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Research Article

THE PERFORMANCE OF MULTISTAGE SEQUENTIAL SAMPLING PROCEDURES FOR THE MEAN OF A NORMAL POPULATION: A MONTE CARLO SIMULATION STUDY

Ali YOUSEF*[1], Hosny HAMDY[2]

[1]*Faculty of Engineering, Department of Mathematics, Kuwait College of Science and Technology, KUWAIT;* ORCID: 0000-0002-8824-5947
[2]*Faculty of Management Sciences, October University for Modern Sciences and Arts, 12566 6th October City, Cairo, EGYPT;* ORCID: 0000-0001-9667-5472

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ABSTRACT

This paper studies the performance of multistage sequential sampling procedures in chronological order, starting from Stein's two-stage procedure, the one-by-one purely sequential procedure, Hall's three-stage procedure, and the accelerated sequential procedure for estimating the mean of the normal distribution under a moderate sample size using Monte Carlo simulation. We also introduce and discuss the performance of a new sequential sampling procedure called the progressive procedure that starts with a bulk stage and ends by one-by-one purely sampling under moderate and large sample sizes (asymptotic) based on Monte Carlo simulation. The simulation results show that the new procedure competes with other procedures and attains all targeted asymptotic characteristics except the exact consistency property.
Keywords: Accelerated sequential scheme, asymptotic characteristics of multistage sampling procedures, one-by-one procedure, progressive sampling procedure, three-stage procedure, two-stage procedure.

1. INTRODUCTION

Multistage sampling procedures were developed over the past few decays to achieve several popular characteristics that were lacked in fixed sample size sampling procedures. The idea started from Abraham Wald in 1945, who first developed a sequential test during World War II as a tool to establish more efficient quality control in equipment inspection. His sequential procedure is known as a sequential probability ratio test (SPRT), and the objective was to minimize the cost of inspection during the war [1]. Although Wald's test was devised to treat a particular specific problem in testing hypotheses, the idea of multistage sampling came out to inspire researchers to device other techniques to perform statistical inference.

We emphasize that, contrary to the fixed sample size sampling procedure, where the inference is made after the sampling phase, in multistage sampling, the inference is made within the sampling framework. See also Stein and Wald for confidence interval estimation problem [2].

* Corresponding Author: e-mail: a.yousef@kcst.edu.kw, tel: 965 24972835

In general, multistage sampling procedures are used in statistical inference when no suitable fixed sample size procedure is available, especially when the optimal fixed sample size needed to meet certain specific goals depends on unknown nuisance parameters. The word "*optimal*" refers to the minimum fixed sample size needed to satisfy certain criteria had the nuisance parameters been known.

Assume that our objective is to make inference for the population normal mean under some optimal criteria. Like minimizing the cost associated with point estimation when we propose to estimate the population mean μ by the corresponding sample mean $\bar{X}_n$, $n(\geq 2)$ or constructing a fixed-width confidence interval for μ with a prescribed width $2d\,(>0)$, where the coverage probability exceeds the nominal value $100(1-\alpha)\%$, $0<\alpha<1$.

Regardless of the type of inference we seek, the optimal sample size required to perform inference about the population mean takes the following form, for more details see, Sen [3] and Ghosh, Mukhopadhyay, and Sen [4]

$$n^* = \lambda g(\theta) \tag{1}$$

where $\lambda(>0)$ is a known constant with the characteristic that as $\lambda \to \infty$ $n^* \to \infty$. Moreover, it depends on some predetermined constants, which may, for example, appear in a loss function incurred in point estimation of the mean μ or arise from consideration of a fixed-width confidence interval for the mean μ with a prescribed coverage probability. The function $g(>0)$ is a real-valued continuously differentiable function (of a proper order) of the unknown θ. For more details, see Sen [3] and Ghosh, Mukhopadhyay, and Sen [4].

Since n^* in (1) is numerically unknown because θ is unknown, then no fixed sample size procedure provides the above point estimation of μ uniformly for $\forall\ \theta>0$. Therefore, we resort to multistage procedures to estimate μ via estimation of n^*.

Before we state the multistage sampling procedures, we summarize the required *optimal criteria* of an efficient multistage sampling that were developed over the years regarding both point and interval estimation.

