THE EXISTENCE OF GLOBAL ATTRACTORS FOR SUSPENSION BRIDGE EQUATIONS

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ABSTRACT

The aim of this paper is to establish a well-posedness result and the existence of finite-dimensional global attractors for a model of a coupled suspension bridge as well as the regularity of global attractor is achieved. This result generalizes the previous result in [6].

Keywords: Suspension bridge, global attractors, bounded absorbing set.

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1. INTRODUCTION

Tacoma Narrows bridge collapse is certainly the most impressive setback in the history. The crucial event in the collapse was a sudden change from vertical to torsional oscillations.

From the physical point of view, the suspension bridge equation describes the transverse deflection of road bed in the vertical plane. On the other hand, from the mathematical point the suspension bridge model describes the vibration of the vertical plane.

The mathematical model appears necessarily a precise description of the instability and the structural behavior of suspension bridge under the action of the load which reveals its lifelong, the nonlinear behavior of suspension bridge, which is by now well established, also plays a crucial role in causing oscillations. The reliable model for suspension should be nonlinear and it should have enough degrees of freedom to display torsional oscillations.

To motivate our work let us start with some works for example a single equation of suspension bridge, Messaoudi et al [8] suggested the following problem:

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\[
\begin{align*}
\left\{ \begin{array}{l}
ut_t(x,y,t) + \delta u_t(x,y,t) + \mu A u(x,y,t) - \int_0^t g(t-s) A u(x,y,s) \, ds + h(u) = f & \quad \in \Omega \times (0,T) \\
u(0,y,t) = u_{xx}(0,y,t) = 0 & \quad \text{for } (y,t) \in (-L,L) \times (0, +\infty) \\
u(\pi,y,t) = u_{xx}(\pi,y,t) = 0 & \quad \text{for } (y,t) \in (-L,L) \times (0, +\infty) \\
u_{yy}(x,\pm L,t) + \delta u_{xx}(x,\pm L,t) = 0 & \quad \text{for } (y,t) \in (0,\pi) \times (0, +\infty) \\
u_{yyy}(x,\pm L,t) + (2 - \delta) u_{xx}(x,\pm L,t) = 0 & \quad \text{for } (y,t) \in (0,\pi) \times (0, +\infty) \\
u(x,y,0) = u_0(x,y), u_t(x,y,0) = u_1(x,y) & \quad \text{in } \Omega \\
\end{array} \right.
\]

where \( \delta, \mu > 0 \) are constants, \( u(x,y,t) \) is the vertical displacement of the plate in the downward direction, \( h(u) \) is a restoring force due to hangers of the suspension bridge, \( f \in \mathbb{L}^2(\Omega) \) is an external force which also includes the gravity. The memory kernel \( g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is an absolutely continuous function which may blow up at 0. They gave a rigorous well-posedness result and established the existence of a global attractor.

Recently, Lazer and McKenna [5] studied the nonlinear oscillation problems in suspension bridge and presented a (one-dimensional) mathematical model for a suspension as a new problem of nonlinear analysis where they modeled a suspension bridge as a rectangular plate since the plate is perfectly correct and corresponds mechanically to a vibrating suspension bridge. Gazzola [3] suggested an equation with linearized stretching term
\[
\Delta^2 u - \delta \Delta u = f \quad \text{in } \Omega.
\]

Here \( u = u(x,t) \) represents the vertical displacement of the plate, and \( f \) is an external force including the gravity.

The plate is assumed to be suspended by its vertical edges
\[
u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0,
\]
and the horizontal edges are free
\[
u_{y}(x,\pm l) + \sigma u_{xx}(x,\pm l) = u_{yyy}(x,\pm l) + (2 - \sigma) u_{xx}(x,\pm l) - u_t(x,\pm l) = 0
\]
for all \( y \in (-L,l) \) and all \( x \in (0,\pi) \), where \( 0 < \sigma < \frac{1}{2} \) is the Poisson ratio.

It is well known that Ma and Wang [7] presented the following nonlinear problem which describes a vibrating beam equation coupled with a vibrating string equation
\[
\begin{align*}
\left\{ \begin{array}{l}
ut_t + \alpha u_{xxxx} + \delta_1 u_t + k(u - v) + f_s(u) = h_B(x,t) & x \in [0, L] \\
\nu_{tt} - \beta v_{xx} + \delta_2 \nu_{tt} - K(u - v) + f_s(v) = h_S(x,t) & x \in [0, L]
\end{array} \right.
\end{align*}
\]
with the simply supported boundary-value conditions
\[
u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \quad \nu(0,t) = v(L,t) = 0, \quad t \geq \tau
\]
such that
\[
(u-v)^+ = \begin{cases} u - v, & \text{if } u - v > 0 \\ 0, & \text{if } u - v \leq 0 \end{cases}
\]

where \( k > 0 \) denotes the spring constant of the ties, \( \alpha > 0 \) and \( \beta > 0 \) are the flexural rigidity of the structure and coefficient of tensile strength of the cables, respectively
\( \delta_1, \delta_2 > 0 \) are constants, the forces term \( h_B, h_S \in \mathbb{L}^2_{loc}[\mathbb{R},\mathbb{L}^2(0,L)] \) the nonlinear functions \( f_s(u), f_s(v) \in C^2(\mathbb{R}) \) are the source terms.

They proved the existence of pullback D-attractors for the non-autonomous coupled suspension bridge equations with suspended and clamped ends. Similar models have been studied by several authors, we refer the readers to [16],[9],[10] and the references therein. For example Jum-Ran Kang [4] investigated the long-time behavior of a solution to the following thermoelastic suspension bridge equation with linear memory.

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\[
\begin{aligned}
\left\{ \begin{array}{l}
\ddot{u} + \alpha \Delta^2 u - \Delta u + K u + \int_0^\infty \mu(s) \Delta^2 u(t-s) ds + \beta \Delta \phi \\
+f(u) = h(x) \quad \text{in } \Omega \times (0, \infty)
\end{array} \right.
\end{aligned}
\]

\[
\begin{aligned}
\ddot{\phi} - \Delta \phi - \beta \Delta u - \int_0^\infty k(s) \Delta \phi(t-s) ds \\
\text{in } \Omega \times (0, \infty)
\end{aligned}
\]

such that

\[(u-v)_+ = \begin{cases} 
\begin{array}{ll}
u - v, & \text{if } u - v > 0 \\
0, & \text{if } u - v \leq 0
\end{array}
\end{cases}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \). Here \( \alpha \) is the flexural rigidity of the structure, and \( \beta > 0 \) provides connection between deflection and temperature and depends on mechanical and thermal properties of the material. They showed the existence of a compact global attractor.

