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Research Article

THE NEW SUMUDU TRANSFORM ITERATIVE METHOD FOR STUDYING THE RANDOM COMPONENT TIME-FRACTIONAL KLEIN-GORDON EQUATION

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ABSTRACT

In this study, the solutions of the random component time-fractional Klein-Gordon equation is obtained as approximately or exactly. The initial condition of this Klein-Gordon equation is studied by Gamma distribution. The fractional derivatives are defined in the Caputo sense. An example is shown to illustrate the influence of the solutions obtained by the new Sumudu transform iterative method (NSTIM). The expected value and variance of these solutions of this Klein-Gordon equation are obtained. The approximate analytical solution of this equation obtained by NSTIM and VIM are compared. NSTIM is applied to analyze the solution of this equation. Solution and figures are obtained by using MAPLE software. The formulas for the expected values and variances and results from the simulations of this Klein-Gordon equation are compared and the efficiency of this method is investigated.
Keywords: Expected value, random component time-fractional Klein-Gordon equation, the new Sumudu transform iterative method, variance.
2010 Mathematics Classification: 35R11, 35R60.

1. INTRODUCTION

Fractional calculus is a quite important topic in several scientific areas [19, 25, 30, 31, 32, 35]. Methodologies of fractional calculus have been widely used in the modeling of a lot of real matters in applied mathematics. Specially, fractional partial differential equations (FPDEs) describe certain a lot of phenomena in several scientific areas such as damping laws, diffusion equations, heat transfer modeling, electrostatics, fluid flow, elasticity and many others [1, 2, 3, 4, 5, 21, 22, 23, 24, 27, 28].

In the literature, there are very few studies on random fractional partial differential equations (RFPDEs). Random fractional partial differential equations (RFPDEs) are described as fractional partial differential equations with random inputs that can be a random variable or a stochastic process. These equations have a enormous significance in a lot of applications in engineering, biology, physics, mathematics and many other applied sciences. It is generally not possible to find

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random nonlinear fractional partial differential equations analytically. Thus, several numerical methods and approximation schemes for RFPDEs and FPDEs have been improved. There are a lot of numerical methods and approximation schemes such as adomian decomposition method (ADM) [41], homotopy perturbation method (HPM) [17], differential transformation method (DTM) [29], variational iteration method (VIM) [18], fractional variational iteration method (FVIM) [45], random finite difference scheme [9, 14] and many other methods. The main motivation in writing this paper is to analyze random nonlinear fractional partial differential equations using the new Sumudu transform iterative method (NSTIM) which is the reliable computational method.

The nonlinear Klein–Gordon equations are used to model a lot of problems in physics and these equations has been solved by different numerical methods in mathematics [6, 7, 13, 38, 42]. Recently, the fractional nonlinear Klein-Gordon equations have been solved by many numerical methods [15, 16, 20, 36]. For example, Golmankhaneh et al. successfully implemented the HPM for obtaining approximate analytical solutions of these equations [16].

This paper studies the random component time-fractional Klein-Gordon equation solve numerically by NSTIM. Wang and Liu established this method. They successfully applied this method to acquire the solutions of time-fractional Cauchy reaction-diffusion equations approximately and analytically [39]. There aren't enough research and articles on the power series transformation like Sumudu transform in the literature. The Sumudu transform method (STM) proposed by G. K. Watugala, was applied to solve engineering problems [40]. The method was applied to partial differential equations by Weerakoon [43]. Weerakoon found the inverse formula of this transform [44]. Demiray et al. used the Sumudu transform method (STM) to find the solutions of fractional differential equations exactly [11]. Kumar and Daftardar-Gejji extended Sumudu transform iterative method (STIM) to solve various both FPDEs and systems of FPDEs [26]. Prakash et al. suggested a new iterative Sumudu transform method (NISTM) to acquire solutions of the nonlinear time fractional Zahkarov-Kuznetsov equations numerically [33]. In this paper, the solutions of this Klein-Gordon equation are approximately obtained with NSTIM and VIM. Projected technique is used for error analysis. In addition, approximate solutions obtained by two methods with exact solution of this equation are shown in comparison tables. The difference of this work from Prakash et al. (2018) and Wang and Liu (2016) studies is to examine the random component fractional partial differential equation. The aim of this study is to present the application of NSTIM for obtaining the approximate analytical solution of the random component time-fractional Klein-Gordon equation with Caputo derivative and for calculating the expected value and variance of this solution. It is observed that the numerical solution obtained by NSTIM for this Klein-Gordon equation is almost similar to exact solution for this Klein-Gordon equation. Tables indicate that absolute error is negligible. It is observed that NSTIM is superior than VIM for this Klein-Gordon equation.

