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Research Article

∗-HERMITE-HADAMARD-FEJER INEQUALITY AND SOME NEW INEQUALITY VIA ∗- CALCULUS

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ABSTRACT

In this paper, firstly it is proved ∗-Hermite-Hadamard-Fejer inequality in terms of ∗-calculus. Secondly, some theorems are generalized by this calculi. Finally, it is obtained some corollaries depend on these theorems.
Keywords: Generators, Non-Newtonian calculus, ∗-Hermite-Hadamard-Fejer inequality.
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1. INTRODUCTION

Definition 1 : Let I interval in $\mathbb{R}$ and $f: I \to \mathbb{R}$ a function. For all $x, y \in I$ and for $\alpha \in [0,1]$. Then f function said to be convex function, if the following inequality holds.

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) \tag{1.1}$$

Definition 2 [17]: Let $f: [a,b] \to \mathbb{R}$ be a ∗-convex function, $g: [a,b] \to \mathbb{R}$ function is integrable on $[a,b]$, nonnegative and symmetric to $(\frac{a+b}{2})$. Then, for all $x \in \mathbb{R}$

$$f(\tfrac{a+b}{2}) \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \tfrac{f(a)+f(b)}{2} \int_a^b g(x)dx$$

Hermite-Hadamard-Fejer Inequality holds.

The Non-Newtonian calculi were developed by Michael Grossman and Robert Katz, and it were written to nine books related to the Non-Newtonian calculi. Grossman and Katz published first book concerning with Non-Newtonian calculus at 1972 [1].

This calculi, which are mentioned above, are geometric calculus, bigeometric calculus, harmonic calculus, biharmonic calculus, quadratic calculus and biquadratic calculus. In the geometric calculus and the bigeometric calculus from within of these calculi, the derivative and integral are both multiplicative. The geometric derivative and the bigeometric derivative are closely related to the wellknown logarithmic derivative and elasticity, respectively. Also, the linear functions of classical calculus are the functions which having a constant derivative and besides the exponential functions in the geometric calculus are the functions which having a

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constant derivative, the power functions in the bigeometric calculus are the functions which having a constant derivative. Among the Non-Newtonian calculi, geometric and bigeometric calculi have been o ften used.

Since this calculi has emerged, it has become a seriously alternative to the classical analysis developed by Newton and Leibnitz. Just like the classical analysis, Non-Newtonian calculi have many varieties as a derivative, an integral, a natural average, a special class of functions having a constant derivative and two fundamental theorems which reveal that the derivative and integral are inversely related. However, the results of obtained by Non-Newtonian calculus has also significantly different from the classical analysis. For example, infinitely many Non-Newtonian calculi have a nonlinear derivative or integral.

The Non-Newtonian calculi are useful mathematical tools in science, engineering and mathematics and provide a wide variety of possibilities, as a different perspective. Specific fields of application include: fractal theory, image analysis (e.g., in bio-medicine), growth/decay processes (e. g., in economic growth, bacterial growth and radioactive decay), finance (e.g., rates of return), the theory of elasticity in economics, marketing, the economics of climate change, atmospheric temperature, wave theory in physics, quantum physics and gauge theory, signal processing, information technology, pathogen counts in treated water, actuarial science, tumor therapy in medicine, materials science/engineering, demographics, differential equations etc.

Recently, studies related with Non-Newtonian calculus have increased. Especially, these studies are emerging in the field of applied mathematics.[2]–[14].

2. BASIC DEFINITIONS

Now we give a short brief of Non-Newtonian Calculus [1].

2.1. Systems of Arithmetic

Arithmetic is any system that satisfies the whole of the ordered field axiom whose domain is a subset of $\mathbb{R}$. There are many types arithmetic, all of which are isomorphic, that is, structurally equivalent.

A generator α is a one-to-one function whose domain is $\mathbb{R}$ and whose range is a subset $\mathbb{R}_\alpha$ of $\mathbb{R}$ where $\mathbb{R}_\alpha = \{\alpha(x): x \in R\}$. Each generator generates exactly one arithmetic, and conversely each $I(x) = x$ arithmetic is generated by exactly one generator. If, for all $x \in \mathbb{R}$, the identity function's inverse is itself. In the special cases $\alpha = I$ and $\alpha = exp$, α generates the classical and geometric arithmetic, respectively.

2.1.1. α -Arithmetics

By α -arithmetic, we mean the arithmetic whose domain is $\mathbb{R}$ and whose operations are defined as follows: for $\mathbb{R}_\alpha$ and generator α,

$$\begin{aligned} &\alpha - addition, \quad x \dot{+} y = \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\},\\ &\alpha - subtraction, \quad x \dot{-} y = \alpha\{\alpha^{-1}(x) - \alpha^{-1}(y)\},\\ &\alpha - multiplication, \quad x \dot{\times} y = \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(y)\},\\ &\alpha - division, \quad x \dot{/} y = \alpha\{\alpha^{-1}(x) / \alpha^{-1}(y)\},\\ &\alpha - order, \quad x \dot{<} y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y). \end{aligned}$$

As a generator, we choose exp function acting from $\mathbb{R}$ into the set $\mathbb{R}_{exp} = (0, \infty)$ as follows:

$$\alpha: \mathbb{R} \to \mathbb{R}_{exp}$$
$$x \to y = \alpha(x) = e^x .$$

It is obvious that α arithmetic reduces to the geometric arithmetic as follows:

$$geometric - addition,\ x\dot{+}y = e^{\{lnx+lny\}} = x.y,$$
$$geometric - subtraction,\ x\dot{-}y = e^{\{lnx-lny\}} = x/y,$$
$$geometric - multiplication,\ x \dot{\times} y = e^{\{lnx\times lny\}} = x^{lny} = y^{lnx},$$
$$geometric - division,\ x\dot{/}y = e^{\{lnx/lny\}} = x^{\frac{1}{lny}},$$
$$geometric - order,\ x \dot{<} y \Leftrightarrow ln(x) < ln(y),,$$

Definition 3 *[1] Let $\alpha(p) = \dot{p}$ for all $p \in \mathbb{Z}$. If $y \in \mathbb{R}_\alpha$ for, $y\dot{+}\dot{0} = y$ and $y \dot{\times} \dot{1} = y$, then according to α -addition $\dot{0}(\alpha - zero)$ and $\dot{1}(\alpha - one)$ numbers are called identity and unit elements, respectively.*