In [12] the authors suggested the following problem of suspension bridges:

\[
\begin{aligned}
\begin{cases}
\mathcal{M} u + \mathcal{E} u + \mathcal{L} u + f(u) = 0 & \text{in } (0, L) \times (0, +\infty) \\
\mathcal{E} v + \mathcal{L} v + g(v) = 0 & \text{on } (0, L) \times (0, +\infty)
\end{cases}
\end{aligned}
\]

(1)

such that \( \mathcal{M}, \mathcal{E}, \mathcal{L}, \mathcal{A}, \mathcal{C}_1, \mathcal{C}_2 \) 1 and \( I \) are well determined in [12], by using a continuous model of the suspension bridge and by a quasi stationary approach, a simple formula of the combined vertical/torsional flutter wind speed is given. A good agreement is obtained comparing the predictions from the proposed formula with the flutter speeds of three modern suspension or cable stayed bridges. A more slightly sophisticated and complicated string-beam model was suggested by Lazer-McKenna [5]. They treated the cable as a vibrating string coupled with the vibrating beam of the roadway by piecewise linear springs having a spring constant \( k \) if expanded, but no restoring force if compressed. The sustaining cable is subject to some forcing term such as the wind or the motions in the towers. This leads to the following system:

\[
\begin{aligned}
\begin{cases}
\ddot{u} - \delta_1 u + \mathcal{K}_1 (u - v) = f(x, t) & \text{in } (0, L) \times \mathbb{R}^+ \\
\ddot{v} + \delta_2 u + \mathcal{K}_2 (u - v) = \mathcal{W}_0 & \text{on } (0, L) \times \mathbb{R}^+
\end{cases}
\end{aligned}
\]

such that

\[(u-v)_+ = \begin{cases} 
\begin{array}{ll}
u - v, & \text{if } u - v > 0 \\
0, & \text{if } u - v \leq 0
\end{array}
\end{cases}
\]

where \( v \) is the displacement from equilibrium of the cable and \( u \) is the displacement of the beam, both measured in the downward direction. \( \delta_1, \delta_2 \) are respectively positive constants and the constants \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are the spring constants and satisfy \( \mathcal{K}_2 \leq \mathcal{K}_1 \). The two damping terms can possibly be set to 0, while \( \mathcal{W}_0 \) are the forcing terms. They proved the existence and multiplicity of periodic solutions of mathematical model of nonlinearly supported bending beams, and they showed also some nonlinear behaviors as observed in large-amplitude flexings in suspension bridges.

In the present paper, we consider a plate model that better describes torsional oscillations in suspension bridges, we consider a variant of (1), we add to the equation the term \( h_1(u), h_2(v) \) which represent the restoring force due to the hangers of the suspension bridge, and a convolution term which means that the stress at any instant \( t \) depends on the whole history of strains, here \( f_1 \) and \( f_2 \) are nonlinear source terms.

We omit the space variables \( x, y \) of \( u(x,t), v(x,y,t) \), \( u(x,y,t) \) and \( v(x,y,t) \) and for simplicity denote \( u(x,y) = u \), \( v(x,y) = v \), \( u(x,y,t) = u_t \) and \( v(x,y,t) = v_t \), when no confusion arises also the functions considered are all real valued, here \( u_t = \partial u(t)/\partial t \), \( u_{tt} = \partial^2 u(t)/\partial t^2 \), \( v_t = \partial v(t)/\partial t \) and \( v_{tt} = \partial^2 v(t)/\partial t^2 \).

We consider the modified suspension bridge problem.
In memories terms and more general form of source terms, we establish several results concerning knowledge, there are no results concerning coupled suspension bridge in the presence of influence of the viscoelastic, source on the behavior of bridge.

We identify some parameters that arise in the equation (2), for example, \( x \) and \( y \) are the space variables along the beam in the bounded domain \( \Omega \).

- \( t \) denotes the time variable.
- \( u \) and \( v \) denote respectively the vertical and torsional components of the oscillation of the bridge.
- \( ut, vt \) represent the damping terms, the damping are produced by processes that dissipate the energy stored in the oscillation.
- \( f_1(x,y) \) and \( f_2(x,y) \) are the lift and the moment for unit girder length of the self-excited forces
- \( h_1(u) \) and \( h_2(v) \) represent restoring force due to the hangers of the suspension bridge
- \( \mu_1(\cdot), \mu_2(\cdot) \) represent the viscoelastic materials are a kind of materials that have the properties of keeping past information (memories) and which will be used in the future.
- \( E \) and \( I \) are, respectively, the elastic modulus and the moment of inertia of the stiffening girder so that \( EI \) is the stiffness of the girder
- \( m \) denotes the mass per unit length
- \( I_0 \) The polar moment of inertia of the girder section
- \( 2\ell \) the roadway
- \( w = mg \) is the weight which produces a cable stress whose horizontal component is \( H_w \)
- \( C_1 \) and \( C_2 \) are, respectively the warping and torsion.

Motivated by the previous works, in the present paper, it is interesting to analyze the influence of the viscoelastic, source on the behavior of (2). Under suitable assumptions on the functions \( \mu_i(\cdot), f_i(\cdot)(i = 1,2) \), the initial data and the parameters in the equations, to the best of our knowledge, there are no results concerning coupled suspension bridge in the presence of memories terms and more general form of source terms, we establish several results concerning existence and regularity of global attractor.

The scope of this paper is as follows: In Section 2, we give some preliminaries and main result. In Section 3, we prove the existence of global attractor, firstly, we prove the existence of
an absorbing set, then, establish the smoothness. In Section 4, we verify the regularity of global attractors.

2. MAIN RESULT

For simplicity, one denote \( m = E = I = w = H_w = 1, C_1 = 1 \) and \( C_2 = \ell = \frac{1}{2} \)

Now, we present the following conditions about memory kernel

Assumption (H)

i) \( \mu_1, \mu_2 \in C(0, +\infty) \cap L_1(0, +\infty) \)
\[ \mu_1(s) \leq 0 \leq \mu_1(s), \quad \forall s \in (0, +\infty) \]
\[ \mu_2(s) \leq 0 \leq \mu_2(s), \quad \forall s \in (0, +\infty) \]

(ii) \( \mu_1 \in C(0, +\infty) \cap L_1(0, +\infty) \)
\[ \mu_2(s) = 1 - \mu_0 > 0, \quad \forall s \in (0, +\infty) \]
\[ \mu_2(s) = 1 - \mu_0 > 0, \quad \forall s \in (0, +\infty) \]

Assumption (G)

Concerning the forcing term \( h_i : \mathbb{R} \to \mathbb{R}, \ i = 1, 2 \), we assume that

(i) \( h_i(0) = 0, \ |h_i(u) - h_i(\bar{u})| \leq K_0 (1 + |\mu|^p + |\bar{u}|^p) u - \bar{u} | \forall u, \bar{u} \in (0, +\infty) \)

where \( \delta > 0 \) and \( p > 0 \). The condition \( p > 0 \) implies that such that

\[
\lim_{|s| \to \infty} \inf \frac{h_i(s)}{s} \geq \delta, \quad i = 1, 2
\]
\[\lim_{|s| \to \infty} \sup \frac{|h_i(s)|}{s} = 0, \quad i = 1, 2\] (4)

Where \( 0 \leq P < \infty \)

As in Dafermos [2], we introduce the relative displacement past history functions as

\[
\phi_1(t, y, s) = u(x, y, t) - u(x, y, t - s), \quad \forall s \geq 0
\]
\[\phi_2(t, y, s) = u(x, y, t) - u(x, y, t - s), \quad \forall s \geq 0\] (5)

