Hermite-Hadamard and Simpson Type Inequalities for Multiplicatively Harmonically P-Functions

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ABSTRACT

In this paper, using the Hölder-İşcan and improved power-mean integral inequalities together with an identity, we obtain new estimates on generalization of Hadamard and Simpson type inequalities for multiplicatively harmonically P-functions. The obtained results are compared with the previous ones.

Keywords: Multiplicatively P-function, multiplicatively harmonically P-functions, Hermite–Hadamard type inequalities, Simpson type inequality, Hölder–İşcan inequality, improved power-mean inequality.

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1. INTRODUCTION

Let real function \( f \) be defined on some nonempty interval \( I \) of the real line \( \mathbb{R} \). The function \( f \) is said to be convex on interval \( I \) if the following inequality

\[
 f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]

holds for all \( x, y \in I \) and \( t \in [0,1] \).

The following inequalities are well known in the literature as Hermite-Hadamard inequality [2] and Simpson inequality [1] respectively:

**Theorem 1.** Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a convex function defined on the interval \( I \) and \( a, b \in I \). The following double inequality holds

\[
 f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.\]

**Theorem 2.** Let \( f : [a,b] \rightarrow \mathbb{R} \) be a four times continuously differentiable mapping on \( (a,b) \) and \( \|f^{(4)}\|_\infty = \sup_{x \in (a,b)} |f^{(4)}(x)| < \infty \). Then the following inequality holds:

\[
 \frac{1}{3} f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b-a)^4.
\]

Let \( H = H(a,b) = 2ab/(a+b) \), \( G = G(a,b) = \sqrt{ab} \), \( L = L(a,b) = (b-a)/(\ln b - \ln a) \), \( I = I(a,b) = (1/e)(b^b/a^a)^{1/(b-a)} \), \( A = A(a,b) = \frac{a+b}{2} \), \( A_1 = A_1(a,b) := \lambda b + (1 - \lambda)a, \lambda \in [0,1] \).

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functions:

Remark 1. The range of the multiplicatively function (or function) as

Definition 3. Let $L_p = L_p(a, b) = \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}}$, $p \in \mathbb{R}\{\{-1, 0\}\}$, be the geometric, logarithmic, identric, arithmetic, weighted arithmetic and $p$-logarithmic means of $a$ and $b$, respectively. Then

$$
\min\{a, b\} < H < G < L < I < A = \max\{a, b\}.
$$

In [3], S.S. Dragomir et al. defined the following new class of functions.

**Definition 1.** We recall that a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be $P$-function on $I$ or belong to the class $P(I)$ if it is nonnegative and,

$$
f(tx + (1 - t)y) \leq f(x) + f(y)
$$

for all $x, y \in I$ and $t \in [0, 1]$. Note that $P(I)$ contains all nonnegative convex and quasi-convex functions [1].

**Example 1.** [11] A non-negative function $f : I \to \mathbb{R}$ is called trigonometrically convex function on interval $[a, b]$, if for each $x, y \in [a, b]$ and $t \in [0, 1],$

$$
f(tx + (1 - t)y) \leq \left( \sin \frac{\pi t}{2} \right) f(x) + \left( \cos \frac{\pi t}{2} \right) f(y).
$$

Clearly, if $f(x)$ is a nonnegative function, then every trigonometric convex function is a $P$-function. Indeed,

$$
f(tx + (1 - t)y) \leq \left( \sin \frac{\pi t}{2} \right) f(x) + \left( \cos \frac{\pi t}{2} \right) f(y) \leq f(x) + f(y).
$$

In recent years, many authors have studied errors estimations for Hermite-Hadamard and Simpson inequalities; for refinements, counterparts, generalizations see [5, 6, 14] and references therein.

In [4], İşcan gave the definition of harmonically convexity as follows:

**Definition 2.** Let $I \subseteq \mathbb{R}\{\{0\}\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$
f\left( \frac{x y}{tx + (1 - t)y} \right) \leq tf(y) + (1 - t)f(x) \quad (1.1)
$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.1) is reversed, then $f$ is said to be harmonically concave.

The following result of the Hermite-Hadamard type inequality holds.

**Theorem 3.** Let $f : I \subseteq \mathbb{R}\{\{0\}\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ then the following inequalities hold

$$
f\left( \frac{2ab}{a+b} \right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \leq \frac{f(a) + f(b)}{2}.
$$

In [10], Kadakal gave the definition of multiplicatively $P$-function (or log-$P$-function) as follows:

**Definition 3.** Let $I \neq \emptyset$ be an interval in $\mathbb{R}$. The function $f : I \to [0, \infty)$ is said to be multiplicatively $P$-function (or log-$P$-function), if the inequality

$$
f(tx + (1 - t)y) \leq f(x) f(y)
$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

We will denote by $MP(I)$ the class of all multiplicatively $P$-functions on interval $I$. Clearly, $f : I \to (0, \infty)$ is multiplicatively $P$-function if and only if $\log f$ is $P$-function.