Definition 4 *[1] Let $\dot{-}\dot{n} = \dot{0} \dot{-} \dot{n} = \alpha(-n)$ for all $n \in \mathbb{Z}_\alpha$. Set of α -integers is defined and denoted by $\mathbb{Z}_\alpha$ as can be seen in the figure below:*

$\mathbb{Z} = \{\dots, \dot{-}\dot{2}, \dot{-}\dot{1}, \dot{0}, \dot{1}, \dot{2}, \dots\}$
$=\{\dots, \alpha(-2), \alpha(-1), \alpha(0)\alpha(1), \alpha(2), \dots\}$

Namely, $\mathbb{Z}_\alpha = \{\dot{n}\colon n = \dot{\alpha}(n), n \in \mathbb{Z}\}$

2.1.2. β-Arithmetics

β generator is a one-to-one function whose domain is $\mathbb{R}$ and whose range is a subset of $\mathbb{R}_\beta$ of $\mathbb{R}$. We choose the function β such that its basic algebraic operations are defined as follows:

$$\beta - addition, \qquad x\ddot{+}y = \beta\{\beta^{-1}(x) + \beta^{-1}(y)\},$$
$$\beta - subtraction, \qquad x\ddot{-}y = \beta\{\beta^{-1}(x) - \beta^{-1}(y)\},$$
$$\beta - multiplication, \qquad x \ddot{\times} y = \beta\{\beta^{-1}(x) \times \beta^{-1}(y)\},$$
$$\beta - division, \qquad x\ddot{/}y = \beta\{\beta^{-1}(x)/\beta^{-1}(y)\},$$
$$\beta - order, \qquad x \ddot{<} y \Leftrightarrow \beta^{-1}(x) < \beta^{-1}(y),$$

for all$x, y \in \mathbb{R}_\beta$, where the non-Newtonian real field $\mathbb{R}_\beta = \{\beta(x)\colon x \in \mathbb{R}\}$. as in [8].

The β-positive real numbers, $\mathbb{R}_\beta^+$ denoted by , are the numbers x in $\mathbb{R}$ such that $\ddot{0} \ddot{<} x$; the β-negative real numbers, denoted by $\mathbb{R}_\beta^+$ are those for which $x \ddot{<} \ddot{0}$. The beta-zero $\ddot{0}$ and the beta-one $\ddot{1}$, turn out to be $\beta\{0\}$ and $\beta\{1\}$. Further,$\beta(-1) = \ddot{-}\ddot{1}$. Thus the set of all β -integers turns out to be the following:

$$\mathbb{Z}_\beta = \{\dots, \ddot{-}\ddot{2}, \ddot{-}\ddot{1}, \ddot{0}, \ddot{1}, \ddot{2} \dots\}$$
$$=\{\dots, \beta(-2), \beta(-1), \beta(0), \beta(1), \beta(2), \dots\}.$$

2.2. *-Calculus(Star-Calculus)

α and β are arbitrarily selected generators and *-(star) is the ordered pair of arithmetics, that is α -arithmetics and β-arithmetics. Along this paper the following notations will be used.

	$\alpha - Arithmetic$	$\beta - Arithmetic$
$Realm$	A	B
$Addition$	$\dot{+}$	$\ddot{+}$
$Subtraction$	$\dot{-}$	$\ddot{-}$
$Multiplication$	$\dot{\times}$	$\ddot{\times}$
$Division$	$\dot{/}$	$\ddot{/}$
$Order$	$\dot{<}$	$\ddot{<}$

It should be understood that all definition in α -arithmetics apply equally to β -arithmetics. In the $*$-Calculus, α-arithmetics is used on arguments and β-arithmetics is used on values: in particular, changes in arguments and values are measured by α-differences and β-differences, respectively. The operators of the $*$-calculus are applied only to functions with arguments in A and values in B. Accordingly unless indicated or implied otherwise, all functions are assumed to be of that character.

Remark 1 [1] We can choose α and β are identical, but if they are not? So, M. Grossman and R. Katz gave the definition of isomorphism about this problem.

Definition 5 *[1] The isomorphism from α -arithmetic to β -arithmetic is the unique function ι (iota) that possesses the following three properties.*

1) ι is one-to-one,
2) ι is on $\mathbb{R}_\alpha$ and onto $\mathbb{R}_\beta$,
3)For any number u and v in $\mathbb{R}_\alpha$
$\iota(u\dot{+}v) = \iota(u)\ddot{+}\iota(v)$,
$\iota(u\dot{-}v) = \iota(u)\ddot{-}\iota(v)$,
$\iota(u\dot{\times}v) = \iota(u)\ddot{\times}\iota(v)$,
$\iota(u\dot{/}v) = \iota(u)\ddot{/}\iota(v)$ and
$u \dot{<} iff\ \iota(u) \ddot{<} \iota(v)$

It turns out that

$$\iota(x) = \beta\{\alpha^{-1}(x)\},$$

for every number and that

$$\iota(\dot{n}) = \ddot{n}$$

for every integer n. Since, for example,

$$u\dot{+}v = \iota^{-1}\{\iota(u) \ddot{+} \iota(v)\},$$

Definition 6 [23] ($*$-limit) The $*$-limit of a function f at an element a in $\mathbb{R}_\alpha$ is, if it exist, the unique number b in $\mathbb{R}_\beta$ such that

$$* \lim_{x\to a} = b \Leftrightarrow For\ all\ \varepsilon \ddot{>} \ddot{0}: for\ all\ x \in \mathbb{R}_\alpha, |x\dot{-}a|_\alpha \dot{<} \delta,\ |f(x)\ddot{-}b|_\beta \ddot{<} \varepsilon$$

And is denoted by $* \lim_{x\to a} = b$.