Then

\[
\phi_1(t, y, s) + \phi_1(s, y, s) - ut = 0, \phi_1(0, y, 0) = 0
\]
\[\phi_0(t, y, s) = u(0; x, y) - u(x, y, t - s) = w_1(s)\]
\[\phi_2(t, y, s) + \phi_1(s, y, s) - Vt = 0, \phi_2(0, y, 0) = 0\]
\[\phi_2(t, y, s) = V0(x, y) - V(x, y, t - s) = w_2(s)\] (6)

where \( w_1, w_2 \) represents the history of \( u, v \). Consequently, the problem equivalent to

\[
\begin{align*}
\uptt + \int_0^s \mu_1(s) \Delta^2 u - \Delta u + \upt + \int_0^\infty \mu_1(s) \Delta^2 \phi_1(t, y, s) ds + h_1(s) &= f_1(x, y) \quad \text{in} \ \Omega \times (0, \infty) \\
Vtt + \int_0^s \mu_2(s) \Delta^2 V - \Delta V + Vt + \int_0^\infty \mu_2(s) \Delta^2 \phi_2(t, y, s) ds + h_2(s) &= f_1(x, y) \quad \text{in} \ \Omega \times (0, \infty)
\end{align*}
\] (7)

with the boundary conditions
We consider the relative displacement
$$\phi_i(t, x) = u_i(t, x) - v_i(t, x), \quad \phi_2(t, x) = u_2(t, x) - v_2(t, x)$$
and the norm
$$\|\phi\|_p = \left( \int_0^T \int_{\Omega} |\phi|^p \, dx \, dt \right)^{1/p}$$
where $\Omega$ is the domain of the problem. The solution $u(t, x)$ and $v(t, x)$ are assumed to be in $L^2(\Omega)$.

We will use the standard functional space and denote $(\cdot, \cdot)$ be a $L^2(\Omega)$- inner product and $\| \cdot \|_p$ be $L^p(\Omega)$ norm. Especially, we take
$$H = V = L^2(\Omega), \quad V = V_1 = V_2 = H^2(\Omega)$$
with
$$H^2(\Omega) = \{ \zeta \in H^2(\Omega) : \zeta = 0 \text{ on } [0, \pi] \times \{-L, L\}$$
equipped with the inner product and norm respectively
$$(u, v) = (Au, Av), \quad \|u\|_V = \|Au\|_2$$
Define
$$D(A) = \{ u, v \in H^2(\Omega) : (Au, Av) \text{ hold } \}$$
such that, $Au = \Delta^2 u$, and equip this space with the inner product $(Au, Av)$, and the norm $\|Au\|_2 = (Au, Au)$. We have the following continuous dense injections
$$D(A) \subset V \subset H \subset H^* \subset V^*$$
the dual spaces of $H$, $V$ respectively.

We consider the relative displacement $\phi_1$, $\phi_2$ as new variables and we introduce the weighted $L^2$-space as follows
$$L^2(R^2, \mu i) = \{ \phi : R^2 \to V_i / \int_0^\infty \mu_i(s)|\phi_i(s)|^2 ds < \infty \}$$
which is a Hilbert space endowed with the inner product and norm
$$(u, v)_i = \int_0^\infty \mu_i(r)(u(r)V(r))_i \, dr$$
$$\|u\|_{i, V_i} = \int_0^\infty \mu_i(r)|u_i(r)|^2 dr, \quad i = 1, 2$$
respectively, where $V_i = D(A^2).$ Finally, we introduce the following Hilbert spaces
$$H_0 = V \times V \times H \times L^2_1(R^2; V) \times L^2_2(R^2; V)$$
$$H_1 = D(A) \times D(A) \times V \times V \times L^2_{1,1}(IR^2; D(A)) \times L^2_{2,2}(IR^2; D(A)),$$
equipped with the norms.
The Existence of Global Attractors for ...

\[ \left| \mathbf{u},\mathbf{v},\mathbf{u}_t,\phi_1,\phi_2 \right|_{H^0} \leq \left| \mathbf{u},\mathbf{v},\mathbf{u}_t,\phi_1,\phi_2 \right|_{L^2}^2 + \left| \mathbf{u},\mathbf{v},\mathbf{u}_t,\phi_1,\phi_2 \right|_{V^2}^2 + \left| \mathbf{u},\mathbf{v},\mathbf{u}_t,\phi_1,\phi_2 \right|_{L^2(A)}^2 \]

and

\[ \left| \mathbf{u},\mathbf{v},\mathbf{u}_t,\phi_1,\phi_2 \right|_{H^0} = \left| \mathbf{u},\mathbf{v},\mathbf{u}_t,\phi_1,\phi_2 \right|_{L^2}^2 + \left| \mathbf{u},\mathbf{v},\mathbf{u}_t,\phi_1,\phi_2 \right|_{V^2}^2 + \left| \mathbf{u},\mathbf{v},\mathbf{u}_t,\phi_1,\phi_2 \right|_{L^2(A)}^2 \]

We will assume a Poincaré inequality

\[ \tau \left| \mathbf{v} \right|_{L^2}^2 \leq \left| \mathbf{u}_t \right|_{V}^2 \quad \forall \mathbf{v} \in V \]

where \( \tau \) denotes the first eigenvalue of \( \Delta^2 \nu = \tau \nu \) in \( \Omega \).

Using the semigroup theory (see [11]) we can conclude the following theorem

**Theorem 2.1** Assume that assumption (H) hold and \( f_1, f_2 \in \mathbb{L}^2(\Omega) \). Then the problem (7) – (9) has a weak solution \( (u, v, u_t, \phi_1, \phi_2) \in C([0, T]; H^0) \) with the initial data \( u_0, v_0, u_1, \phi_1, \phi_2 \in H^0 \), satisfying

\[ (u, v) \in L^\infty(0, T; \mathbb{L}^2) \; ; \; (u_t, v_t) \in L^\infty(0, T; \mathbb{V}) \; ; \; \phi' \in L^\infty(0, T; \mathbb{L}^2) \]

and the mapping \( \{ u_0, v_0, u_1, \phi_1, \phi_2 \} \rightarrow \{ (u(t), v(t), u_1(t), v_1(t), \phi_1(t), \phi_2(t)) \} \) is continuous in \( H^0 \). In addition, if \( z^n(t) = (u^n(t), v^n(t), u_1^n(t), v_1^n(t), \phi_1^n(t), \phi_2^n(t)) \) is a weak solution of the problem (7) – (9) corresponding to the initial data \( z^n(0) = (u^n_0, v^n_0, u_1^n, v_1^n, \phi_1^n, \phi_2^n) \), then one has

\[ \left| z_1(t) - z_2(t) \right|_{H^0} \leq e^c \left| z_1(0) - z_2(0) \right|_{H^0}, \quad t \in [0, T], \]

for some constant \( c \geq 0 \).

The well-posedness of the problem (7) – (9) implies that the family of operators \( S(t) : H^0 \rightarrow H^0 \) defined by

\[ S(t)(u_0, v_0, u_1, \phi_1, \phi_2) = (u, v, u_t, v_t, \phi_1 t, \phi_2 t), \quad t \geq 0, \]

where \( (u, v, u_t, v_t, \phi_1 t, \phi_2 t) \) is the unique weak solution of the problem (7) – (9), satisfies the semigroup properties and defines a nonlinear \( C_0 \)-semigroup, which is locally Lipschitz continuous on \( H^0 \).

Now, we recall some basic definitions and theorems concerning a global attractor.