2. THE BASIC DEFINITIONS

In this section, a few main definitions of fractional calculus, Sumudu transform and Gamma distribution are presented.

Definition 2.1. For $\mu \in \mathbb{R}$, a real function $f(x), x > 0,$ is said to be in the space $C_\mu,$ if there exists a real number $p, (p > \mu),$ such that $f(x) = x^p f_1(x),$ where $f_1(x) \in C[0, \infty)$ and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu, m \in N$ [12, 39].

Definition 2.2. The Caputo fractional derivative of $f(x)$ is defined by [19, 37, 39]

$$D^a f(x) = I^{a-n} D^n f(x) = \frac{1}{\Gamma(n-a)} \int_0^x (x-t)^{n-a-1} f^{(n)}(t) dt \quad (1)$$

where $n - 1 < a \leq n,\ n \in N,\ x > 0, f \in C_{-1}^n.$

Some properties of the operator D^a that will be used in this study are given below:

(1) $D^{a}I^{a}f(x) = f(x)$,

(2) $I^{a}D^{a}f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^{+})\frac{x^k}{k!}, x > 0$.

Definition 2.3. The Riemann-Liouville fractional integral operator of order $a \geq 0$, of a function $f \in C_{\mu}, \mu \geq -1$ is given in the following [19, 39]

$$I^{a}f(x) = \begin{cases} \frac{1}{\Gamma(a)}\int_0^x (x-t)^{a-1}f(t)dt, & a > 0, x > 0, \\ I^{0}f(x) = f(x), & a = 0, \end{cases} \quad (2)$$

where $\Gamma(.)$ is the Gamma function.

Some features of the operator I^{a} that will be used in this study are given below:

For $f \in C_{\mu}, \mu, \gamma \geq -1, \alpha, \beta \geq 0$,

(3) $I^{a}I^{\beta}f(x) = I^{\beta}I^{a}f(x) = I^{a+\beta}f(x)$,

(4) $I^{a}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}$.

Definition 2.4. For $a > 0$, the Mittag-Leffler function E_a is given as follows [8, 39]

$$E_a(z) = \sum_{n=0}^{\infty}\frac{z^a}{\Gamma(na+1)}. \quad (3)$$

Definition 2.5. The Sumudu transform on the set of functions $A = \left\{ f(t) \middle| \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{t}{\tau_j}}, if\ t \in (-1)^j \times [0, \infty) \right\}$ is given as follows [8, 39]

$$S[f(t)] = \int_0^{\infty} e^{-t}f(vt)dt, \quad v \in (\tau_1, \tau_2). \quad (4)$$

Definition 2.6. The Sumudu transform of the Caputo fractional derivative is as follows [39]

$$S[D_x^{na}u(x,t)] = v^{-na}S[u(x,t)] - \sum_{k=0}^{n-1} v^{(-na+k)}u^{(k)}(0,t), n-1 < na \leq n. \quad (5)$$

Definition 2.7. If the probability density function of a random variable X has the following form, then this random variable has the Gamma distribution and is called a Gamma random variable. For $x, \alpha, \beta > 0$,

$$f(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}} \quad (6)$$

If the random variable X has a Gamma distribution with parameters α and β, then the expected value and variance of the random variable X are given as follows [34]

$$E(X) = \alpha\beta, Var(X) = \alpha\beta^2. \quad (7)$$

In this study, Gamma distribution in the example is chosen as Gamma$(\alpha = 4, \beta = 5)$.

3. THE NEW SUMUDU TRANSFORM ITERATIVE METHOD

Consider the following equation with the initial condition

$$\begin{cases} D_t^{na}u(x,t) + Lu(x,t) + Ru(x,t) = g(x,t), \\ n-1 < na \leq n, \\ u(x,0) = k(x). \end{cases} \quad (8)$$

where D_t^{na} is the Caputo fractional derivative operator, $D_t^{na} = \frac{\partial^{na}}{\partial t^{na}}$, $g(x,t)$ is a continuous function, L is linear operator and R is nonlinear operator [39].