**Remark 1.** The range of the multiplicatively $P$-functions is greater than or equal to 1.

The following result of the Hermite-Hadamard type inequalities holds for multiplicatively $P$-functions:
Theorem 4. Let the function \(f: I \to [1, \infty)\) be a multiplicatively \(P\)-function and \(a, b \in I\) with \(a < b\). If \(f \in L[a, b]\), then the following inequalities hold:

i) \(f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \left( a + b - x \right) dx \leq [f(a)f(b)]^2\)

ii) \(f \left( \frac{a+b}{2} \right) \leq f(a)\int_a^b \frac{1}{b-a} f(x) dx \leq [f(a)f(b)]^2\)

Definition 4. A function \(f:I \subseteq (0, \infty) \to \mathbb{R}\) is said to be harmonically \(P\)-function on \(I\) or belong to the class \(HP(I)\) if it is nonnegative and,

\[
f \left( \frac{xy}{ty+(1-t)x} \right) \leq f(x) + f(y),
\]

for all \(x, y \in I\) and \(t \in [0,1]\).

Hermite-Hadamard inequalities can be represented for harmonically \(P\)-function as follows.

Theorem 5. Let \(f:I \subseteq (0, \infty) \to \mathbb{R}\) be a function such that \(f \in L[a, b]\), where \(a, b \in I\) with \(a < b\). If \(f\) is a harmonically \(P\)-function on \([a, b]\), then the following inequalities hold:

\[
f \left( \frac{2ab}{a+b} \right) \leq \frac{2ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \leq 2[f(a) + f(b)].
\]

In [9], İşcan and Olucak gave the definition of multiplicatively harmonically \(P\)-function as follows:

Definition 5. Let \(I \neq \emptyset\) be an interval in \(\mathbb{R}\setminus\{0\}\). The function \(f:I \to [0, \infty)\) is said to be multiplicatively harmonically \(P\)-function, if the inequality

\[
f \left( \frac{xy}{ty+(1-t)x} \right) \leq f(x)f(y)
\]

holds for all \(x, y \in I\) and \(t \in [0,1]\).

Example 2. The function \(f: [1, \infty) \to [1, \infty), f(x) = x\) is a multiplicatively harmonically \(P\)-function. Really, for any \(x, y \in (1, \infty)\) with \(x < y\), we have

\[
f \left( \frac{xy}{ty+(1-t)x} \right) = \frac{xy}{ty+(1-t)x} \leq y \leq xy = f(x)f(y).
\]

Example 3. The function \(f: (0, \infty) \to (0, \infty), f(x) = e^x\) is a multiplicatively harmonically \(P\)-function. Since, for any \(x, y \in (0, \infty)\) with \(x < y\), we have

\[
f \left( \frac{xy}{ty+(1-t)x} \right) = e^{ty+(1-t)x} \leq e^y \leq e^x e^y = f(x)f(y).
\]

We will denote by \(MHP(I)\) the class of all multiplicatively harmonically \(P\)-functions on interval \(I\).

The following result of the Hermite-Hadamard type inequalities hold for multiplicatively harmonically \(P\)-function:

Theorem 6. Let the function \(f:I \subseteq \mathbb{R}\setminus\{0\} \to [1, \infty)\) be a multiplicatively harmonically \(P\)-function and \(a, b \in I\) with \(a < b\). If \(f \in L[a, b]\), then the following inequalities hold:

i) \(f \left( \frac{2ab}{a+b} \right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)(a^{-1}b^{-1}x^{-1} - 1)}{x^2} dx \leq [f(a)f(b)]^2\)

ii) \(f \left( \frac{2ab}{a+b} \right) \leq f(a)\int_a^b \frac{ab}{b-a} \frac{f(x)}{x^2} dx \leq [f(a)f(b)]^2\).

A refinement of Hölder integral inequality better approach than Hölder integral inequality can be given as follows [7]:

Theorem 7 (Hölder-İşcan Integral Inequality [7]). Let \(p > 1\) and \(\frac{1}{p} + \frac{1}{q} = 1\). If \(f\) and \(g\) are real functions defined on \([a, b]\) and if \(|f|^p, |g|^q\) are integrable functions on \([a, b]\) then
\[ \int_a^b |f(x)g(x)| \, dx \leq \frac{1}{b-a} \left( \left( \int_a^b (b-x)|f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_a^b (b-x)|g(x)|^q \, dx \right)^{\frac{1}{q}} \right)^{\frac{1}{2}} + \left( \int_a^b (x-a)|f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_a^b (x-a)|g(x)|^q \, dx \right)^{\frac{1}{q}} \]  