Let N be the final random sample size generated by a multistage sampling procedure, and let n^* be as in (1). Then a procedure attains *first-order asymptotic efficiency* if the ratio of the expected final sample size required by the multistage procedure to perform inference relative to the fixed sample size required to perform inference had θ been known, say n^* approaches one. That is as $\lambda \to \infty$ $E\left(N/n^*\right) \to 1$. The procedure is *second-order asymptotic efficient* if as n^* increases, the quantity $E\left(N-n^*\right) < \infty$ remains bounded in the sense of Ghosh and Mukhopadhyay [5].

Regarding the confidence interval estimation, let I_N be the fixed-width confidence interval constructed via a multistage procedure. Then the procedure is *consistent* or *exactly consistent* if $P\left(\mu \in I_N\right) \geq 1-\alpha$, uniformly $\forall\ \mu$ and θ, while it is *asymptotically first-order consistent* if as, $\lambda \to \infty$ $P\left(\mu \in I_N\right) \to 1-\alpha$ uniformly $\forall\ \mu$ and θ, in the sense of Stein [6], Mukhopadhyay [7], and Chow and Robbins [8].

Additional criteria are required for each specific estimation objective, for example in the case of point estimation problem; it is required that the *asymptotic regret*, $\omega(n^*)$ which is the difference between the expected loss of the multistage sampling from the optimal loss had n^* been known should be bounded, that is let R_N be the multistage risk encountered in estimating the

population mean μ by the corresponding sample measure and let R_{n^*} be the optimal fixed sample size risk had θ been known. Then, the multistage procedure is called *first-order asymptotically risk efficient* if as $\lambda \to \infty$ $R_N / R_{n^*} \to 1$, while it is *asymptotically second-order risk efficient* if as $\lambda \to \infty$ $\omega(n^*) = R_N - R_{n^*}$ remains bounded in the sense of Ghosh and Mukhopadhyay [5].

In addition to the above asymptotic measures, other factors may be used to compare the procedures; the practical implementations in real-life problems, the insensitivity to changes in the underline distribution and the sensitivity to depict any potential changes in the parameter under consideration.

2. MULTISTAGE SAMPLING PROCEDURES FOR ESTIMATING THE POPULATION NORMAL MEAN

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables from a normal distribution with unknown mean $\mu \in \Re$ and unknown variance $\theta > 0$. Based on a fixed sample of size $n(\geq 2)$ say $(X_1, \ldots, X_n)$ we propose $\bar{X}_n = \sum_{i=1}^{n} \frac{X_i}{n}$ and $S_n^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X}_n)^2}{(n-1)}$ as point estimates for μ and θ respectively.

Suppose we need to estimate the mean μ by a confidence interval such that the width of the interval is $2d\,(>0)$ and $P\left(\left|\bar{X}_n - \mu\right| \leq d\right) \geq (1-\alpha)$, uniformly for $\forall\ (\mu, \theta) \in \mathbb{R} \times \mathbb{R}^+$, where α $0 < \alpha < 1$ and d are previously known and fixed to the experimenter. If θ is known, then this fixed-width confidence interval problem has a simple fixed sample size solution given by (1) with $g(\theta) = \theta$ and $\lambda = (a/d)^2$ where a is an upper $\alpha/2$ quantile point of standard normal distribtuion $N(0,1)$. The quantity $n^* = a/d^{\,2}\theta$ is referred to as the optimal fixed sample size required constructing a fixed-width confidence interval for μ given that θ is known.

Since θ is unknown, then n^* is unknown, and it was shown by Dantzig [9] that there is exist no fixed sample size procedure which allows us to construct a size $(1-\alpha)$ confidence interval for μ with a fixed width $2d$. Therefore, we resort to multistage procedures to estimate n^* via estimation of θ.

In the following sub-sections, we list in chronological order the multistage procedures as shown in the literature.

2.1. Stein's Two-Stage Procedure

Stein, in 1945, introduced the two-stage sequential procedure to estimate the mean of the normal distribution by a confidence interval with assigned width and confidence coefficient [6]. Regarding Stein's two-stage procedure, the sampling starts by taking a pilot sample of size $m(\geq 2)$ from the distribution, and calculate the initial estimates of μ and θ respectively.

We define the stopping rule as follows

$$N = \max\{m, N_1\},\ N_1 = \left[\lambda S_m^2\right] + 1 \tag{2}$$

where $\lambda = (t_{m-1}/d)^2$ and t_{m-1} is the upper 50α % point of the student t - *distribution* with $(m-1)$ degrees of freedom, and $[x]$ is the largest integer less than x. If $m \geq N_1$ stop sampling at this stage, otherwise continue to sample the difference $(N_1 - m)$ samples and perform the required inference based on N samples.