Definition 7 [23] ($*$-Differentiation) I f the following $*$-limit in (2.1) exist, we denote it by $f^*(a)$, call it the $*$-derivative of f at a and say that f is $*$-differentiable at a:

$$* \lim_{x\to a}(f(x)\ddot{-}f(a)\ddot{/}((x)\ddot{-}(a)) = \lim_{x\to a}\beta\left\{\frac{\beta^{-1}(f(x))-\beta^{-1}(f(a))}{\alpha^{-1}(x)-\alpha^{-1}((a))}\right\} \tag{2.1}$$
$$= \lim_{x\to a}\beta\left\{\frac{\beta^{-1}(f(x))-\beta^{-1}(f(a))}{x-a}\frac{x-a}{\alpha^{-1}(x)-\alpha^{-1}((a))}\right\} = \beta\left\{\frac{(\beta^{-1}\circ f(x))\prime(a)}{(\alpha^{-1})(a)}\right\}.$$

Definition 8 [18] ($*$-Integration) Let α and β be generators and $r, s \in \mathbb{R}_\alpha$,
$r \dot{<} s$. If $f: I_\alpha \subseteq \mathbb{R}_\alpha \to \mathbb{R}_\beta$ be a $*$-continuous function,
*$I_\alpha \coloneqq [r,s]_\alpha$ **,** then we say the followin equality $*$-integration of f*

$$* \int_r^s f(\alpha(t) \dot{\times} a\dot{+}\alpha(1-t) \dot{\times} b)dt = \beta\left\{\int_r^s(\beta^{-1} \circ f)(\alpha(t) \dot{\times} a\dot{+}\alpha(1-t) \dot{\times} b)dt\right\}. \tag{2.2}$$

Definition 9 *[22] ($*$-convex function) Let I_α be an interal $\mathbb{R}_\alpha$. Then $f: I_\alpha \to \mathbb{R}_\beta$ is said to be $*$-convex if for all $t \in [0,1]$,*

$$f(\alpha(t) \dot{\times} a\dot{+}\alpha(1-t) \dot{\times} b) \ddot{<} \beta\{t\} \ddot{\times} f(x)\ddot{+}\beta\{1-t\} \ddot{\times} f(y) \tag{2.3}$$

Hold. Therefore, by combining thiswith the genarators α and β, we deduce that

$$f(\alpha\{t\alpha^{-1}(x)+(1-t)\alpha^{-1}(y)\}) \ddot{<} \beta\{t\beta^{-1}(f(x))+(1-t)\beta^{-1}(f(y))\} \quad (2.4)$$

3. MAIN RESULT

Definition 10 *Let α and β are any two generators and* $g: \dot{[}a,b\dot{]} \subseteq \mathbb{R}_\alpha \to \mathbb{R}_\beta$ *be a function. Then g function is said to be ∗-symmetric to* $\left(\frac{a\dot{+}b}{2}.\right)$*, if the following inequality holds.*

$$g\left(\frac{a\dot{\times}b}{x}.\right) = g\left(\frac{a\dot{\times}b}{a\dot{+}b\dot{-}x}.\right)$$

Remark 2 Along this paper, we show ∗-integration as follow,

$$*\int_r^s f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)d_\alpha t = \beta\left\{\int_{\alpha^{-1}(r)}^{\alpha^{-1}(s)}(\beta^{-1}\circ f)(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)dt\right\}. \quad (3.1)$$

Theorem 1 Let $f: \dot{[}a,b\dot{]} \subseteq \mathbb{R}_\alpha \to \mathbb{R}_\beta$ be a ∗-convex function, $a \dot{<} b$, let $g: \dot{[}a,b\dot{]} \subseteq \mathbb{R}_\alpha \to \mathbb{R}_\beta$ is a ∗-nonnegative, ∗-integrable and ∗-symmetric to $\left(\frac{a\dot{+}b}{2}.\right)$, and

$$*\int_a^b (f\ddot{\times}g)(x)\,d_\alpha x = *\int_a^b f(x)\,d_\alpha x \ddot{\times} *\int_a^b g(x)\,d_\alpha x$$

Then for all x ∈ ℝ,

$$f\left(\frac{a\dot{+}b}{2}.\right)\ddot{\times}*\int_a^b(g\circ\alpha)(x)\,d_\alpha x \,\ddot{\leqq}\, *\int_a^b(f\circ\alpha)(x)\,d_\alpha x \,\ddot{\times}\, *\int_a^b(g\circ\alpha)(x)\,d_\alpha x \,\ddot{\leqq}\, \frac{f(a)\ddot{+}f(b)}{\ddot{2}}..\ddot{\times}* \int_a^b(g\circ\alpha)(x)\,d_\alpha x \quad (3.2)$$

The inequality holds.

Proof. For all $t \in [0,1]$, we can write below equality. Firstly we prove the left hand of (3.2).

$$\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b\dot{+}\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a = [\alpha(t)\dot{+}\alpha(1-t)]\dot{\times}a\dot{+}[\alpha(t)\dot{+}\alpha(1-t)]\dot{\times}b = [\alpha(t)\dot{+}\alpha(1-t)]\dot{\times}(a\dot{+}b)$$

$$\alpha(t)\dot{+}\alpha(1-t) = \alpha\{\alpha^{-1}(\alpha(t))+\alpha^{-1}(\alpha(1-t))\} = \alpha(1) = \dot{1}.$$

Hence

$$(a\dot{+}b)\dot{\times}\alpha(1) = a\dot{+}b \Rightarrow a\dot{+}b = \alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b\dot{+}\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a$$

then from we get,

$$f\left(\frac{a\dot{+}b}{2}.\right) = f\left(\frac{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b\dot{+}\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}{\dot{2}}\right)$$

since f is a ∗-convex function, we have

$$f(\lambda_1\dot{\times}a\dot{+}\lambda_2\dot{\times}b) \ddot{\leqq} \theta_1\ddot{\times}f(a)\ddot{+}\theta_2\ddot{\times}f(b).$$

$$\lambda_1,\lambda_2 \in [0,1]_\alpha,\ \theta_1,\theta_2 \in [0,1]_\beta,\ \lambda_1\dot{+}\lambda_2 = \dot{1}\ and\ \theta_1\ddot{+}\theta_2 = \ddot{1}.$$

For $\lambda_1 = \alpha\left(\frac{1}{2}\right)$ and $\theta_1 = \beta\left(\frac{1}{2}\right)$.

$$f\left(\frac{a\dot{+}b}{2}.\right) \ddot{\leqq} \frac{f(a)\ddot{+}f(b)}{\ddot{2}}.. \quad (3.3)$$

hence if we take

$$\alpha = \alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b \quad (3.4)$$

and

$$b = \alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a \quad (3.5)$$

then from (3.3), (3.4) and (3.5) we can write

$$f\left(\frac{a\dot{+}b}{\dot{2}}\right) = f\left(\frac{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b\dot{+}\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}{\dot{2}}\right)$$
$$\ddot{\leq} \frac{f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)\ddot{+}f(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)}{\ddot{2}} \qquad (3.6)$$