**Definition 2.1** A dynamical system \( (H, S(t)) \) is dissipative if it possesses a bounded absorbing set, that is, a bounded set \( B \subset H \) such that for any bounded set \( B \subset H \) there exists \( t_B \geq 0 \) satisfying

\[ S(t)B \subset B, \quad \forall \quad t \geq t_B. \]

**Definition 2.2** [10] Let \( X \) be Banach space and \( B \) a bounded subset of \( X \). We call a function \( \Phi(.,.) \) which is defined on \( X \times X \) a contractive function on \( B \times B \) if for any sequence \( \{ x_n \}_{n \in \mathbb{N}} \subset B \), there is a subsequence \( \{ x_{n_k} \}_{k \in \mathbb{N}} \subset \{ x_n \}_{n \in \mathbb{N}} \), such that

\[ \lim_{k \rightarrow \infty, l \rightarrow \infty} \Phi(x_{nk}, x_{nl}) = 0. \quad (10) \]

**Denote all such contractive functions on** \( B \times B \) **by** \( C \).

**Definition 2.3** [10] Let \( \{ S(t) \}_{t \geq 0} \) be semigroup on a Banach space \( (X, \| \cdot \|) \) that has a bounded absorbing set \( B_0 \). Moreover, assume that for \( \epsilon > 0 \) there exist \( T = T(B_0, \epsilon) \) and \( \Phi T(.,.) \subset C(B_0) \) such that

\[ \| S(T)x - S(T)y \| \leq \epsilon + \Phi T(x, y), \quad \forall \; (x, y) \in B_0 \]

where \( \Phi T \) depends on \( T \). Then \( \{ S(t) \}_{t \geq 0} \) is asymptotically compact in \( X \), i.e., for any bounded sequence \( \{ y_n \}_{n \in \mathbb{N}} \subset X \) and \( \{ t_n \} \) with \( t_n \rightarrow \infty \), \( \{ S(t_n)y_n \}_{n \in \mathbb{N}} \) is precompact in \( X \).

**Theorem 2.2** [6] A dissipative dynamical system \( (H, S(t)) \) has a compact global attractor if and only if it is asymptotically smooth.

Our main result is the following:

\[ \text{1355} \]
Theorem 2.3 Assume that assumptions (H), (G) hold and \( h_1, h_2 \in C^1(\mathbb{R}, \mathbb{R}) \), \( f_1, f_2 \in L^2(\Omega) \), then the dynamical system \((H_0, S(t))\) corresponding to the system (7) - (9) has a compact global attractor \( \mathcal{A} \subset H_0 \) which attracts any bounded set in \( H_0 \) with \( l_{l_0} \).

3. GLOBAL ATTRACTOR IN \( H_0 \)

In this section, we would like to prove Theorem 2.3 by showing that the dynamical system \((H_0, S(t))\) is dissipative, and verify the asymptotic compactness. Therefore, we get the existence of compact global attractor by Theorem 2.2.

3.1. Existing of absorbing set

We formally take the scalar product in \( H \) of the first equation of (7) with \( \varphi = ut + \theta u \) and the second \( w = vt + \theta v \) after a computation we find

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int \left( \varphi l_1^2 + l_2^2 + 1 \Delta u l_2^2 + y l_2^2 + y^2 u l_2^2 + \nabla v l_2^2 \right) + 2 \int H_1(v) dx - 2 \int f_1(x, y) dx & - 2 \int f_1(x, y) dx + \Theta l_1 \Delta u l_2^2 + \Theta l_2 \Delta v l_2^2 + \Theta l_2 \Delta v l_2^2 \\
&+ \int \nabla v (l_1(1 - \theta)(ut, \varphi)) + (1 - \theta)(vt, \varphi) + (\phi_1, ut)_{\mu_1 l_1 v} + (\phi_2, vt)_{\mu_2 v} + \Theta (\phi_2, vt)_{\mu_2 v} + \Theta \int h_1(u) dx + \Theta \int h_2(v) v dx & - 2 \int f_1(x, y) u dx - 2 \int f_2(x, y) v dx
\end{align*}
\]

Exploiting (H) and Hölder inequality, we have

\[
(1 - \theta)(ut, \varphi) = (1 - \theta) \varphi l_2^2 - \theta(1 - \theta)(u, \varphi),
\]

\[
(1 - \theta)(vt, \varphi) = (1 - \theta) \varphi l_2^2 - \theta(1 - \theta)(v, \varphi),
\]

And

\[
(\phi_1, ut) = (\phi_1, \phi_1 t + \phi_1) = \frac{1}{2} \frac{d}{dt} \left( \phi_1 l_2^2 + \phi_1 l_2^2 \right) + \int_0^\infty \mu(s) (\phi_1, \phi_1 s) v ds
\]

\[
= \frac{1}{2} \frac{d}{dt} \left( \phi_1 l_2^2 + \phi_1 l_2^2 \right) + \int_0^\infty \mu(s) \frac{d}{ds} \phi_1 l_2^2 ds
\]

\[
= \frac{1}{2} \frac{d}{dt} \left( \phi_1 l_2^2 + \phi_1 l_2^2 \right) + \int_0^\infty \mu(s) \left( \phi_1 l_2^2 \right) ds = \frac{1}{2} \frac{d}{dt} \left( \phi_1 l_2^2 + \phi_1 l_2^2 \right) + \phi_1 l_2^2
\]

\[
\begin{align*}
\theta(\phi_1, u) & \geq \frac{1}{\mu} \left( \phi_1 l_2^2 + \frac{1}{2} \phi_2 l_2^2 \right) \\
\theta(\phi_2, v) & \geq \frac{1}{\mu} \left( \phi_1 l_2^2 + \frac{1}{2} \phi_2 l_2^2 \right)
\end{align*}
\]

We choose \( \theta \) small enough, such that

\[
\frac{1}{\mu} \left( \frac{1 - \theta}{\delta} + \frac{1 - \theta}{\delta} \right) \geq 1 - \theta, \quad \frac{1}{\mu} \left( \frac{1 - \theta}{\delta} \right) \geq 1 - \theta, \quad \frac{1}{\mu} \left( \frac{1 - \theta}{\delta} \right) \geq \frac{1}{\mu} \geq 1 - \theta
\]

hence, we conclude from Hölder, Young and Poincaré inequalities

\[
\begin{align*}
\theta l_1 & \left( \frac{1 - \theta}{\delta} \right) l_2^2 + (1 - \theta) \left( \frac{1 - \theta}{\delta} \right) l_2^2 \\
\theta & \left( \frac{1 - \theta}{\delta} \right) l_2^2 + (1 - \theta) \left( \frac{1 - \theta}{\delta} \right) l_2^2
\end{align*}
\]

\[
\begin{align*}
\theta l_1 & \left( \frac{1 - \theta}{\delta} \right) l_2^2 + (1 - \theta) \left( \frac{1 - \theta}{\delta} \right) l_2^2 \\
& + \left( \frac{1 - \theta}{\delta} \right) l_2^2 \\
& \geq \theta l_1 \left( \frac{1 - \theta}{\delta} \right) l_2^2 + \frac{1}{4} l_2^2
\end{align*}
\]

Analogously
\[ \theta l^2(1-(1-2\theta)^2) \, l|\Delta v_1|^2 + (1-\theta) \, w|\Delta v_1^2 - \theta(1-\theta)(v, w) \geq \theta l^2 \, l|\Delta v_1|^2 + \frac{1}{4} \, l|v_1|^2 \]

(15)

