{\partial}{\partial u}.$$

Case 2. If $A(u) = u^{-1}$ we have the infinitesimal generator in a form

$$X = c_1 \frac{\partial}{\partial x} + (c_2 t + c_3)\frac{\partial}{\partial t} + \alpha c_2 u \frac{\partial}{\partial u}. \tag{18}$$

After the same operation as in case 1 we have $c_1 = c_3 = 0$. For $t = 0$ we get $0 = \alpha c_2 g(x)$, which means the initial condition has a form $u(0, x) = 0$. For $x = 0$, $f(t)$ is a solution of $c_2 t \frac{\partial}{\partial t} f(t) = \alpha c_2 f(t)$ in a form $f(t) = t^\alpha k_3$ with k_3 an arbitrary constant. Thus, our IBVP (15)-(16) with $A(u) = u^{-1}$ is invariant according to infinitesimal generator (18) if the boundary and initial conditions have above form and there is an infinitesimal operator

$$X_2 = t\frac{\partial}{\partial t} + \alpha u \frac{\partial}{\partial u}.$$

The infinitesimal operator X_2 gives us a transformation $u = t^\alpha \varphi(x)$. After applying this transformation to our equation (15) we get

$$\varphi(x)\Gamma(\alpha+1) = 2\varphi(x)^{-3}\varphi'(x)^3 - 3\varphi(x)^{-3}\varphi'(x)\varphi''(x) + \varphi(x)^{-1}\varphi'''(x).$$

The initial condition $u(0, x) = 0$ keeps $u = t^\alpha \varphi(x)$, and the boundary condition $u(t, 0) = t^\alpha k_3$ gives us $\varphi(0) = k_3$

Therefore, the IBVP (15)-(16) have a reduced Cauchy problem

$$\begin{cases} \varphi(x)\Gamma(\alpha+1) = 2\varphi(x)^{-3}\varphi'(x)^3 - 3\varphi(x)^{-3}\varphi'(x)\varphi''(x) + \varphi(x)^{-1}\varphi'''(x), \\ \varphi(0) = k_3. \end{cases}$$

Case 3. In case if $A(u) = 1$, then we obtain an infinitesimal generator

$$X = (\alpha c_1 x + c_2)\frac{\partial}{\partial x} + (3c_1 t + c_3)\frac{\partial}{\partial t} + \left(\left(\alpha + \frac{3}{2(1-\alpha)}\right)c_1 u + c_4\right)\frac{\partial}{\partial u}. \quad (19)$$

The invariance of $x = 0$ and $t = 0$ gives us $c_2 = c_3 = 0$. For $t = 0$ we have an equation

$$\alpha c_1 x \frac{\partial}{\partial x} g(x) = \left(\alpha + \frac{3}{2(1-\alpha)}\right)c_1 g(x) + c_4.$$

Suppose $c_4 = 2c_1$, then the equation has a solution $g(x) = -\frac{4(1-\alpha)}{4\alpha(1-\alpha)+3} + x^{2+\frac{3}{2\alpha(1-\alpha)}} k_4$, k_4 is a constant. And for $x = 0$ we have

$$3c_1 t \frac{\partial}{\partial t} f(t) = \left(\alpha + \frac{3}{2(1-\alpha)}\right)c_1 f(t) + c_4,$$

with solution $f(t) = -\frac{4(1-\alpha)}{4\alpha(1-\alpha)+3} + t^{\frac{2\alpha(1-\alpha)+3}{6(1-\alpha)}} k_5$, k_5 is a constant. In a like manner, our IBVP (15)-(16) with $A(u) = 1$ is invariant under (19) if the boundary and initial conditions have above form and there is an infinitesimal operator

$$X_3 = \alpha x \frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} + \left(\left(\alpha + \frac{3}{2(1-\alpha)}\right)u + 2\right)\frac{\partial}{\partial u}.$$

5. CONCLUSION

In this work, we present the applications of Lie group analysis to study the initial and boundary value problems for time-fractional nonlinear PDE given by Caputo sense. We give the definition of invariance of the initial and boundary value problems for time-fractional PDE and proved some theorems. By using Lie symmetry method and its analysis for the initial and boundary value problems with time-fractional PDE we obtained reduced invariant IBVP.

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