Since the publication of Stein's paper, a large body of research was directed to develop multistage techniques to improve the quality of the inference, especially in point and interval estimation. Cox, in 1952 generalized Stein's two-stage procedure to another class of distributions; he developed *Double-sampling* methods for estimating an unknown parameter by a confidence interval so that the variance of the estimate is a function of the unknown parameter [10]. Seelbinder [11] provided a method to determine the size of the first part of Stein's two-stage sample for estimating the population mean with a given accuracy. Cox [10], Ghosh and Mukhopadhyay[5] have shown that the two-stage procedure suffers from the lack of asymptotic efficiency. That is, the procedure is asymptotically first-order inefficient, which leads to oversampling that is $\overline{N} > n^*$, where $\overline{N}$ is the average sample size, and hence second-order asymptotic inefficient.

Robustness of Stein's two-stage procedure was first considered by Bhattacharjee [12], where he used the first four terms of the Edgeworth series to approximate the underline distribution, while Blumenthal and Govindarajulu [13] studied the robustness of the two-stage procedures under a mixture of two normal distributions. Ramkaran[14] considered the same problem and showed that Stein's two-stage procedure is robust under the Edgeworth first four terms series. Sook and DasGupta [15] studied the robustness of the two-stage procedure against possible departure from normality only via the expectation of the random final sample size. They found that the procedure is robust for certain classes of distributions, including the *t –distribution* with 3 and 5 degrees of freedom and double exponential distribution.

2.2. Purely Sequential Procedure

In order to tackle oversampling in the two-stage procedure, one may control the sampling process by estimating the population variance θ successively in a sequential manner. Anscombe [16] introduced the idea of the purely sequential procedure, also known as one-by-one procedure, and discussed some of its asymptotic characteristics. The procedure proceeds as follows: Start with a pilot sample as in the previous case, but instead of taking a bulk sample we take the observations one-by-one and check with the following stopping rule

$$N = \inf\left\{n \geq m : n \geq (a/d)^2 S_n^2\right\} \tag{3}$$

That is, if $m \geq (a/d)^2 S_m^2$ stop sampling right here and the final random sample size is $N = m$, this means we have enough information in the pilot phase. Otherwise, take one additional observation say, X_{m+1} and update the sample variance by S_{m+1}^2. If $m+1 \geq (a/d)^2 S_{m+1}^2$ stop sampling here and the final sample size is $N = m+1$, if not, draw an additional observation say X_{m+2} and update the sample variance by S_{m+2}^2. Repeat the process until (3) holds.

Regarding the purely sequential sampling, the seminal work of Robbins [17], Chow and Robbins[8] in one-by-one sequential sampling schemes were set forward to satisfy all the requirements of an efficient multistage sampling procedure, which enjoys all the above optimal criteria for both point and confidence interval estimation except the exact consistency property, while it attains the targeted nominal value only asymptotically.

The asymptotic characteristics of the one-by-one sampling procedure were given in Anscombe [18], and recently by Woodroofe [19]. Although the one-by-one sequential sampling procedure enjoys all the required optimal characteristics, which also guarantee a genuine stopping rule, still it can take quite some time to terminate the procedure, which is considered a drawback especially when time and cost are a matter of concern.

2.3. Three-stage Procedure

Hall [20] introduced the three-stage sampling procedure to achieve two primary objectives; the operational savings made possible by sampling in batches and the asymptotic efficiency attained by the one-by-one purely sequential sampling. In the three-stage sampling procedure, sampling is performed in three bulks, the pilot study phase, the main study phase, and the fine-tuning phase. Similar to the two-stage and the one-by-one sequential, we start with a pilot sample $m \geq 2$ and calculate both $\bar{X}_m$ and S_m^2 as point estimates of μ and θ respectively. During the main study phase and contrary to the two-stage sampling, we estimate only a portion γ (design factor), $0 < \gamma < 1$ of n^*. The decision to stop sampling or proceed to the next phase depends on the following stopping rule,

$$N_2 = \max\{m, N_1\}, \quad N_1 = \left[\gamma (a/d)^2 S_m^2\right] + 1 \tag{4}$$