Combining (12) and (15), we have

\[ \frac{1}{2} \frac{d}{dt} (\theta l^2 + l|\Delta v_1|^2 + l|\Delta u_1|^2 + l|\Delta v_2|^2 + l|\Delta v|^2 + l|w|^2 + l|\phi l_1|^2 + l|\phi l_2|^2 + l|w|^2) \]

(2) \int H1(V) \, dx + 2 \int H2(V) \, dx - 2 \int f1(x, y) \, dx - 2 \int f1(x, y) \, dx

\[ + 2 \theta \, l|\Delta u_1|^2 + 2 \theta \, l|\Delta v_1|^2 + \frac{\delta}{2} \, l|\phi l_1|^2 + \frac{\delta}{2} \, l|\phi l_2|^2 \]

(16)

+ 2 \theta \int h1(u) \, dx + 2 \theta \int h2(v) \, v \, dx

- 2 \int f1(x, y) \, u \, dx - 2 \int f2(x, y) \, v \, dx \leq 0

Put

\[ \Theta_0 = \min\left\{ 2 \, \theta \, l|l_1|^{-\frac{1}{2}}; 2 \, \theta \, l|l_1|^{-\frac{1}{2}}; \frac{1}{2}; \frac{\delta}{2} \right\} \]

(19)

which implies that

\[ E(t) \leq \Theta_0 \int t(\xi) \, d\xi + E(0) \]

(20)

Hence

\[ E(t) = l(1 + \Theta \, u|l_1|^2 + l(u + \Theta \, v|l_1|^2 + l(w + \Theta \, v|l_1|^2 + l|\Delta u_0| + l|\Delta v_0|^2 + l|\nabla u|^2 + l|\nabla v|^2 + l|\phi l_1|^2 + l|\phi l_2|^2 + l|w|^2) + 2 \int H1(u) \, dx + 2 \int H2(v) \, v \, dx - 2 \int f1(x, y) \, u \, dx - 2 \int f1(x, y) \, v \, dx \]

(17)

Noticing that (4) and (17) (18), and using the compact Sobolev embedding theorem we get

\[ E(t) \geq l(1 + \Theta \, u|l_1|^2 + l(u + \Theta \, v|l_1|^2 + l(w + \Theta \, v|l_1|^2 + l|\Delta u_0| + l|\Delta v_0|^2 + l|\nabla u|^2 + l|\nabla v|^2 + l|\phi l_1|^2 + l|\phi l_2|^2 - M) \]

(21)

Similarly

\[ I(t) \geq l(1 + \Theta \, u|l_1|^2 + l(u + \Theta \, v|l_1|^2 + l(w + \Theta \, v|l_1|^2 + l|\Delta u_0| + l|\Delta v_0|^2 + l|\nabla u|^2 + l|\nabla v|^2 + l|\phi l_1|^2 + l|\phi l_2|^2 - M) \]

(22)

where

\[ M = \frac{2}{\epsilon}(l|f1|^2 + (l|f2|^2 + 2Kl|\Omega|), therefore \frac{\epsilon + 2\theta}{2\xi} < l, and 0 < \theta < \frac{\xi}{l}(1 - \frac{1}{2}) \]

(23)

\[ \frac{\epsilon + 2\theta}{2\xi} < l, and 0 < \theta < \frac{\xi}{l}(1 - \frac{1}{2}), we have \]

\[ L1 \geq \frac{\epsilon + 2\theta}{2\xi} > 0 \]

(24)

And
Proof. Multiplying (25) by $\|\varphi\|_2^2 + \|\nabla\varphi\|_2^2 + \|\Delta\varphi\|_2^2$, we get

$$E(t) \geq \|\varphi\|_2^2 + \|\nabla\varphi\|_2^2 + \|\varphi\|_2^2$$

(25)

$$I(t) \geq \|\varphi\|_2^2 + \|\nabla\varphi\|_2^2 + \|\Delta\varphi\|_2^2$$

(26)

So, we deduce from (25) and (26) that

$$C_1(\|\varphi\|_2^2 + \|\varphi\|_2^2 + \|\nabla\varphi\|_2^2 + \|\Delta\varphi\|_2^2)$$

Thus, for any $\delta > \frac{M}{C_1}$, there exists $t_0$, such that

$$\|\varphi(t_0)\|_2^2 + \|\nabla\varphi(t_0)\|_2^2 + \|\Delta\varphi(t_0)\|_2^2 \leq \delta^2$$

(27)

And we end up to

**Lemma 3.1.** Assume that assumptions (H) and (G) hold and $h, l_2 \in C^1(\mathbb{R}, \mathbb{R})$, $f_1, f_2 \in L^2(\Omega)$, then the ball of $H_0$, $B_0 = B_{\rho_0}(0, \rho_0)$, centered at 0 of radius $\rho_0$, is an absorbing set in $H_0$ for the group $S(t)$. For any bounded subset $B$ in $H_0$, $S(t)B \subset B_0$ for $t \geq t_0$, there exists a positive constant $\rho_2 > \rho_1$, such that $\forall t \geq t_0$ we have

$$\|\varphi(t)\|_2^2 + \|\nabla\varphi(t)\|_2^2 + \|\Delta\varphi(t)\|_2^2 \leq \delta^2$$

(29)

3.2. Attractor

First, we prove the following important Lemma:

**Lemma 3.2.** Under the hypotheses of Theorem 2.3, there exists a constant $\rho_3 > \delta$, such that

$$\|\varphi(t)\|_2^2 + \|\nabla\varphi(t)\|_2^2 + \|\Delta\varphi(t)\|_2^2 \leq \rho_3$$

(30)

Proof. Multiplying (7) by $-\Delta \varphi = -\Delta u - \theta \Delta u$, and (7) by $-\Delta \varphi = -\Delta v - \theta \Delta v$ and integrating over $\Omega$, we get

$$\|\varphi(t)\|_2^2 + \|\nabla\varphi(t)\|_2^2 + \|\Delta\varphi(t)\|_2^2 \leq \rho_3$$

(31)

Similar to previous estimates, we see that

$$(1 - \theta)(v_t, -\Delta \varphi) = (1 - \theta)(\nabla u^2 - \theta(1 - \theta)(\nabla u, \nabla \psi),$$

$$(1 - \theta)(v_t, -\Delta \varphi) = (1 - \theta)(\nabla v^2 - \theta(1 - \theta)(\nabla v, \nabla \varphi).$$

And

$$\langle \varphi, v \rangle_{D(A^2)} \geq \frac{1}{2} \frac{d}{dt} \|\varphi\|_{D(A^2)}^2 + \frac{\delta}{2} \|\varphi\|_{D(A^2)}^2$$

We have
Lemma 3.3 (Stabilizability inequality) Under the hypotheses of Theorem 2.3, given a bounded set $B \subset H_0$, let $z_1 = (u, v, u_t, v_t, \phi_1, \phi_2)$ and $z_2 = (\bar{u}, \bar{v}, \bar{u}_t, \bar{v}_t, \zeta_1, \zeta_2)$ be two weak solutions of problem (7) – (9) such that $z_1(0) = (u_0, v_0, u_1, v_1, \phi_{10}, \phi_{20})$ and $z_2(0) = (\bar{u}_0, \bar{v}_0, \bar{u}_1, \bar{v}_1, \zeta_{10}, \zeta_{20})$ are in $B$. Then, for all $t \geq 0$, we have

$$F(t) \leq \exp \left( -\Theta_0 t \right) F(0) + \frac{\bar{c}}{\Theta_0}$$

Using the fact that

$$F(t) \geq \Delta u(\bar{u}_t) + \Delta v(\bar{v}_t) + \Delta u(\bar{u}_t) + \Delta v(\bar{v}_t) + \Delta u(\bar{u}_t) + \Delta v(\bar{v}_t) + \phi_{11}(\bar{u}_t) + \phi_{21}(\bar{u}_t)$$

we obtain (30).