If Sumudu transform is implemented to both sides of Eq. (8), then it can be found that

$$S[D_t^{na}u(x,t) + Lu(x,t) + Ru(x,t)] = S[g(x,t)]. \tag{9}$$

The following equation is obtained with the feature of the Sumudu transform [39]

$$S[u(x,t)] - v^{na}\sum_{k=0}^{n-1} u^{(k)}(x,0) + v^{na} S[Lu(x,t) + Ru(x,t) - g(x,t)] = 0. \tag{10}$$

From Eq. (10), it can be found that

$$S[u(x,t)] = v^{na}\sum_{k=0}^{n-1} u^{(k)}(x,0) - v^{na} S[Lu(x,t) + Ru(x,t) - g(x,t)]. \tag{11}$$

If the inverse Sumudu transform is implemented to Eq. (11), then Eq. (12) is obtained

$$u(x,t) = S^{-1}\left[v^{na}\sum_{k=0}^{n-1} u^{(k)}(x,0)\right] - S^{-1}\left[v^{na}S[Lu(x,t) + Ru(x,t) - g(x,t)]\right]. \tag{12}$$

Assume the following equalities hold:

$$\begin{cases} f(x,t) = S^{-1}\left[v^{na}\sum_{k=0}^{n-1} u^{(k)}(x,0) + v^{na}S[g(x,t)]\right], \\ N\big(u(x,t)\big) = -S^{-1}\left[v^{na}S[Ru(x,t)]\right], \\ K\big(u(x,t)\big) = -S^{-1}\left[v^{na}S[Lu(x,t)]\right]. \end{cases}$$

So Eq. (12) becomes Eq. (13):

$$u(x,t) = f(x,t) + K\big(u(x,t)\big) + N\big(u(x,t)\big), \tag{13}$$

where f is a known function, K is a linear operator of u and N is a nonlinear operator of u. The solution of Eq. (13) is given by the series form as follows [39]

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t). \tag{14}$$

Since K is a linear operator, the following equation is written as

$$K\left(\sum_{i=0}^{\infty} u_i\right) = \sum_{i=0}^{\infty} K(u_i). \tag{15}$$

The nonlinear operator N is written as [10, 39]

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=0}^{\infty}\left[N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right]. \tag{16}$$

Thus, Eq. (13) is given as [39]

$$\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} K(u_i) + N(u_0) + \sum_{i=0}^{\infty}\left[N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right]. \tag{17}$$

If the following recurrence is defined

$$\begin{cases} u_0 = f, \\ u_1 = K(u_0) + N(u_0), \\ u_{m+1} = K(u_m) + N(u_0 + \dots + u_m) - N(u_0 + u_1 + \dots + u_{m-1}). \end{cases} \tag{18}$$

then the Eq. (19) is obtained [39]

$$(u_1 + \dots + u_{m+1}) = K(u_0 + \dots + u_m) + N(u_0 + \dots + u_m). \tag{19}$$

Thus, Eq. (20) is obtained as

$$\sum_{i=0}^{\infty} u_i = f + K\left(\sum_{i=0}^{\infty} u_i\right) + N\left(\sum_{i=0}^{\infty} u_i\right). \tag{20}$$

The m-term solution of Eq. (13) is approximately obtained as

$$u = u_0 + u_1 + u_2 + u_3 + \dots + u_{m-1}. \tag{21}$$

In this work, the approximate analytical solution of random component time-fractional Klein-Gordon equation is obtained with the NSTIM. It is illustrated in the numerical experiment.

4. ERROR ANALYSIS OF PROJECTED TECHNIQUE

The error analysis of used technique acquired by NSTIM is given as follows.

Theorem 4.1. For all m values, a real number $0 < k < 1$ satisfies $\|u_{m+1}(x,t)\| \leq k\|u_m(x,t)\|$. Furthermore, if the truncated series $\sum_{m=0}^{l} u_m(x,t)$ is used to be an approximate solution $u(x,t)$, then maximum absolute truncated error is found with [33]

$$\left\|u(x,t) - \sum_{m=0}^{l} u_m(x,t)\right\| \leq \frac{k^{l+1}}{(1-k)}\|u_0(x,t)\|.$$

5. NUMERICAL EXPERIMENT

Consider the random component time-fractional Klein-Gordon equation

$$\begin{cases} u_t^{\alpha}(x,t) = u_{xx}(x,t) + u(x,t), \\ \qquad 0 < \alpha \leq 1, \\ \quad u(x,0) = Asinx + B, \end{cases} \tag{22}$$

where A and B are Gamma distributed random variable with parameters $\alpha = 4$ and $\beta = 5$, i.e. $A, B \sim G(\alpha = 4, \beta = 5)$.