If $m \geq N_1$, we stop at this stage. Otherwise, we continue sampling an extra sample of size $(N_1 - m)$ from the distribution function to bring the total sample to $X_1,\ldots,X_m,X_{m+1},\ldots,X_{N_1}$. Hence, we update the estimates, $\bar{X}_{N_1}$ and $S_{N_1}^2$ for both μ and θ respectively and proceed to the next phase. The fine tuning phase stopping rule

$$N = \max\left\{N_1, \left[(a/d)^2 S_{N_1}^2\right] + 1\right\} \tag{5}$$

If additional samples of size $(N - N_1)$ are to be taken according to the fine-tuning stopping rule, we continue to sample $X_{N_1+1},\ldots,X_N$, then we terminate the sampling process and propose $\bar{X}_N$ and S_N^2 for both μ and θ respectively to perform the required inference.

The three-stage procedure enjoys all the asymptotic characteristics required by any multistage sampling procedure, including the asymptotic efficiency of the first and second-order, as well as the asymptotic consistency of Robbins [17] and Chow and Robbins [8] of the one-by-one-sequential procedure. Although Hall's three-stage procedure was specifically designed to treat the confidence estimation problem for the normal mean, he mentioned in page 1230 that" *However, It does not seem possible to give a global theory for all such applications, like that given in Cox* [10] *for double sampling*."

Mukhopadhyay [21], Hamdy [22], Hamdy and Pallotta [23], Mukhopadhyay et al. [24], and Mukhopadhyay and Mauromoustakos [25] extended Hall's three-stage sampling procedure to other distributions rather than the normal distribution and adopted the procedure to treat the point estimation as well, for more details see Ghosh, Mukhopadhyay, and Sen [4].

Yousef, Kimber, and Hamdy [26] derived the asymptotic theory of the three-stage sampling for the mean of the unknown distribution provided the first six moments are unknown but finite. They studied the robustness of the procedure to the underline distribution and found that the procedure is generally non-robust, and robust only for a limited class of distributions.

Yousef [27-28] studied the sensitivity of the normal-based three-stage procedure for estimating the mean of an unknown distribution to departure from normality. He found two compact forms of the three-stage fixed-width confidence interval for the mean. The first when the

explicit form of the underlying distribution is analytically known and the second when the underlying distribution can be approximated by the first four terms of the Edgeworth series. He showed that the three-stage procedure always produces coverage probabilities less than the nominal value and attains it only asymptotically. Also, the performance of the coverage probability depends on the characteristics of the underlying distribution, mainly the kurtosis; thus, the good or bad behavior of the performance is controlled by the behavior of the Edgeworth approximation for the standardized underlying distribution.

2.4. Accelerated Sequential Scheme

Hall [29] introduced the accelerated sequential schemes where the bulk sampling in the main study phase of the three-stage procedure is replaced by purely sequential sampling in the main study phase. The fine-tuning phase, however, remains the same as bulk sampling. The procedure starts with a pilot sample as before to initiate the sampling procedure and calculate $\bar{X}_m$ and S_m^2 as initial estimates of both μ and θ respectively. The following defines the main study phase

$$N_1 = \inf\left\{n \geq m : n \geq \gamma\left(a/d\right)^2 S_n^2\right\} \tag{6}$$

If the decision is to continue sampling, an extra sample of size $N_1 - m$ is to be randomly drawn from the population to update the estimates of both μ and θ by $\bar{X}_{N_1}$ and $S_{N_1}^2$ respectively. The fine-tuning phase based on bulk sampling,

$$N = \max\left\{N_1, \left[\left(a/d\right)^2 S_{N_1}^2\right] + 1\right\} \tag{7}$$

If we continue sampling, an extra sample of size $N - N_1$ is to be randomly selected and augmented with previous samples. Whenever sampling is terminated, we update the estimates of $\bar{X}_N$ and S_N^2 then proceed to perform the required inference. The accelerated sequential schemes enjoy the same asymptotic characteristics of the three-stage sampling procedure and achieve both the asymptotic efficiency and consistency.

2.5. Progressive Sequential Procedure

In this study, we propose a new sampling procedure called the *progressive* sequential sampling procedure, which is very much the same as the accelerated sequential scheme where we interchange the main study phase by bulk sampling and ends by the fine-tuning phase. The asymptotic characteristics of the progressive sampling technique will appear somewhere else shortly.