A refinement of power-mean integral inequality better approach than power-mean inequality as a result of the Hölder-İşcan integral inequality can be given as follows [12]:

**Theorem 8 (Improved power-mean integral inequality [12]).** Let \( q \geq 1 \). If \( f \) and \( g \) are real functions defined on \([a, b]\) and if \(|f|, |g| \) are integrable functions on \([a, b]\) then

\[
\int_a^b |f(x)g(x)| \, dx \leq \frac{1}{b-a} \left( \left( \int_a^b (b-x)|f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_a^b (b-x)|g(x)|^q \, dx \right)^{\frac{1}{q}} \right)^{\frac{1}{2}} + \left( \int_a^b (x-a)|f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_a^b (x-a)|g(x)|^q \, dx \right)^{\frac{1}{q}} \]

In this paper, using Hölder-İşcan integral inequality better approach than Hölder integral inequality and improved power-mean integral inequality better approach than power-mean inequality and together with an integral identity, authors obtain a generalization of Hadamard and Simpson inequalities for functions whose derivatives in absolute value at certain power are

\[ \int_0^1 f(t) \, dt \]

In order to prove our main results we need the following identity [8].

**Lemma 1.** Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^* \), the interior of \( I \). Throughout this section we will take

\[ I_f(\lambda, \mu, a, b) = (\lambda - \mu) f \left( \frac{2ab}{a+b} \right) + (1 - \lambda) f(a) + \mu f(b) = \frac{ab}{b-a} \int_a^b f(\nu) \, d\nu \]

where \( a, b \in I \) with \( a < b \) and \( \lambda, \mu \in \mathbb{R} \).

In order to prove our main results we need the following identity [8].

**Theorem 9.** Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^* \) such that \( f' \in L[a, b] \), where \( a, b \in I \) with \( a < b \). If \(|f'|^q \) is multiplicatively harmonically \( P \)-function on the interval \([a, b]\) for some fixed \( q \geq 1 \) and \( 0 \leq \mu \leq \frac{1}{2} \leq \lambda \leq 1 \), then the following inequality holds for \( q > 1 \)

\[ I_f(\lambda, \mu, a, b) \leq 2ab(b-a)|f'(a)||f'(b)| \left[ C_1^{\frac{1-q}{q}}(\mu)D_1^{\frac{1}{q}}(\mu, a, b) + C_2^{\frac{1-q}{q}}(\mu)D_2^{\frac{1}{q}}(\mu, q, a, b) \right] + C_3^{\frac{1-q}{q}}(\lambda)D_3^{\frac{1}{q}}(\lambda, q, a, b) + C_4^{\frac{1-q}{q}}(\lambda)D_4^{\frac{1}{q}}(\lambda, q, a, b) \]

where

\[ I_0^{1/2} (\frac{1}{2} - t) |\mu - t| \, dt = C_1(\mu) = -\frac{\mu^2}{3} + \frac{\mu^2}{2} - \frac{\mu}{8} + \frac{1}{48} \]

\[ I_0^{1/2} t |\mu - t| \, dt = C_2(\mu) = \frac{\mu^3}{3} - \frac{\mu}{8} + \frac{1}{24} \]
\[ f_{1/2}^{1/2}(t - \lambda) = C_4(\lambda) = -\frac{\lambda^3}{3} + \lambda^2 - \frac{7\lambda}{8} + \frac{1}{4} \]
\[ f_{1/2}^{1/2}(t - \frac{1}{2} \lambda) = C_4(\lambda) = \frac{\lambda^3}{3} - \frac{\lambda^2}{2} + \frac{\lambda}{8} + \frac{1}{16} \]