We present important lemmas to prove Theorem 2.3.
\[
\|z_t(t) - z_2(t)\|_{H^0} \leq \exp \left( -\nu t \right) \|z_t(0) - z_2(0)\|_{H^0} + C_3 \int_0^t \exp \left( -\nu (t - s) \right) \left( l w(s) - \bar{u}(s) \right)_{L^2(p, 1)} + l v(s) - v^{-}(s)_{L^2(p, 1)} ds
\]

where \( \nu > 0 \) is a small constant and \( p, C_3 \) are positive constants.

**Proof.** Let us fix a bounded set \( B \subset H_0 \). We set \( w = u - \bar{u} \), \( V = v - \bar{v} \) and \( \zeta = \phi I - \zeta_1 \), \( \rho = \phi 2 - \zeta_2 \). Then \( (w, \zeta) \) and \( (V, \rho) \) satisfy

\[
\left\{
\begin{array}{l}
wt + l 1 \Delta 2 w - \Delta w + wt + \int_0^s \mu 1(s) \Delta 2 \zeta(t) ds + h 1(u) - h 1(\bar{u}) = 0 \\
Vt + l 1 \Delta 2 V - \Delta V + Vt + \int_0^s \mu 1(s) \Delta 2 \rho t(s) ds + h 2(u) - h 2(\bar{u}) = 0 \\
\end{array}
\right.
\]

(36)

With initial condition

\[
W(0) = u(0) - \bar{u}(0), \quad \text{wt}(0) = u_1 - \bar{u}_1, \quad \zeta(0) = \phi 10 - \zeta_1 10 \\
V(0) = v(0) - \bar{v}(0), \quad Vt(0) = V_1 - \bar{V}_1, \quad \rho(0) = \phi 20 - \zeta_2 20
\]

We take the scalar product in \( H \) of (37), with \( \zeta = w + \theta w \), and (37) \( _2 \) with \( \psi = Vt + \theta V \), we get

\[
\frac{1}{2} \frac{d}{dt} \left( \|l \Delta w l_2^2 + l \Delta v l_2^2 + (l \zeta l_2^2 + l \psi l_2^2) + l \nabla w l_2^2 + l \nabla V l_2^2 + \Theta l \Delta w l_2^2 + \Theta l \Delta V l_2^2 + (1 - \Theta)(w, \zeta) + (1 - \Theta)(Vt, \psi) + \Theta (\zeta, \psi) + \Theta (w, \psi) \right)
\]

(38)

The same as the previous calculations

\[
\Theta (\zeta, \psi) \geq \frac{\delta}{\frac{d}{dt} l \zeta l_2^2 + \frac{(1 - \Theta)\delta}{\delta} l \psi l_2^2}
\]

And

\[
\Theta (w, \psi) \geq \frac{\delta}{\frac{d}{dt} l \psi l_2^2 + \frac{(1 - \Theta)\delta}{\delta} l \psi l_2^2}
\]

So, we have

\[
\Theta l (1 - \frac{(1 - \Theta)\delta}{\delta}) l \Delta w l_2^2 + (1 - \Theta) l \zeta l_2^2 - \Theta l (1 - \Theta) (w, \zeta)
\]

\[
\geq \Theta l (1 - \Theta) l \Delta w l_2^2 + \frac{\delta}{\delta} l \zeta l_2^2
\]

\[
\Theta l (1 - \frac{(1 - \Theta)\delta}{\delta}) l \Delta V l_2^2 + (1 - \Theta) l \psi l_2^2 - \Theta l (1 - \Theta) (V, \psi)
\]

\[
\geq \Theta l (1 - \Theta) l \Delta V l_2^2 + \frac{\delta}{\delta} l \psi l_2^2
\]

Then, we have

\[
\frac{1}{2} \frac{d}{dt} \left( \|l \Delta w l_2^2 + l \Delta v l_2^2 + (l \zeta l_2^2 + l \psi l_2^2) + l \nabla w l_2^2 + l \nabla V l_2^2 + l \zeta l_2^2 + l \psi l_2^2 \right)
\]

\[
+ \Theta l (1 - \Theta) l \Delta w l_2^2 + \Theta l (1 - \Theta) l \Delta V l_2^2 + \frac{\delta}{\delta} l \zeta l_2^2 + \frac{\delta}{\delta} l \psi l_2^2
\]

\[
\leq h 1(u) - h 1(\bar{u}), \quad \zeta + (h 2(V) - h 2(\bar{V}), \psi)
\]

And
\[ \int_{\Omega} (h_1(u) - h_1(\bar{u})) (w^T \Omega w) \, dx \leq K_1 \int_{\Omega} (1 + |u|^p + |v|^p) \, \|w^T \Omega w\| \, dx \]
\[ \leq K_1 \int_{\Omega} (\Omega \frac{1}{2p+1} + l_2(2p+1)^p) \, \|lw(2p+1)\| \leq (KC + \Theta) \|lw\|_{2p+1}^2 \]  
(40)

By the same technique, we get
\[ \|f(\Omega, (s)) \|_{2p+1} \leq (KC + \Theta) \|V\|_{2p+1}^2 \]
\[ \text{where } s = \min \left\{ \gamma \delta \frac{L}{C} \right\} \text{ and } C = KC + \Theta \text{ which implies that} \]
\[ E(t) = \|V\|_{2p+1}^2 \]
\[ \text{Hence} \]
\[ \frac{d}{dt} E(t) + v_1 E(t) \leq C \left( l_2(2p+1)^2 + l_2(2p+1)^2 \right) \]
\[ \text{where } v_1 = \min \left\{ \gamma \delta \frac{L}{C}, \frac{1}{2} \right\} \]  
(41)

Invoking the fact that \( E(t) \geq l_2(\tau(t) - \tau(t)) \) \( l_2(2p+1) \), we easily obtain (36).

**Lemma 3.4** (Asymptotic smoothness) Under assumptions of Theorem 2.3, the dynamical system \((H_0, S(t))\) corresponding to problem (7) - (9) is asymptotically smooth.