In the following lines, we present the progressive sampling procedure in three phases: *The pilot-phase*, draw a pilot sample of size $m \geq 2$ from the distribution function and calculate both $\bar{X}_m$ and S_m^2 as initial estimates of μ and θ respectively. Second, the *Main-study phase*, for a given γ , $0 < \gamma < 1$, (design factor) we have

$$N_1 = \max\left\{m, \left[\gamma\left(a/d\right)^2 S_m^2\right] + 1\right\} \tag{8}$$

The *Fine-tuning phase*, once we stop sampling from the previous stage, we continue sampling based on

$$N = \inf\left\{n \geq N_1 : n \geq \left(a/d\right)^2 S_n^2\right\} \tag{9}$$

Upon termination of the sampling process, we propose to estimate μ and θ by the corresponding sample measures $\bar{X}_N$ and S_N^2, respectively and proceed to perform the required inference.

Table 1 below shows the asymptotic characteristics of multistage procedures, as shown and discussed in the literature.

As one can see, all the results of multistage sampling procedures are asymptotic. Therefore it will be of interest to study the moderate sample size performance of all procedures using Monte Carlo simulation. Other criteria can be considered in our comparison, including the practical implementation of the procedure in real-life problems.

Table 1. Asymptotic Characteristics of Multistage procedures

Procedure	First-order	Second-order	Consistent	Asy. Consistent	Point estimation	Confidence estimation	Oversample
Two-stage	No	No	Yes	Yes	No	Yes	Yes
One-by-one	Yes	Yes	No	Yes	Yes	Yes	No
Three-stage	Yes	Yes	No	Yes	Yes	Yes	Yes
Accelerated	Yes	Yes	No	Yes	Yes	Yes	Yes

We have to emphasize here that both the three-stage and the accelerated procedure may lead to oversampling if the choice of m is somehow small relevant to n^*. Table 2 demonstrates this behavior at $n^* = 500$.

Table 2. Asymptotic Characteristics of the three-stage procedure at $m = 5$, $\gamma = 0.5$, $1-\alpha = 0.95$

n^*	$\overline{N}$	SE($\overline{N}$)	μ	SE(μ)	$1-\alpha$
24	19.98	0.05068	0.00139	0.00126	0.86712
43	37.90	0.08094	-0.00090	0.00094	0.88144
61	55.63	0.10364	-0.00119	0.00077	0.89650
76	70.88	0.12036	0.00070	0.00067	0.90408
96	91.05	0.13958	0.00056	0.00058	0.91578
125	120.52	0.16532	-0.00056	0.00049	0.92310
171	168.11	0.19977	-0.00025	0.00040	0.93072
246	244.98	0.24990	0.00034	0.00032	0.93988
500	506.01	0.41009	0.00017	0.00021	0.94606

3. MONTE CARLO SIMULATION RESULTS

Monte Carlo simulation is a computer-based method that depends on repeated random sampling and statistical analysis to compute approximate results generated from the procedure. These results are indications of the performance of the procedure. Although the results are not exact and depend on the number of repeated runs, it is still reliable in a sense it gives a wide view of how likely the estimates should be and determines whether the procedure is applicable or not. For more details, see [30].

3.1. Simulation Results

We proceed with the simulation study by coding FORTRAN programs using Microsoft Developer Studio to generate a series of simulations. For each experimental situation, 50,000 replicate samples were used. Random samples from the standard normal distribution were generated and a two-stage sampling rule (2), purely sampling rule (3), three-stage sampling rule

(4) - (5), the accelerated sequential sampling scheme (6)-(7) and the progressive sampling rule (8)-(9) were implemented to estimate the mean μ and its standard error s$\left(\mu\right)$, $\overline{N}$ the estimated values of n^* and its standard errors s$\left(\overline{N}\right)$ and finally, the estimated coverage probability $1-\alpha$. The starting sample size ranges from $m = 5$, 10 and 15, while n^* ranges from small, moderate to large sample sizes $n^* = 24$, 43, 61, 76, 96, 125, 171, 246, and 500 as recommended by Hall [20], the design factor are chosen $\gamma =$ 0.3, 0.4, 0.5, 0.6, 0.7 and 0.8. Regarding the coverage probability, we take $(1-\alpha) = 0.95$. For brevity, we report the case at $m = 10$.

Table 3 below shows the simulation results for the first four multistage procedures. Regarding the two-stage procedure, we noticed that the procedure achieves both the first and the second order asymptotic efficiency, that is $\overline{N} \simeq n^*$ and $\overline{N} - n^*$ is bounded by a finite number of observations contrary to the theoretical results. Also, the standard errors $\overline{N}$ are slightly higher than the other multistage procedures due to the high variability of the final sample size. The estimates are nearly unbiased with standard errors decrease as n^* increases. Regarding the estimated coverage probability, the two-stage procedure satisfies the requirement of exact consistency. It yields coverage probabilities exceed the required nominal value.