If we multiply both sides of (3.6) by

$$g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)$$

and then if we take $*$-integration of (3.6) with respect to t over $[\dot{0}\dot{1}]$, we obtain

$$*\int_{\dot{0}}^{\dot{1}} f\left(\frac{a\dot{+}b}{\dot{2}}\right)\ddot{\times} g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)d_\alpha t$$
$$\ddot{\leq} *\int_{\dot{0}}^{\dot{1}}\left[\frac{f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)}{\ddot{2}}\ddot{\times} g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)d_\alpha t\right]$$
$$\ddot{+} *\int_{\dot{0}}^{\dot{1}}\left[\frac{f(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)}{\ddot{2}}\ddot{\times} g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)d_\alpha t\right]$$
$$\Rightarrow f\left(\frac{a\dot{+}b}{\dot{2}}\right)\ddot{\times}\int_{\dot{0}}^{\dot{1}} g(\alpha(t)\dot{\times}b\dot{+}\alpha(\ddot{1}-t)\dot{\times}a)d_\alpha t$$
$$\ddot{\leq} *\int_{\dot{0}}^{\dot{1}}\frac{f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)}{\ddot{2}}d_\alpha t\ddot{\times}\int_{\dot{0}}^{\dot{1}} g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)d_\alpha t$$
$$\ddot{+} *\int_{\dot{0}}^{\dot{1}}\frac{f(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)}{\ddot{2}}d_\alpha t\ddot{\times}\int_{\dot{0}}^{\dot{1}} g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)d_\alpha t$$

We can write below inequality due to properties of β generator and (3.1)

$$\beta\left\{\beta^{-1}\left(f\left(\frac{a\dot{+}b}{\dot{2}}\right)\right)\beta^{-1}\left(\beta\left\{\int_0^1(\beta^{-1}\circ g)(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)dt\right\}\right)\right\}$$
$$\ddot{\leq}\beta\left(\frac{1}{2}\right)\ddot{\times}\left[\beta\left\{\beta^{-1}\left(\beta\left\{\int_0^1(\beta^{-1}\circ f)(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)dt\right\}\right)\times\right.\right.$$
$$\left.\left.\left(\beta\left\{\int_0^1(\beta^{-1}\circ g)(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)dt\right\}\right)\right\}\right]$$
$$\ddot{+}\beta\left(\frac{1}{2}\right)\ddot{\times}\left[\beta\left\{\beta^{-1}\left(\beta\left\{\int_0^1(\beta^{-1}\circ f)(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)dt\right\}\right)\times\right.\right.$$
$$\left.\left.\left(\beta\left\{\int_0^1(\beta^{-1}\circ g)(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)dt\right\}\right)\right\}\right]$$

$$\beta\left\{\beta^{-1}\left(f\left(\frac{a\dot{+}b}{\dot{2}}\right)\right)\beta^{-1}\left(\beta\left\{\int_0^1(\beta^{-1}\circ g\circ\alpha)\big(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)\big)dt\right\}\right)\right\}$$
$$\ddot{\leq}\beta\left(\frac{1}{2}\right)\ddot{\times}\left[\beta\left\{\int_0^1(\beta^{-1}\circ f\circ\alpha)\big(t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)\big)dt\times\int_0^1(\beta^{-1}\circ g\circ\alpha)\big(t\alpha^{-1}(b)+\right.\right.$$
$$\left.\left.(1-t)\alpha^{-1}(a)\big)dt\right\}\right]$$
$$\ddot{+}\beta\left(\frac{1}{2}\right)\ddot{\times}\left[\beta\left\{\int_0^1(\beta^{-1}\circ f\circ\alpha)\big(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)\big)dt\times\int_0^1(\beta^{-1}\circ g\circ\alpha)\big(t\alpha^{-1}(b)+\right.\right.$$
$$\left.\left.(1-t)\alpha^{-1}(a)\big)dt\right\}\right] \qquad (3.7)$$

If we take $x = t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)$ (3.7) inequality becomes as follows

$$\beta\left\{\beta^{-1}\left(f\left(\frac{a\dot{+}b}{\dot{2}}\right)\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}\circ g\circ\alpha)(x)\frac{dx}{\alpha^{-1}(b)-\alpha^{-1}(a)}\right\}$$
$$\ddot{\leq}\beta\left(\frac{1}{2}\right)\ddot{\times}\left[\beta\left\{\int_0^1(\beta^{-1}\circ f\circ\alpha)\big(t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)\big)dt\times\right.\right.$$
$$\left.\left.\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}\circ g\circ\alpha)(x)\frac{dx}{\alpha^{-1}(b)-\alpha^{-1}(a)}\right\}\right]$$
$$\ddot{+}\beta\left(\frac{1}{2}\right)\ddot{\times}\left[\beta\left\{\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}\circ f\circ\alpha)\frac{dx}{\alpha^{-1}(b)-\alpha^{-1}(a)}dt\times\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}\circ g\circ\alpha)(x)\frac{dx}{\alpha^{-1}(b)-\alpha^{-1}(a)}\right\}\right]$$

Since g function be $*$-symmetric to $\left(\frac{a\dot{+}b}{\dot{2}}\right)$, we have

$$\beta\left\{\beta^{-1}\left(f\left(\frac{a\dot{+}b}{\dot{2}}\right)\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}\circ g\circ\alpha)(x)\frac{dx}{\alpha^{-1}(b)-\alpha^{-1}(a)}\right\}$$
$$\ddot{\leq}\beta\left(\frac{1}{2}\right)\ddot{\times}\left[\beta\left\{\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}\circ f\circ\alpha)\frac{dx}{\alpha^{-1}(b)-\alpha^{-1}(a)}dt\times\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}\circ g\circ\alpha)(x)\frac{dx}{\alpha^{-1}(b)-\alpha^{-1}(a)}\right\}\right]$$
$$\ddot{+}\beta\left(\frac{1}{2}\right)\ddot{\times}\left[\beta\left\{\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}\circ f\circ\alpha)\frac{dx}{\alpha^{-1}(b)-\alpha^{-1}(a)}dt\times\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}\circ g\circ\alpha)(x)\frac{dx}{\alpha^{-1}(b)-\alpha^{-1}(a)}\right\}\right]$$

and

$$f\left(\frac{a\dot{+}b}{\ddot{2}}.\right) \ddot{\times} * \int_a^b (g\circ\alpha)(x)\, d_\alpha x \ddot{\leq} * \int_a^b (f\circ\alpha)(x)\, d_\alpha x \ddot{\times} \ * \int_a^b (g\circ\alpha)(x)\, d_\alpha x. \tag{3.8}$$