If Sumudu transform is applied to Eq. (22) and the differential feature of Sumudu transform is used, then the Eq. (23) is found

$$S[u] = u(x,0) + v^{\alpha}S[u_{xx} - u]. \tag{23}$$

If the inverse Sumudu transform is applied to Eq. (23), then Eq. (24) is obtained

$$u(x,t) = S^{-1}[Asinx + B] + S^{-1}\left[v^{\alpha}S[u_{xx} - u]\right]. \tag{24}$$

From Eq. (24), it is obtained as

$$u(x,t) = Asinx + B + S^{-1}\left[v^{\alpha}S[u_{xx} - u]\right]. \tag{25}$$

For NSTIM, Eq. (26) holds:

$$\begin{cases} \qquad u_0 = Asinx + B, \\ K[u(x,t)] = S^{-1}\left[v^{\alpha}S[u_{xx} - u]\right]. \end{cases} \tag{26}$$

By iteration, the results are obtained as follows

$$u_0 = Asinx + B,$$
$$u_1 = \frac{Bt^{\alpha}}{\Gamma(1+\alpha)},$$
$$u_2 = \frac{Bt^{2\alpha}}{\Gamma(1+2\alpha)},$$
$$u_3 = \frac{Bt^{3\alpha}}{\Gamma(1+3\alpha)},$$
$$\cdots$$
$$u_n = \frac{Bt^{n\alpha}}{\Gamma(1+n\alpha)}.$$

Thus, the approximate solution of Eq. (22) is found as follows

$$u(x,t) = Asinx + B + \frac{Bt^{\alpha}}{\Gamma(1+\alpha)} + \frac{Bt^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{Bt^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots + \frac{Bt^{n\alpha}}{\Gamma(1+n\alpha)}$$
$$= Asinx + B\left(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots + \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\right)$$
$$= Asinx + BE_{\alpha}[t^{\alpha}]. \tag{27}$$

The form $u(x,t) = Asinx + Be^{t}$ is the approximate solution of the Eq. (22) for $\alpha = 1$. Also, this form is the exact solution of this equation for $\alpha = 1$.

Table 1. Comparison of the exact solution, approximate solution obtained with the sixth-order NSTIM and the VIM solution for α=1, $A=2, B=3$

x	t	**Exact Sol.**	**NSTIM**	**VIM**
0.5	0.2	4.623059351	4.623059345	4.623059077
0.5	0.4	5.434325171	5.434325171	5.434307078
0.5	0.6	6.425207477	6.425189477	6.424995080
0.5	0.8	7.635473861	7.635335345	7.634243080
0.5	1.0	9.113696561	9.113017745	9.108851080
1.0	0.2	5.347150244	5.347150238	5.347149970
1.0	0.4	6.158416064	6.158415038	6.158397970
1.0	0.6	7.149298370	7.149280370	7.149085970
1.0	0.8	8.359564754	8.359426238	8.358333970
1.0	1.0	9.837787454	9.837108638	9.832941970
1.5	0.2	5.659198247	5.659198247	5.659198241
1.5	0.4	6.470464067	6.470463041	6.470445970
1.5	0.6	7.461346373	7.461328373	7.461133970
1.5	0.8	8.671612757	8.671474241	8.670381970
1.5	1.0	10.149835460	10.149156640	10.144989970

Table 2. Comparison of the sixth- order NSTIM and VIM solution for α=0.9, $A=2$ and $B=3$

x	t	**NSTIM**	**VIM**
0.5	0.2	4.800450449	4.798719554
0.5	0.4	5.740077833	5.734086881
0.5	0.6	6.865282529	6.849157273
0.5	0.8	8.227008434	8.189855214
0.5	1.0	9.880842554	9.804132230
1.0	0.2	5.524541342	5.522810447
1.0	0.4	6.464168726	6.458177768
1.0	0.6	7.589373422	7.573248163
1.0	0.8	8.951099327	8.913946104
1.0	1.0	10.604933450	10.528223110
1.5	0.2	5.836589345	5.834858450
1.5	0.4	6.770216729	6.770225768
1.5	0.6	7.901421425	7.885296163
1.5	0.8	9.263147330	9.225994106
1.5	1.0	10.916981450	10.840271110