Regarding the purely sequential procedure, the three-stage and the accelerated sequential procedure all generate estimates that satisfy all measures except the exact consistency. They generate estimate coverage probabilities that are always less than the prescribed nominal value while they attain the nominal values only asymptotically.

Table 3. Moderate and large sample size performance of multistage procedures

Two-stage						*Purely*					
n^*	$\overline{N}$	SE($\overline{N}$)	μ	SE(μ)	$1-\alpha$	n^*	$\overline{N}$	SE($\overline{N}$)	μ	SE(μ)	$1-\alpha$
24	24.64	0.0498	0.0010	0.0010	0.9570	24	22.34	0.0328	0.0003	0.0010	0.9256
43	43.36	0.0907	-0.0000	0.0008	0.9504	43	40.87	0.0481	0.0004	0.0007	0.9296
61	61.58	0.1288	-0.0006	0.0006	0.9512	61	59.21	0.0555	-0.0005	0.0006	0.9392
76	76.57	0.1604	-0.0001	0.0006	0.9515	76	74.34	0.0600	-0.0000	0.0005	0.9430
96	96.28	0.2027	0.0005	0.0005	0.9488	96	94.37	0.0662	0.0001	0.0005	0.9456
125	125.47	0.2630	-0.0000	0.0004	0.9514	125	123.72	0.0733	-0.0006	0.0004	0.9448
171	171.70	0.3616	0.0004	0.0004	0.9512	171	169.66	0.0849	0.0006	0.0004	0.9472
246	246.40	0.5173	-0.0001	0.0003	0.9506	246	244.58	0.1008	-0.0001	0.0003	0.9489
500	500.92	1.0569	-0.0005	0.0002	0.9511	500	498.72	0.1426	-0.0001	0.0002	0.9497
Three Stage						*Accelerated*					
n^*	$\overline{N}$	SE($\overline{N}$)	μ	SE(μ)	$1-\alpha$	n^*	$\overline{N}$	SE($\overline{N}$)	μ	SE(μ)	$1-\alpha$
24	20.39	0.0447	-0.0012	0.0011	0.8917	24	22.46	0.0366	-0.0013	0.0010	0.9205
43	38.38	0.0680	0.0003	0.0008	0.9050	43	38.73	0.0615	-0.0002	0.0008	0.9155
61	56.23	0.0839	0.0002	0.0007	0.9176	61	55.70	0.0798	-0.0001	0.0006	0.9179
76	71.15	0.0949	0.0003	0.0006	0.9234	76	70.52	0.0910	0.0001	0.0006	0.9249
96	91.31	0.1067	0.0004	0.0005	0.9309	96	90.68	0.1011	0.0001	0.0005	0.9298
125	120.45	0.1221	-0.0005	0.0004	0.9336	125	120.13	0.1117	0.0005	0.0004	0.9371
171	166.82	0.1421	0.0003	0.0004	0.9419	171	166.60	0.1265	0.0003	0.0004	0.9416
246	242.58	0.1706	-0.0001	0.0003	0.9438	246	242.08	0.1460	0.0003	0.0003	0.9456
500	497.72	0.2469	-0.0001	0.0002	0.9488	500	496.39	0.2029	-0.0002	0.0002	0.9468

3.2. Simulation Results Regarding the Progressive Sampling Procedure

The simulation process performs as follow;

First: draw a pilot sample of size $m = 10$ from the standard normal distribution.

Second: compute $\bar{X}_m = m^{-1}\sum_{i=1}^{m} X_i$ and $S_m^2 = (m-1)^{-1}\sum_{i=1}^{m}\left(X_i - \bar{X}_m\right)^2$.

Third: apply the progressive sampling rule given in (8) with a specific combination of m, n^* α and γ. That is

$$N_1 = \max\left\{10, \left[\gamma\,(1.96)^2\left(S_m/d\right)^2\right]+1\right\},$$

If $m \geq N_1$, stop sampling at this stage, else draw a bulk sample of size $N_1 - m$. The resultant sample at this stage is $\left(X_1,\ldots,X_m,X_{m+1},\ldots,X_{N_1}\right)$, update the estimates $\bar{X}_{N_1}$ and $S_{N_1}^2$ and proceed to rule (9). That is,

$$N = \inf\left\{n > N_1 : n \geq (1.96/d)^2 S_{N_1}^2\right\}.$$

If $N_1 \geq N$ a stop at this stage else take additional observation and update the sample until the above rule satisfied, the final random sample is $\left(X_1,\ldots,X_m,X_{m+1},\ldots,X_{N_1},X_{N_1+1},\ldots,X_N\right)$.