So the left hand of (3.2) proof completes. Now we prove the second part. Since, f is a $*$-convex function we can write

$$f(\lambda_1 \dot{\times} a \dot{+} \lambda_2 \dot{\times} b) \ddot{\leq} \theta_1 \ddot{\times} f(a) \ddot{+} \theta_2 \ddot{\times} f(b). \tag{3.9}$$

and

$$f(\lambda_1 \dot{\times} b \dot{+} \lambda_2 \dot{\times} a) \ddot{\leq} \theta_1 \ddot{\times} f(b) \ddot{+} \theta_2 \ddot{\times} f(a). \tag{3.10}$$

If we add (3.9) and (3.10), we get

$$f(\lambda_1 \dot{\times} a \dot{+} \lambda_2 \dot{\times} b) \ddot{+} f(\lambda_1 \dot{\times} b \dot{+} \lambda_2 \dot{\times} a) \ddot{\leq} f(a) \ddot{+} f(b). \tag{3.11}$$

If we take $\lambda_1 = \alpha(t)$ and $\lambda_2 = \alpha(1-t)$ in (3.11) we get

$$f(\alpha(t) \dot{\times} a \dot{+} \alpha(1-t) \dot{\times} b) \ddot{+} f(\alpha(t) \dot{\times} b \dot{+} \alpha(1-t) \dot{\times} a) \ddot{\leq} f(a) \ddot{+} f(b) \tag{3.12}$$

and multiplying both sides of (3.12) by

$g(\alpha(t) \dot{\times} b \dot{+} \alpha(1-t) \dot{\times} a)$

$$\begin{aligned} & f(\alpha(t) \dot{\times} a \dot{+} \alpha(1-t) \dot{\times} b) \ddot{\times} g(\alpha(t) \dot{\times} b \dot{+} \alpha(1-t) \dot{\times} a) \\ & \ddot{+} f(\alpha(t) \dot{\times} b \dot{+} \alpha(1-t) \dot{\times} a) \ddot{\times} g(\alpha(t) \dot{\times} b \dot{+} \alpha(1-t) \dot{\times} a) \\ & \ddot{\leq} [f(a) \ddot{+} f(b)] \ddot{\times} g(\alpha(t) \dot{\times} b \dot{+} \alpha(1-t) \dot{\times} a). \end{aligned} \tag{3.13}$$

and then if we take $*$-integrating of (3.13) with respect to t over $[\dot{0}, \dot{1}]$ we get,

$$\begin{aligned} & * \int_{\dot{0}}^{\dot{1}} f(\alpha(t) \dot{\times} a \dot{+} \alpha(1-t) \dot{\times} b) \ddot{\times} g(\alpha(t) \dot{\times} b \dot{+} \alpha(1-t) \dot{\times} a)\, d_\alpha t \\ & \ddot{+} * \int_{\dot{0}}^{\dot{1}} f(\alpha(t) \dot{\times} b \dot{+} \alpha(1-t) \dot{\times} a) \ddot{\times} g(\alpha(t) \dot{\times} b \dot{+} \alpha(1-t) \dot{\times} a)\, d_\alpha t \\ & \ddot{\leq} * \int_{\dot{0}}^{\dot{1}} [f(a) \ddot{+} f(b)] \ddot{\times} g(\alpha(t) \dot{\times} b \dot{+} \alpha(1-t) \dot{\times} a)\, d_\alpha t \end{aligned}$$

Hence we can write following inequality.

$$\beta\{A.B\} \ddot{+} \beta\{C.D\} \ddot{\leq} \beta\left\{\beta^{-1}\{f(a) \ddot{+} f(b)\} \times \int_0^1 (\beta^{-1}\circ g \circ \alpha)\left(t\alpha^{-1}(b) + (1-t)\alpha^{-1}(a)\right)dt\right\} \tag{3.14}$$

where

$$\begin{aligned} A &:= \int_0^1 (\beta^{-1}\circ f \circ \alpha)\left(t\alpha^{-1}(a) + (1-t)\alpha^{-1}(b)\right)dt \\ B &:= \int_0^1 (\beta^{-1}\circ g \circ \alpha)\left(t\alpha^{-1}(b) + (1-t)\alpha^{-1}(a)\right)dt \\ C &:= \int_0^1 (\beta^{-1}\circ f \circ \alpha)\left(t\alpha^{-1}(b) + (1-t)\alpha^{-1}(a)\right)dt \\ D &:= \int_0^1 (\beta^{-1}\circ g \circ \alpha)\left(t\alpha^{-1}(b) + (1-t)\alpha^{-1}(a)\right)dt \end{aligned}$$

If we choose $x = t\alpha^{-1}(b) + (1-t)\alpha^{-1}(a)$ in (3.14) inequality becomes as follows

$$* \int_a^b (f\circ\alpha)(x)\, d_\alpha x \ddot{\times} * \int_a^b (g\circ\alpha)(x)\, d_\alpha x \ddot{\leq} \frac{f(a) \ddot{+} f(b)}{\ddot{2}}..\ddot{\times} * \int_a^b (g\circ\alpha)(x)\, d_\alpha x \tag{3.15}$$

The following inequality holds from (3.8) and (3.15). Thus proof of second part of theorem completes. Consequently we obtain desired result.

$$f\left(\frac{a\dot{+}b}{\ddot{2}}.\right) \ddot{\times} * \int_a^b (g\circ\alpha)(x)\, d_\alpha x \ddot{\leq} * \int_a^b (f\circ\alpha)(x)\, d_\alpha x \ddot{\times} \ * \int_a^b (g\circ\alpha)(x)\, d_\alpha x \ \ddot{\leq} \ \frac{f(a) \ddot{+} f(b)}{\ddot{2}}..\ddot{\times} * \int_a^b (g\circ\alpha)(x)\, d_\alpha x$$

Definition 11 *We say the (3.2) inequality $*$-Hermite-Hadamard-Fejer inequality.*

Corollary 1 *If we choose $\alpha = \beta = I$ in (3.2)*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx.$$

Corollary 2 If we choose $\beta = \alpha$ in (3.2)

$$f\left(\frac{a\dot{+}b}{\dot{2}}.\right)\dot{\times}\int_a^b g(x)^{dx} \ \dot{\leq} \int_a^b [f(x)\dot{\times} g(x)]^{dx}$$
$$\dot{\leq}\frac{f(a)\dot{+}f(b)}{\dot{2}}.\dot{\times}\int_a^b g(x)^{dx}.$$