**Proof.** Let \( B \) be a bounded subset of \( H_0 \) positively invariant with respect to \( S(t) \). Denote by \( C \) several positive constants that are dependent on \( B \) but not on \( t \). For \( (Z10, Z20) \in B \), \( S(t) \) \( Z10 = (U, U_t, \bar{u}) \) \( S(t) \) \( Z20 = (V, V_t, \xi) \) are the solutions of (7) - (9). Then given \( \epsilon > 0 \) from inequality (41), we can choose \( T > 0 \) such that
\[ \|S(t)Z10 - S(t)Z20\|_{H_0} \leq \epsilon + C \int_0^T \left( \|u(s) - \bar{u}(s)\|_{2(p+1)}^2 + \|V(s) - \bar{V}(s)\|_{2(p+1)}^2 \right)^{\frac{1}{2}} \, ds \]
\[ \text{where } C_\beta > 0 \text{ is a constant which depends only on the size of } B. \text{ The condition } p > 0 \text{ implies that } 2 < (p + 1) < \infty. \text{ Taking } \alpha = \frac{1}{2} \left( 1 - \frac{1}{p+1} \right) \text{ and applying Gagliardo-Nirenberg interpolation inequality, we have} \]
\[ \|u(t) - \bar{u}(t)\|_{2(p+1)}^2 \leq C_\beta \Delta u(t) - \bar{u}(t)\|_{2(p+1)}^2 \leq \|W(t) - \bar{W}(t)\|_{2(p+1)}^2 \]
\[ \|u(t) - \bar{u}(t)\|_{2(p+1)}^2 \leq C_\beta \Delta u(t) - \bar{u}(t)\|_{2(p+1)}^2 \leq \|W(t) - \bar{W}(t)\|_{2(p+1)}^2 \]
\[ \text{Since } l_2(\Omega) \text{ and } l_2(\Omega) \text{ are uniformly bounded, there exists a constant } C > 0 \text{ such that} \]
\[ l_2(\Omega) - \bar{l_2(\Omega)}^2 \leq C \leq l_2(2p+1) \]
\[ l_2(\Omega) - \bar{l_2(\Omega)}^2 \leq C \leq l_2(2p+1) \]  
(42)

(43)
Then, from (43) and (44) – (45) we obtain
\[ |S(t)Z10 - S(t)Z20|_{H0} \leq \varepsilon + |\Phi T(Z10, Z20)| \]

With
\[ \Phi T(Z10, Z20) \leq \varepsilon + C \int_0^T |(u(s) - \bar{u}(s)|_{L_{2(p+1)}^2}^2 + |V(s) - \bar{V}(s)|_{L_{2(p+1)}^2}^{2(1+\varepsilon)} + \rho(s)|_{H_{p+1}^2}^{2(1+\varepsilon)} ds \]

The following proof \( \Phi T \in C \) namely \( \Phi T \) satisfies (10). Indeed, give a sequence \( Zn0= (U0n, \ U1n, \ \phi 0n) \in B \), let us write \( S(t)(Zn0) = (Un, Unt, \ \phi nt) \) is uniformly bounded in \( H_{p+1} \). On the other hand, \( (Un, Unt) \) is bounded in \( C([0,T], V \times H), T > 0 \). By the compact embedding \( V \subset H \), the Aubin lemma implies that there exists a subsequence \( (u^{nk}) \) that converges strongly in \( C([0,T], H) \). Therefore,

\[ \lim_{k \to \infty} \int_0^T |Unk(s) - Un1(s)|_{L_{2(p+1)}^2}^{2(1+\varepsilon)} + |Vnk(s) - Vn1(s)|_{L_{2(p+1)}^2}^{2(1+\varepsilon)} ds = 0 \]

This completes the proof.

4. THE REGULARITY

Our main result is the following theorem

**Theorem 4.1** Under assumptions of Theorem 2.3, then the global attractor \( A \) is a bounded subset \( H_1 \).

4.1. The semigroup decomposition

We fix a bounded set \( B \subset H0 \) and for \( Z = (U0, U1, \phi 0) \in B \), we decompose the solution \( S(t)Z = (U, Ut, \phi) \) of problem (7) – (9) into the sum

\[ S(t)Z = D(t)Z + K(t)Z, \]

where

\[ D(t)Z = z1(t), \quad K(t)Z = Z(t), \]

and

\[ z = (U, Ut, \phi) = Z1 + Z2, \]

furthermore,

\[ U = \bar{w} + w, \quad V = \bar{v} + v, \quad \phi1 = \zeta1 + \zeta1t, \quad \phi2 = \rho t + \zeta2t \]

where \( Z(t) \) satisfies

\[
\begin{cases}
\bar{w}tt + l1\Delta 2 \bar{w} - \Delta \bar{w} + \bar{w}t + \int_0^\infty \mu 1(s)\Delta 2 \zeta(t)ds = 0 \\
\bar{v}tt + l2\Delta 2 \bar{v} - \Delta \bar{v} + \bar{v}t + \int_0^\infty \mu 1(s)\Delta 2 \rho t(s)ds = 0 \\
\zeta t = -\zeta s + \bar{w}t \\
\rho t = -\rho s + \bar{v}t
\end{cases}
\]

(46)

\[
\begin{cases}
wtt + l1\Delta 2 w - \Delta w + wt + \int_0^\infty \mu 1(s)\Delta 2 \zeta(t)ds = 0 \\
vtt + l2\Delta 2 v - \Delta v + vt + \int_0^\infty \mu 1(s)\Delta 2 \rho t(s)ds = 0 \\
\zeta t = -\zeta s + wt \\
\rho t = -\rho s + vt
\end{cases}
\]

(47)

The well-posedness of the problem (46) and (47) can be obtained by Faedo-Galerkin methods. Combining with the previous estimate about the solution \( Z1(t) \) of equation (46) we obtain the exponential decay of \( D(t)Z \)
Lemma 4.1 Under assumptions of Theorem 2.3, there exists a constant $K > 0$, such that the solution of (46) satisfies 
\[ |D(t)|_{H^2}^2 \leq C \exp(-k.t) \]
where $C$ is a constant.

About the solution of equation (47), we have the next result that provides the boundedness of $K(t)\zeta$ in a more regular space.

Lemma 4.2 Under the assumptions of Theorem 2.3, there exists a constant $N > 0$ such that the solution of (47) satisfies 
\[ |K(t)|_{H^2}^2 \leq C \exp(-k.t) \]

Proof. Taking the scalar product in $H$ of (47)$_1$ with $A\zeta = Aw + \theta Aw$ and (47)$_2$ with $A\psi = AV + \theta AV$, we obtain

\[
\frac{d}{dt}[|11Aw|^2 + |12Av|^2 + |1\Delta \zeta_2^2 + 1\Delta \psi_2^2 + 1\nabla Aw|^2 + |1\nabla \psi|^2] + \theta (1 - \Theta)(Vt, Aw) + (1 - \Theta)(Vt, Aw) + \theta \nabla \psi_1^2 + \theta \nabla \psi_1^2 = 0 \tag{48}
\]

We find from (48)

\[
\frac{d}{dt}[|11Aw|^2 + |12Av|^2 + |1\Delta \zeta_2^2 + 1\Delta \psi_2^2 + 1\nabla Aw|^2 + |1\nabla \psi|^2] + \theta (1 - \Theta)\nabla \psi_1^2 + \theta \nabla \psi_1^2 = 0 \tag{49}
\]

\[
\frac{\delta}{4} [11t_{1}|D(A)|^2 + \frac{2}{4} \nabla \psi_1^2 - \Theta(1 - \Theta)(AV, \psi) + (h1(u), A\zeta) + (h2(V), A\psi) = (f1, A\zeta) + (f2, A\psi) \tag{50}
\]

Furthermore

\[
\Theta I I (1 - \Theta)(1 - \Theta) |1\Delta \zeta_2^2 + \Theta(1 - \Theta)(Aw, \zeta) \geq \Theta I I (1 - \Theta)(1 - \Theta)^2 + \frac{1}{4} |1\Delta \zeta_2^2 \geq (51)
\]