Table 3. Comparison between absolute error when $\alpha = 1$,$A = 2$ and $B = 3$ for this example

		t					
	x	**0.0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
NSTIM	**0.0**	0.0 0.0	0.0	0.6×10^{-8}	0.1×10^{-6}	0.1×10^{-5}	0.4×10^{-5}
VIM		0.0	0.4×10^{-8}	0.2×10^{-6}	0.3×10^{-5}	0.1×10^{-4}	0.7×10^{-4}
NSTIM	**0.1**	0.0	0.0	0.6×10^{-8}	0.1×10^{-6}	0.1×10^{-5}	0.4×10^{-5}
VIM		0.0	0.4×10^{-8}	0.2×10^{-6}	0.3×10^{-5}	0.1×10^{-4}	0.7×10^{-4}
NSTIM	**0.2**	0.0	0.0	0.6×10^{-8}	0.1×10^{-6}	0.1×10^{-5}	0.4×10^{-5}
VIM		0.0	0.4×10^{-8}	0.2×10^{-6}	0.3×10^{-5}	0.1×10^{-4}	0.7×10^{-4}
NSTIM	**0.3**	0.0	0.0	0.6×10^{-8}	0.1×10^{-6}	0.1×10^{-5}	0.4×10^{-5}
VIM		0.0	0.4×10^{-8}	0.2×10^{-6}	0.3×10^{-5}	0.1×10^{-4}	0.7×10^{-4}
NSTIM	**0.4**	0.0	0.0	0.6×10^{-8}	0.1×10^{-6}	0.1×10^{-5}	0.4×10^{-5}
VIM		0.0	0.4×10^{-8}	0.2×10^{-6}	0.3×10^{-5}	0.1×10^{-4}	0.7×10^{-4}
NSTIM	**0.5**	0.0	0.0	0.6×10^{-8}	0.1×10^{-6}	0.1×10^{-5}	0.4×10^{-5}
VIM		0.0	0.4×10^{-8}	0.2×10^{-6}	0.3×10^{-5}	0.1×10^{-4}	0.7×10^{-4}

It is observed from Table 1 that the solution obtained by NSTIM numerically is very close to exact solution. The comparison of absolute error between approximate solutions acquired from different methods and exact solution different values of x and t is shown in Table 3. Thus, it can be seen in Table 3 that absolute error is negligible. Table 1, 2 and 3 indicate that NSTIM is more efficient than VIM.

Now we get the expected value and variance of the approximate solution.

So the expected values of A, A^2, B and B^2 are obtained as follows

$$E(A) = 4.5 = 20, E(A^2) = 4.25(1+4) = 500, E(B) = 4.5 = 20, E(B^2) = 4.25(1+4) = 500.$$

For $A, B \sim G(\alpha = 4, \beta = 5)$, the expected value of Eq. (27) is given by

$$E[u(x,t)] = E[A]sinx + E[B]E_\alpha(t^\alpha) = 20\left(sinx + E_\alpha(t^\alpha)\right).$$

The graphs of the expected values of the Eq. (27) for different values of α are plotted in Maple software as follows

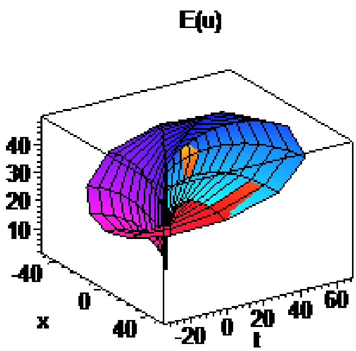

Figure 1. For $\alpha = 0.9$, time-dependent change of expected value of the Eq. (22)

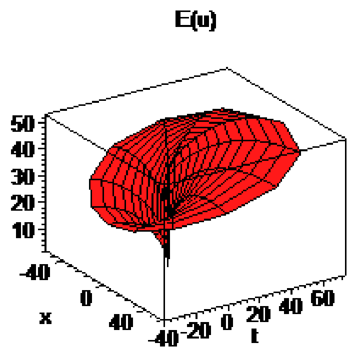

Figure 2. For $\alpha = 0.8$, time-dependent change of expected value of the Eq. (22)