Corollary 3 If we choose $\beta = \alpha = exp$ in (3.2)

$$ln\left(f(e^{\frac{\ln(ab)}{2}})\right)\int_{\ln a}^{\ln b} lng(x)dx \ \leq \int_{\ln a}^{\ln b}\ln(fg)(x)dx \leq \frac{\ln(f(a)+f(b))}{2}\int_{\ln a}^{\ln b} lng(x)dx.$$

Corollary 4 If we choose $\beta = \alpha = exp$ in (3.2)

$$\ln(f\left(\frac{a+b}{2}\right))\int_{\ln a}^{\ln b} lng(x)dx \ \leq \int_a^b \ln(fg)(x)dx\int_a^b lng(x)dx \qquad \leq$$
$$\frac{\ln f(a)+lnf(b)}{2}\int_a^b lng(x)dx.$$

Corollary 5 If we choose $\beta = I, \alpha = exp$ in (3.2)

$$(f\left(e^{\frac{\ln(ab)}{2}}\right))\int_{\ln a}^{\ln b} g(e^x)dx \ \leq \int_{\ln a}^{\ln b} f(e^x)g(e^x)dx$$
$$\leq \frac{f(a)+f(b)}{2}\int_{lna}^{lnb} g(e^x)dx.$$

Theorem 2 Let$I_\alpha := [a,b]_\alpha$, $a \dot{<} b$and $f: I_\alpha \subseteq \mathbb{R}_\alpha \backslash \{\dot{0}\} \to \mathbb{R}_\beta$be a$*$-harmonically convex function. Then we get below the inequality,

$$f\left(\frac{\alpha(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right) \dot{\leq} \frac{\iota(a)\ddot{\times}\iota(b)}{\iota(b)\ddot{-}\iota(a)}..\ddot{\times}^{*}\int_a^b \frac{(f o \alpha)(x)}{x^{\ddot{2}}}..d_\alpha x \,\ddot{\leq}\, \frac{f(a)\ddot{+}f(b)}{\ddot{2}}.. \tag{3.16}$$

Proof. Since f is $*$- harmonically convex function, for all $u, v \in I_\alpha$ and $t \in [0,1]$

$$f\left(\frac{u\dot{\times}v}{\alpha(t)\dot{\times}u\dot{+}\alpha(1-t)\dot{\times}v}.\right).\ddot{\leq}\,\theta_1 \ddot{\times} f(v)\ddot{+}\theta_2 \ddot{\times} f(u)$$

where, $\theta_1, \theta_2 \in [0,1]_\beta$, $\theta_1\ddot{+}\theta_2 = \ddot{1}$ and $\alpha(t)\dot{+}\alpha(1-t) = \ddot{1}$

For $\alpha(t) = \alpha\left(\frac{1}{2}\right)$ and $\theta = \beta\left(\frac{1}{2}\right)$ we can write the inequality.

$$f\left(\frac{\alpha(2)\dot{\times}u\dot{\times}v}{u\dot{+}v}.\right) \ddot{\leq} \frac{f(u)\ddot{+}f(u)}{\ddot{2}}.. \tag{3.17}$$

in other words if we choose,

$$u = \frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}.$$

and

$$v = \frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}.$$

then we have in (3.17),

$$f\left(\frac{\alpha(2)\dot{\times}u\dot{\times}v}{u\dot{+}v}.\right) = f\left(\frac{\alpha(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right) \ddot{\leq}\ \beta\left(\frac{1}{2}\right)\ddot{\times}\left[f\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b}.\right)\right.$$
$$\left.\ddot{+}f\left(\frac{a\dot{\times}\, b}{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b}.\right)\right]. \tag{3.18}$$

if we take the $*$-integrate of (3.18)

$$* \int_{\dot 0}^{\dot 1} f\left(\frac{\alpha(2)\dot\times a\dot\times b}{a\dot+b}.\right) d_\alpha t$$
$$\ddot\leq^* \int_{\dot 0}^{\dot 1} \beta\left(\frac{1}{2}\right) \ddot\times f\left(\frac{a\dot\times b}{\alpha(t)\dot\times a\dot+\alpha(1-t)\dot\times b}.\right) d_\alpha t$$
$$\ddot+^* \int_{\dot 0}^{\dot 1} \beta\left(\frac{1}{2}\right) \ddot\times f(\frac{a\dot\times b}{\alpha(t)\dot\times a\dot+\alpha(1-t)\dot\times b}.)d_\alpha t$$
$$\Rightarrow * \int_{\dot 0}^{\dot 1} f\left(\frac{\alpha(2)\dot\times a\dot\times b}{a\dot+b}.\right) d_\alpha t$$
$$\ddot\leq^* \beta\left(\frac{1}{2}\right) \ddot\times^* \int_{\dot 0}^{\dot 1} (fo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)}.\right) d_\alpha t$$
$$\ddot+^* \beta\left(\frac{1}{2}\right) \ddot\times^* \int_{\dot 0}^{\dot 1} (fo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}.\right) d_\alpha t$$
$$f\left(\frac{\alpha(2)\dot\times a\dot\times b}{a\dot+b}.\right)$$
$$\ddot\leq \beta\left(\frac{1}{2}\right) \ddot\times \left[\beta\left\{\int_{\dot 0}^{\dot 1} (\beta^{-1} o f o\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)}\right) dt\right\}\right]$$
$$\ddot+\beta\left(\frac{1}{2}\right) \ddot\times \left[\beta\left\{\int_{\dot 0}^{\dot 1} (\beta^{-1} o f o\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}\right) dt\right\}\right] \tag{3.19}$$

if we rewritte (3.19) for

$$x = \frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}$$

then we get

$$f\left(\frac{\alpha(2)\dot\times a\dot\times b}{a\dot+b}.\right) \dot\leq \frac{\iota(a)\ddot\times\iota(b)}{\iota(b)\ddot-\iota(a)}..\ddot\times^* \int_a^b \frac{(fo\alpha)(x)}{x^{\ddot 2}}..d_\alpha x$$

and then we write $u = a$ and $v = b$ in (3.17)

$$f\left(\frac{\alpha(2)\dot\times a\dot\times b}{a\dot+b}.\right) \dot\leq \frac{\iota(a)\ddot\times\iota(b)}{\iota(b)\ddot-\iota(a)}..\ddot\times^* \int_a^b \frac{(fo\alpha)(x)}{x^{\ddot 2}}..d_\alpha x \ddot\leq \frac{f(a)\ddot+f(b)}{\ddot 2}..$$

inequality holds. So the proof is completes.