That is, for i-th iteration, we obtain two arrays of size $K = 50{,}000$ (number of simulations), $N_1^*, N_2^*, \ldots, N_K^*$ and $\bar{X}_1^*, \bar{X}_2^*, \ldots, \bar{X}_K^*$.

Let $\overline{N} = \sum_{i=1}^{K} N_i^*/K$ and $\hat{\mu} = \bar{X} = \sum_{i=1}^{K} \bar{X}_i^*/K$, where $\overline{N}$ and $\bar{X}$ are respectively the estimated mean sample size and the estimated mean of the estimator of the population mean across replicates. Thus, $\hat{\mu} = \bar{X}$ it may be regarded as an estimate of the expected value of the estimator of the population mean μ. The standard errors are $SE\ \hat{\mu} = s_1/\sqrt{K}$, and $SE\ \overline{N} = s_2/\sqrt{K}$, where $s_1 = \sqrt{\sum_{i=1}^{K} \bar{X}_i - \hat{\mu}^{\,2} \Big/ K-1}$ and $s_2 = \sqrt{\sum_{i=1}^{K} \overline{N}_i - \overline{N}^{\,2} \Big/ K-1}$.

Regarding the simulated coverage probability, we count the number of times the true mean lies between $\bar{X}_N \pm d$ divided by the total number of simulations.

Table 4 below shows the performance of the progressive sampling procedure under $m = 10$, $1-\alpha = 0.95$, and different values γ. The shaded line reflects the best performance of the procedure at a specified γ. Obviously, for small, n^* we need a high value of γ, while as n^* increases the value of γ decreases. The procedure satisfies all the measures except the exact consistency. Table 5 below demonstrates the moderate sample performance of multistage procedures and shows the significant improvement of multistage procedures under moderate sample sizes.

Figure 1 below shows the performance of the simulated coverage probability for the three-stage, the accelerated procedure, and the progressive procedure. The progressive procedure has better performance than the others even for small values of n^*.