By Lemma 3.1 and the Sobolev embedding theorem we know that $h(u)$ and $h'(u)$ are uniformly bounded in $IL^p$ that there exists a constant $M > 0$, such that 

\[ |h(u)| \leq M \quad \text{and} \quad |h'(u)| \leq M. \]
Combining with the Hölder, Young, Cauchy and (29), (30), it follows that

\[
\begin{align*}
  \frac{d}{dt}(h_1(u), A\zeta) \\
  = & \quad \frac{d}{dt}(h_1(u), Aw) + \Theta (h_1(u), Aw) \\
  \geq & \quad \frac{d}{dt}(h_1(u), Aw) + \Theta (h_1(u), Aw) \\
  \geq & \quad \frac{d}{dt}(h_1(u), Aw) + \Theta (h_1(u), Aw) - \Theta (h_1'(u)ut, Aw) \\
  \geq & \quad \frac{d}{dt}(h_1(u), Aw) + \Theta (h_1(u), Aw) - \int_\Omega \Theta h_1'(u) ut l Aw \, dx \\
  \geq & \quad \frac{d}{dt}(h_1(u), Aw) + \Theta (h_1(u), Aw) - \frac{\theta_{11}}{4} l Aw l^2 + \frac{M\ell}{\theta_{11}} \\
\end{align*}
\]

Analogously

\[
\begin{align*}
  (f_2, A\psi) & \geq \frac{d}{dt}(f_2, AV) + \Theta (f_2, AV) - \frac{\theta_{12}}{4} l AV l^2 + \frac{M\ell}{\theta_{12}} \\
  (f_1, A\zeta) & = (f_1, Aw + \Theta Aw) = \frac{d}{dt}(f_1, Aw) + \Theta (f_1, Aw) \\
  (f_2, A\psi) & = (f_2, AV_t + \Theta AV) = \frac{d}{dt}(f_2, AV) + \Theta (f_2, AV) \\
\end{align*}
\]

Thus, collecting (50) – (55) from (49) yields

\[
\begin{align*}
\frac{d}{dt}(11 l Aw l^2 + 12 l AV l^2 + l\Delta l^2 + l\Delta l^2 + l\Delta l^2 + l\Delta l^2 + l\Delta l^2 + l\Delta l^2)
\geq & \quad \frac{\theta_{11}}{4} l AV l^2 + \frac{M\ell}{\theta_{11}} \\
\geq & \quad \frac{\theta_{12}}{4} l AV l^2 - \Theta (f_2, AV) \\
\leq & \quad \frac{\theta_{13}}{4} l AV l^2 + \frac{M\ell}{\theta_{13}} \\
\end{align*}
\]

Taking \( \Theta_0 = \min\{ \frac{2\Theta(1-\Theta)}{\theta_{2}}, \Theta, \frac{\delta}{2}, \frac{1}{2} \} \) we can obtain from (56)

\[
\begin{align*}
\frac{d}{dt}(11 l Aw l^2 + 12 l AV l^2 + l\Delta l^2 + l\Delta l^2 + l\Delta l^2 + l\Delta l^2 + l\Delta l^2 + l\Delta l^2)
\geq & \quad \frac{\theta_{11}}{4} l AV l^2 + \frac{M\ell}{\theta_{11}} \\
\geq & \quad \frac{\theta_{12}}{4} l AV l^2 - \Theta (f_1, Aw) - 2(f_2, AV) + l l_1 \| l Aw l^2 + l l_2 l AV l^2 + l l_3 l AV l^2 + l l_4 l AV l^2 \\
\leq & \quad \frac{\theta_{13}}{4} l AV l^2 + \frac{M\ell}{\theta_{13}} \\
\end{align*}
\]

On the other hand, by the Hölder inequality, the Sobolev embedding theorem and (29), it follows that

\[
\begin{align*}
\frac{d}{dt} l_{1/2}^1 \quad Aw + \frac{l_{1/2}^2}{2} AV + \frac{\theta_{11}}{4} l h_1(u) + \frac{\sqrt{2}}{l_{1/2}^2} h 2(V) l^2 \\
\geq & \quad \frac{\theta_{11}}{4} l h_1(u) + \frac{\theta_{12}}{4} l h 2(V) + \frac{\theta_{13}}{4} l l_1^2 + \frac{M\ell}{\theta_{11}} - 4M\ell + 4M\ell \\
\geq & \quad \frac{\theta_{11}}{4} l h_1(u) + \frac{\theta_{12}}{4} l h 2(V) + \frac{\theta_{13}}{4} l l_1^2 + \frac{M\ell}{\theta_{11}} - 4M\ell + 4M\ell \\
\geq & \quad \frac{\theta_{11}}{4} l h_1(u) + \frac{\theta_{12}}{4} l h 2(V) + \frac{\theta_{13}}{4} l l_1^2 + \frac{M\ell}{\theta_{11}} - 4M\ell + 4M\ell \\
\geq & \quad \frac{\theta_{11}}{4} l h_1(u) + \frac{\theta_{12}}{4} l h 2(V) + \frac{\theta_{13}}{4} l l_1^2 + \frac{M\ell}{\theta_{11}} - 4M\ell + 4M\ell \\
\end{align*}
\]

Therefore, integrating with (58), we get from (57)
\[
\begin{align*}
\frac{d}{dt} l \left( l^2_1 - 2(f_1, Aw) \right) &= \frac{d}{dt} \sqrt{\frac{l}{11}} \left( Aw - \frac{2}{l_1} f_1 - \frac{2}{l_2} f_2 \right) - 2(f_1, Aw) \\
\left| \Delta \xi_1^2 + \Delta \psi_1^2 + \nabla \Delta \omega l_2^2 + \nabla \Delta V l_2^2 + \kappa t l_{(D(A))}^2 + \kappa^2 t l_{(A)}^2 \right|^2 \\
&+ \Omega_0 \left( \frac{l}{2} \right) \left( l \frac{1}{2} \right) \left( \frac{\text{AV}}{\sqrt{2}} \right) \left( f_1 - \frac{2}{\sqrt{2}} f_2 + \frac{2}{\sqrt{2}} h_1(u) + \frac{2}{\sqrt{2}} h_2(V) \right) \leq C 
\end{align*}
\]

(61)

Where \( C > 0 \)

Applying the Gronwall lemma, there exists a constant \( N \) such that

\[
\left| \Delta \xi_1^2 + \Delta \psi_1^2 + \nabla \Delta \omega l_2^2 + \nabla \Delta V l_2^2 + \kappa t l_{(D(A))}^2 + \kappa^2 t l_{(A)}^2 \right|^2 \leq N 
\]

Completion of the proof of Theorem 4.1

By collecting Lemma 4.1 and Lemma 4.2, we get \((U_0, V_0, U_1, V_1, \phi_{10}, \phi_{20}) \in H_1\) and

\[
\left| \Delta \xi_1^2 + \Delta \psi_1^2 + \nabla \Delta \omega l_2^2 + \nabla \Delta V l_2^2 + \kappa t l_{(D(A))}^2 + \kappa^2 t l_{(A)}^2 \right|^2 \leq N 
\]

As \( u(t, x) \) satisfies (7) - (9) with initial data \((U_0, V_0, U_1, V_1, \phi_{10}, \phi_{20})\), we easily obtain

\[
\left| l(U_0, V_0, U_1, V_1, \phi_{10}, \phi_{20}) \right|_{H_0} \leq N 
\]

Thus \( A \) is a bounded subset of \( H_1 \).

Which completes the proof.

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