Corollary 6 [19] If we choose $\alpha = \beta =$I in (3.16)

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{a-b}\int_a^b \frac{f(x)}{x^2} d\,x \leq \frac{f(a)+f(b)}{\ddot 2}$$

Corollary 7 If we choose $\alpha = \beta =$exp in (3.16)

$$ln(f\left(e^{\frac{2lnalnb}{lna+lnb}}\right)) \leq \frac{lnalnb}{lnb-lna}\int_{lnb}^{\ln b} \frac{lnf(x)}{e^2 lnx} dx \leq \frac{\ln(f(a)f(b))}{2}$$

Corollary 8 If we choose $\beta = \alpha$ in (3.16)

$$f\left(\frac{\alpha(2)\dot\times a\dot\times b}{a\dot+b}.\right) \dot\leq \frac{a\dot\times b}{b\dot-a}.\dot x \int_a^b [\frac{f(x)}{x^{\dot 2}}]^{dx} \dot\leq \frac{f(a)\dot+f(b)}{\dot 2}.$$

Corollary 9 If we choose $\alpha = I,\ \beta = exp$ in (3.16)

$$\ln(f\left(\frac{2ab}{a+b}\right)) \leq \ln(e^{\frac{ab}{b-a}}\int_a^b \frac{\ln f(x)}{\mathrm{e^x lnx}} dx) \leq \frac{\ln(f(a)f(b))}{2}.$$

Corollary 10 If we choose $\alpha = exp, \beta = I$ in (3.16)

$$f\left(e^{\frac{2lnalnb}{lna+lnb}}\right) \leq \frac{lnalnb}{lna-lnb}\int_a^b \frac{f(e^x)}{\mathrm{e^2 lnx}} dx \leq \frac{f(a)+f(b)}{2}.$$

Theorem 3 Let $I_\alpha \coloneqq [a,b]_\alpha,\ a \dot< b$and $f: I_\alpha \subseteq \mathbb{R}_\alpha\backslash\{\dot 0\} \to \mathbb{R}_\beta$ be a ∗- harmonically convex function. w$: I_\alpha \subseteq \mathbb{R}_\alpha\backslash\{\dot 0\} \to \mathbb{R}_\beta$ be a function is ∗-nonnegative, ∗-integrable and -symmetric to $(\frac{a+b}{2})$. Then we get below the inequality,

$$f\left(\frac{\alpha(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right)\ddot{\times}^{*}\int_{a}^{b}\frac{(wo\alpha)(x)}{x^{\dot{2}}}\,..\,d_{\alpha}x\ \ddot{\leq}^{*}\int_{a}^{b}\frac{(fo\alpha)(x)}{x^{\dot{2}}}\,..\,d_{\alpha}x$$
$$\ddot{\leq}^{*}\int_{a}^{b}\frac{(wo\alpha)(x)}{x^{\dot{2}}}\,..\,d_{\alpha}x$$
$$\dot{\leq}\frac{f(a)\dot{+}f(b)}{\dot{2}}.\ddot{\times}^{*}\int_{a}^{b}\frac{(wo\alpha)(x)}{x^{\dot{2}}}\,..\,d_{\alpha} \tag{3.20}$$

Proof. Since f is $*$- harmonically convex function we can write

$$f\left(\frac{\alpha(2)\dot{\times}u\dot{\times}v}{u\dot{+}v}.\right)\ddot{\leq}\frac{f(u)\ddot{+}f(v)}{\ddot{2}}.. \tag{3.21}$$

in other words if we choose

$$u=\frac{a\dot{\times}b}{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b}.$$

and

$$v=\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}.$$

then we have in (3.21),

$$f\left(\frac{\alpha(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right)\ddot{\leq}\ \beta\left(\frac{1}{2}\right)\ddot{\times}\left[f\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b}\right)\right]$$
$$\ddot{+}f\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}.\right)] \tag{3.22}$$

multiplying both sides of (3.22) by

$$w\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}.\right),$$

$$f\left(\frac{\alpha(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right)\ddot{\times}\ w\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}.\right) \tag{3.23}$$
$$\dot{\leq}\beta\left(\frac{1}{2}\right)\ddot{\times}\left[(f\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b}.\right)\right.$$
$$\left.\dot{\times}\,w\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b}.\right)\right]$$
$$\dot{+}\beta\left(\frac{1}{2}\right)\ddot{\times}\left[f\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}.\right)\right.$$
$$\left.\dot{\times}\,w\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}.\right)\right]$$

and then if we take $*$-integrating of (3.23) with respect to t over $[0,1]_{\alpha}$ we obtain,

$$*\int_{\dot{0}}^{\dot{1}}f\left(\frac{\alpha(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right)\ddot{\times}\ w\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}.\right)d_{\alpha}t$$
$$\dot{\leq}\beta\left(\frac{1}{2}\right)\ddot{\times}\left[*\int_{\dot{0}}^{\dot{1}}f\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b}.\right)\right.$$
$$\left.\dot{\times}\,w\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b}.\right)d_{\alpha}t\right]$$
$$\dot{+}\beta\left(\frac{1}{2}\right)\ddot{\times}\left[*\int_{\dot{0}}^{\dot{1}}f\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}.\right)\right.$$
$$\left.\dot{\times}\,w\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b}.\right)d_{\alpha}t\right]$$
$$\Longrightarrow f\left(\frac{\alpha(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right)\ddot{\times}^{*}\int_{\dot{0}}^{\dot{1}}(wo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)}.\right)d_{\alpha}t$$
$$\ddot{\leq}\beta\left(\frac{1}{2}\right)\ddot{\times}\,[\int_{\dot{0}}^{\dot{1}}(fo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}.\right)d_{\alpha}t$$
$$\ddot{\times}^{*}\int_{\dot{0}}^{\dot{1}}\ (wo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}.\right)d_{\alpha}t]$$
$$\ddot{+}\ \beta\left(\frac{1}{2}\right)\ddot{\times}\,[*\int_{\dot{0}}^{\dot{1}}(fo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}.\right)d_{\alpha}t$$

$$\ddot{\times}\ \int_{\dot{0}}^{\dot{1}} (wo\alpha)\ \left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}.\right) d_\alpha t]$$

we can write below inequality due to properties of β generator and (3.1)

$$\beta\{\beta^{-1}\left(f\left(\frac{a(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right)\right)\int_0^1(\beta^{-1}ofo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}\right)dt\}$$
$$\ddot{\leq}\ \beta\left(\frac{1}{2}\right)\ddot{\times}\left[\beta\left\{\int_0^1(\beta^{-1}ofo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)}\right)dt\ \times\right.\right.$$
$$\left.\left.\int_0^1(\beta^{-1}owo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}\right)dt\right\}\right]$$
$$\ddot{+}\beta\left(\frac{1}{2}\right)\ddot{\times}\left[\beta\left\{\int_{\dot{0}}^{\dot{1}}(\beta^{-1}ofo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}\right)d\ \times\right.\right.$$
$$\left.\left.\int_0^1(\beta^{-1}owo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}\right)dt\right\}\right] \tag{3.24}$$