Table 4. The performance of the Progressive sequential procedure

γ	$\overline{N}$	S($\overline{N}$)	μ	S(μ)	$1-\alpha$	$\overline{N}$	S($\overline{N}$)	μ	S(μ)	$1-\alpha$
$n^* = 24$						$n^* = 43$				
0.3	21.5	0.0317	0.0004	0.0010	0.9202	39.79	0.0485	-0.0001	0.0008	0.9261
0.4	21.5	0.0317	0.0013	0.0010	0.9196	39.78	0.0488	0.0011	0.0008	0.9263
0.5	21.6	0.0319	0.0013	0.0010	0.9205	39.88	0.0492	-0.0007	0.0007	0.9278
0.6	21.7	0.0328	-0.0017	0.0010	0.9191	40.51	0.0511	-0.0020	0.0007	0.9296
0.7	22.4	0.0353	-0.0005	0.0010	0.9197	41.64	0.0555	-0.0006	0.0007	0.9315
0.8	23.6	0.0390	-0.0003	0.0010	0.9271	43.51	0.0615	-0.0006	0.0007	0.9367
$n^* = 61$						$n^* = 76$				
0.3	58.1	0.0564	0.0005	0.0006	0.9349	73.24	0.0610	-0.0004	0.0005	0.9397
0.4	58.0	0.0567	0.0004	0.0006	0.9347	73.27	0.0612	0.0005	0.0005	0.9399
0.5	58.3	0.0573	0.0003	0.0006	0.9356	73.58	0.0624	-0.0004	0.0005	0.9389
0.6	59.1	0.0612	0.0006	0.0006	0.9379	74.58	0.0668	-0.0000	0.0005	0.9423
0.7	60.7	0.0671	-0.0009	0.0006	0.9405	76.45	0.0763	-0.0001	0.0005	0.9412
0.8	63.2	0.0775	-0.0004	0.0006	0.9436	79.52	0.0892	0.0004	0.0005	0.9470
$n^* = 96$						$n^* = 125$				
0.3	93.4	0.0665	-0.0000	0.0005	0.9417	122.64	0.0743	-0.0003	0.0004	0.9450
0.4	93.5	0.0668	0.0003	0.0005	0.9431	122.65	0.0744	-0.0006	0.0004	0.9450
0.5	93.9	0.0688	-0.0000	0.0005	0.9428	123.15	0.0768	-0.0000	0.0004	0.9462
0.6	95.1	0.0742	0.0003	0.0005	0.9450	124.66	0.0866	-0.0003	0.0004	0.9449
0.7	97.7	0.0875	-0.0004	0.0005	0.9479	127.92	0.1062	0.0000	0.0004	0.9494
0.8	101.3	0.1071	0.0006	0.0005	0.9489	132.79	0.1323	-0.0002	0.0004	0.9500
$n^* = 171$						$n^* = 246$				
0.3	168.7	0.0850	0.0002	0.0003	0.9453	243.80	0.1011	-0.0001	0.0003	0.9480
0.4	168.6	0.0855	-0.0003	0.0003	0.9458	243.89	0.1019	0.0000	0.0003	0.9489
0.5	169.5	0.0902	-0.0005	0.0003	0.9484	244.98	0.1104	0.0004	0.0003	0.9471
0.6	171.6	0.1045	-0.0003	0.0003	0.9469	248.18	0.1359	-0.0002	0.0003	0.9494
0.7	176.3	0.1345	-0.0003	0.0003	0.9493	254.16	0.1797	-0.0001	0.0003	0.9508
0.8	182.5	0.1715	0.0003	0.0003	0.9543	263.17	0.2369	0.0003	0.0003	0.9524
$n^* = 500$										
0.3	497.9	0.1418	-0.0001	0.0002	0.9506					
0.4	498.3	0.1467	-0.0001	0.0002	0.9500					
0.5	500.5	0.1707	0.0002	0.0002	0.9492					
0.6	507.0	0.2314	-0.0001	0.0002	0.9507					
0.7	519.2	0.3356	-0.0000	0.0002	0.9503					
0.8	537.8	0.4590	0.0000	0.0002	0.9544					

Table 5. Moderate sample size performance of Multistage Procedures

Procedure	First-order	Second-order	Consistent	Asy. Consistent	Point estimation	Confidence estimation	Oversampling
Two-stage	Yes	Yes	Yes	Yes	Yes	Yes	No
Purely	Yes	Yes	No	Yes	Yes	Yes	No
Three-stage	Yes	Yes	No	Yes	Yes	Yes	No
Accelerated	Yes	Yes	No	Yes	Yes	Yes	No
Progressive	Yes	Yes	No	Yes	Yes	Yes	No

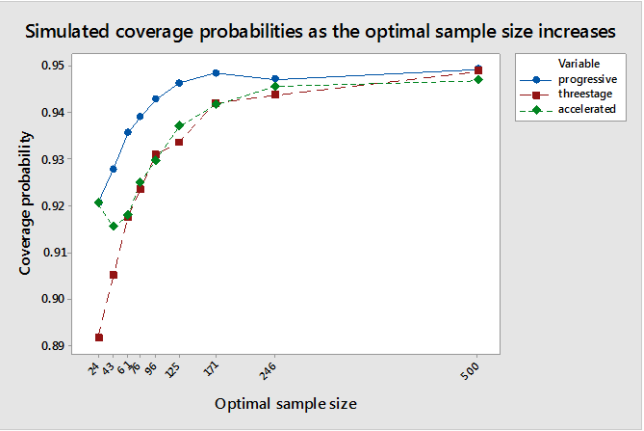

Figure 1. The simulated coverage probability for different multistage procedures at $\gamma =$ **0.5**

4. CONCLUSION

From a practical standpoint, we conclude that the two-stage procedure is more practical and easy to implement in real life situations, while the one-by-one sequential procedure is time-consuming and costly to implement. For the three-stage, accelerated and progressive, we recommend the progressive sampling procedure due to the following: first, both the three-stage and the accelerated procedure are recommended to be applied at $\gamma = 0.5$. Second, both procedures have a final stage that depends on bulk sampling, which may lead to oversampling. The progressive procedure can be applied for different design factors 0.3-0.8 and ends with fine-tuning, one-by-one sampling. We noticed as the optimal sample size increases the design factor decreases to sustain efficient estimates for the mean and the coverage probability.

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