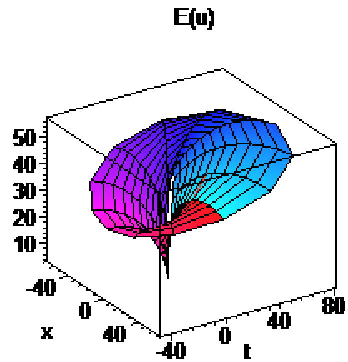

Figure 3. For $\alpha = 0.7$, time-dependent change of expected value of the Eq. (22)

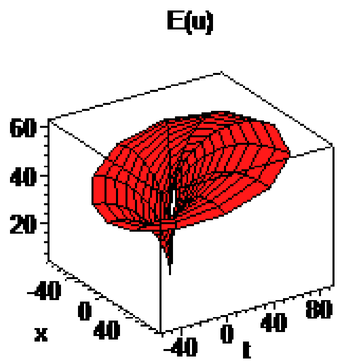

Figure 4. For $\alpha = 0.6$, time-dependent change of expected value of the Eq. (22)

For $A, B \sim G(\alpha = 4, \beta = 5)$, the variance of Eq. (27) is calculated as follows

$$V[u(x,t)] = E[u^2(x,t)] - (E[u(x,t)])^2$$
$$= E[(A\sin x + BE_\alpha[t^\alpha])^2] - (20(\sin x + E_\alpha[t^\alpha]))^2$$
$$= 100(\sin^2 x + (E_\alpha[t^\alpha])^2).$$

The graphs of the variances of the Eq. (27) for different values of α are plotted in Maple software as follows

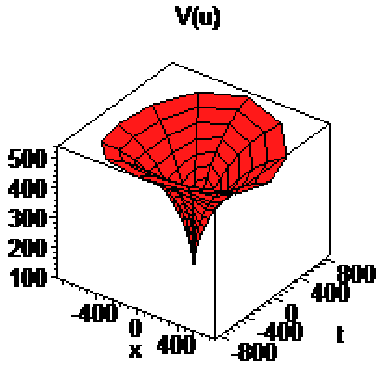

Figure 5. For $\alpha = 0.9$, time-dependent change of variance of the Eq. (22)

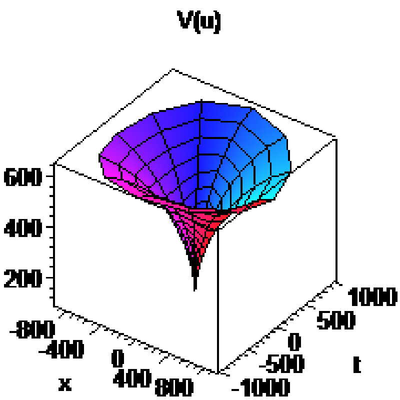

Figure 6. For $\alpha = 0.8$, time-dependent change of variance of the Eq. (22)

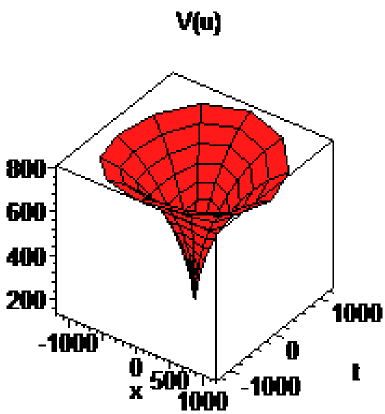

Figure 7. For $\alpha = 0.7$, time-dependent change of variance of the Eq. (22)

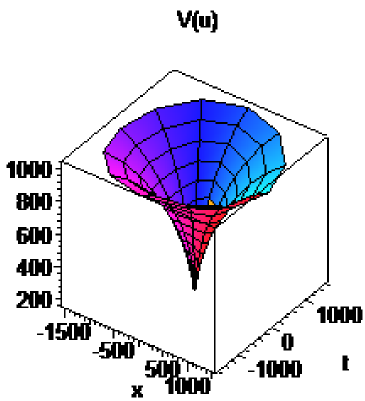

Figure 8. For $\alpha = 0.6$, time-dependent change of variance of the Eq. (22)

6. CONCLUSION

In this study, this Klein-Gordon equation is analyzed by NSTIM. The approximate analytical solution of the random component time-fractional Klein-Gordon equation has been quickly and successfully obtained with NSTIM. NSTIM is more efficent than VIM as shown in Tables 3. Thus, it is concluded that NSTIM is quickly, effective and superior in obtaining the approximate solutions for random component nonlinear fractional partial differential equations.

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