Where if we change the variable in (3.24)

$$x=\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}$$

inequality becomes as follows

$$\beta\{\beta^{-1}\left(f\left(\frac{a(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right)\right)\int_0^1(\beta^{-1}owo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{[\alpha^{-1}(b)-\alpha^{-1}(a)]x^2}\right)dt\}$$
$$\ddot{\leq}\ \beta\left(\frac{1}{2}\right)\ddot{\times}\left[\beta\left\{\int_0^1(\beta^{-1}ofo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)}\right)dt\times\int_0^1(\beta^{-1}owo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{[\alpha^{-1}(b)-\alpha^{-1}(a)]x^2}\right)dt\right\}\right]$$
$$\ddot{+}\beta\left(\frac{1}{2}\right)\ddot{\times}[\beta\{(\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}ofo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{[\alpha^{-1}(b)-\alpha^{-1}(a)]x^2}\right)dt)$$
$$\times(\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}owo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{[\alpha^{-1}(b)-\alpha^{-1}(a)]x^2}\right)dt)\}].$$

Since w function be $*$-symmetric according to $(\frac{a\dot{+}b}{2})$, we have

$$\beta\{\beta^{-1}\left(f\left(\frac{a(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right)\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}owo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{[\alpha^{-1}(b)-\alpha^{-1}(a)]x^2}\right)dx\}$$
$$\ddot{\leq}\ \beta\left(\frac{1}{2}\right)\ddot{\times}\left[\beta\left\{\left(\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}ofo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)}\right)dx\right)\times\right.\right.$$
$$\left.\left.(\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}owo\alpha)\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{[\alpha^{-1}(b)-\alpha^{-1}(a)]x^2})dx\right\}\right]$$
$$\ddot{+}\beta\left(\frac{1}{2}\right)\ddot{\times}[\beta\{(\int_{\alpha^{-1}(a)}^{\alpha^{-\dot{1}}(b)}(\beta^{-1}ofo\alpha)\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{[\alpha^{-1}(b)-\alpha^{-1}(a)]x^2}dx)$$
$$\times(\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(\beta^{-1}owo\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{[\alpha^{-1}(b)-\alpha^{-1}(a)]x^2}\right)dx)\}].$$

Hence we obtain

$$f\left(\frac{a(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right)\ddot{\times}^*\int_a^b\frac{(wo\alpha)(x)}{x^{\dot{2}}}..d_\alpha x$$
$$\ddot{\leq}^*\int_a^b\frac{(fo\alpha)(x)}{x^{\dot{2}}}..d_\alpha x\ \ddot{\times}^*\int_a^b\frac{(wo\alpha)(x)}{x^{\dot{2}}}..d_\alpha x$$

and then we write $u=a$ and $v=b$ in (3.21)

$$f\left(\frac{a(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right)\ddot{\times}^*\int_a^b\frac{(wo\alpha)(x)}{x^{\dot{2}}}..d_\alpha x$$
$$\ddot{\leq}^*\int_a^b\frac{(fo\alpha)(x)}{x^{\dot{2}}}..d_\alpha x\ \ddot{\times}^*\int_a^b\frac{(wo\alpha)(x)}{x^{\dot{2}}}..d_\alpha x$$
$$\ddot{\leq}\frac{f(a)\dot{+}f(b)}{\dot{2}}..\ddot{\times}^*\int_a^b\frac{(wo\alpha)(x)}{x^{\dot{2}}}..d_\alpha x$$

inequality holds. So the proof is completes.

Corollary 11 [20] If we choose $\alpha=\beta=I$ in (3.20)

$$f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{w(x)}{x^2}d\,x\leq\int_a^b\frac{f(x)w(x)}{x^2}d\,x\leq\frac{f(a)+f(b)}{2}\int_a^b\frac{w(x)}{x^2}\ dx$$

Corollary 12 If we choose choose $\beta = \alpha$ in (3.20)

$$f\left(\frac{a(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right)\dot{\times}\int_a^b[\frac{w(x)}{x^{\dot{2}}}.]^{dx} \leq \int_a^b[\frac{f(x)\ddot{\times}w(x)}{x^{\dot{2}}}.]^{dx}\int_a^b\frac{f(x)w(x)}{x^2}d\,x \leq \frac{f(a)\ddot{+}f(b)}{\ddot{2}}.\dot{\times}\int_a^b[\frac{w(x)}{x^{\dot{2}}}.]^{dx}$$

Corollary 13 If we choose $\alpha = I, \beta =$exp in (3.20)

$$ln(f\left(e^{\frac{2lnalnb}{lna+lnb}}\right)\int_{lnb}^{\ln b}\frac{lnw(x)}{e^2lnx}dx) \leq \frac{lnalnb}{lnb-lna}\int_{lnb}^{\ln b}\frac{lnf(x)lnw(x)}{e^2lnx}dx$$
$$\leq \frac{\ln(f(a)f(b))}{2}\int_{lnb}^{\ln b}\frac{lnw(x)}{e^2lnx}dx$$

Corollary 14 If we choose $\alpha = \exp, \beta = I$ in (3.20)

$$\ln[f\left(\frac{2ab}{a+b}\right)\int_{lnb}^{\ln b}\frac{lnw(x)}{x^2}dx] \leq \int_{lnb}^{\ln b}\frac{lnw(x)lnf(x)}{x^2}dx$$
$$\leq \frac{\ln(f(a)f(b))}{2}\int_{lnb}^{\ln b}\frac{lnw(x)}{x^2}dx$$

Corollary 15 If we choose $\alpha = \exp, \beta = I$ in (3.20)

$$ln(f\left(e^{\frac{2lnalnb}{lna+lnb}}\right)\int_{lnb}^{\ln b}\frac{w(e^x)}{x^2}dx) \leq \int_{lnb}^{\ln b}\frac{f(e^x)w(e^x)}{x^2}dx$$
$$\leq \frac{f(a)+f(b)}{2}\int_{lnb}^{\ln b}\frac{w(e^x)}{x^2}dx$$

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