\[
D_1(\mu, q, a, b) = \begin{cases} 
\frac{1}{2(b-a)^2} \left[ \frac{\mu}{b-a} \left[ A \mu \mu L_{-2q}^2(A, a) - (A + A \mu) L_{-2q+1}^2(A, a) + \mu L_{-2q+2}^2(A, a) \right] 
+ \frac{(1-\mu)}{2(b-a)^2} \left[ -A A \mu \mu L_{-2q}^2(A, a) + (A + A \mu) L_{-2q+1}^2(A, a) + L_{-2q+2}^2(A, a) \right] \right] 
& \mu = 0 \\
\frac{1}{2(b-a)^2} \left[ -A L_{-2q}^2(A, a) - 2A L_{-2q+1}^2(A, a) + L_{-2q+2}^2(A, a) \right] 
& 0 < \mu < \frac{1}{2} \\
\frac{1}{2(b-a)^2} \left[ -b A L_{-2q}^2(A, b) + (b + A) L_{-2q+1}^2(A, b) - L_{-2q+2}^2(A, b) \right] 
& \mu = \frac{1}{2} \\
\frac{1}{2(b-a)^2} \left[ -b A L_{-2q}^2(A, b) + (b + A) L_{-2q+1}^2(A, b) - L_{-2q+2}^2(A, b) \right] 
& \frac{1}{2} < \lambda < 1 \\
\frac{1}{2(b-a)^2} \left[ A^2 L_{-2q}^2(A, b) - 2A L_{-2q+1}^2(A, b) + L_{-2q+2}^2(A, b) \right] 
& \lambda = 1 \\
\frac{1}{2(b-a)^2} \left[ -A A \mu \mu L_{-2q}^2(A, a) + (A + A \mu) L_{-2q+1}^2(A, a) + L_{-2q+2}^2(A, a) \right] 
& 0 < \lambda < \frac{1}{2} \\
\frac{1}{2(b-a)^2} \left[ -b A L_{-2q}^2(A, b) + (b + A) L_{-2q+1}^2(A, b) - L_{-2q+2}^2(A, b) \right] 
& \frac{1}{2} < \lambda < 1 \\
\frac{1}{2(b-a)^2} \left[ -A b L_{-2q}^2(A, b) + (A + b) L_{-2q+1}^2(A, b) - L_{-2q+2}^2(A, b) \right] 
& \lambda = 1 \\
\end{cases} \]
\[
D_1(\mu, 1, a, b) = \begin{cases} 
\frac{1}{2(b-a)^2} [(A + a)L^{-1}(A, a) - 2], & \mu = 0 \\
\frac{\mu}{(b-a)^2} \left[ \frac{A+a}{\mu} - (A + A_\mu)L^{-1}(A_\mu, a) \right] + \frac{1}{2(b-a)^2} [(A + A_\mu)L^{-1}(A, A_\mu) - 2], & 0 < \mu < \frac{1}{2} \\
\frac{1}{2(b-a)^2} \left[ \frac{A+a}{\mu} - 2AL^{-1}(A, a) \right], & \mu = \frac{1}{2} \\
\frac{1}{2(b-a)^2} [(A + a)L^{-1}(A, a) - 2], & \mu = 0
\end{cases}
\]

\[
D_2(\mu, 1, a, b) = \begin{cases} 
\frac{1}{2(b-a)^2} [(A + a)L^{-1}(A, a) - 2], & \mu = 0 \\
\frac{\mu}{(b-a)^2} \left[ \frac{A+a}{\mu} - (A + A_\mu)L^{-1}(A_\mu, a) \right] + \frac{1}{2(b-a)^2} [(A + A_\mu)L^{-1}(A, A_\mu) - 2], & 0 < \mu < \frac{1}{2} \\
\frac{1}{2(b-a)^2} [(A + a)L^{-1}(A, a) - 2], & \mu = \frac{1}{2} \\
\frac{1}{2(b-a)^2} [(A + a)L^{-1}(A, a) - 2], & \mu = 0
\end{cases}
\]

\[
D_3(\lambda, 1, a, b) = \begin{cases} 
\frac{1}{2(b-a)^2} [(b + A)L^{-1}(A, b) - 2], & \lambda = 0 \\
\frac{\lambda}{(b-a)^2} \left[ \frac{b+\lambda}{\lambda} - (b + A_\lambda)L^{-1}(A, A_\lambda) \right] + \frac{1}{2(b-a)^2} [(b + A_\lambda)L^{-1}(A, A_\lambda) - 2], & 0 < \lambda < 1, \lambda = \frac{1}{2} \\
\frac{1}{2(b-a)^2} [(b + A)L^{-1}(b, A) - 2], & \lambda = 1
\end{cases}
\]

\[
D_4(\lambda, 1, a, b) = \begin{cases} 
\frac{1}{2(b-a)^2} [(b + A)L^{-1}(A, b) - 2], & \lambda = 0 \\
\frac{\lambda}{(b-a)^2} \left[ \frac{b+\lambda}{\lambda} - (b + A_\lambda)L^{-1}(A, A_\lambda) \right] + \frac{1}{2(b-a)^2} [(b + A_\lambda)L^{-1}(A, A_\lambda) - 2], & 0 < \lambda < 1, \lambda = \frac{1}{2} \\
\frac{1}{2(b-a)^2} [(b + A)L^{-1}(A, b) - 2], & \lambda = 1
\end{cases}
\]

**Proof.** Since \( |f'|^q \) is multiplicatively harmonically \( P \)-function on interval \([a, b] \), \( |f'(a)|^q |f'(b)|^q \) for all \( t \in [0,1] \). Hence, using Lemma 1 and improved power-mean integral inequality we obtain

\[
I_f(\lambda, \mu, a, b) \leq 2ab(b-a) \left\{ \left( \int_0^{1/2} \left( \frac{t}{2} - t \right) |\mu - t| dt \right)^{1-\frac{1}{q}} \left( \int_0^{1/2} \left( \frac{t}{2} - t \right) |\mu - t| \frac{|f'(a)|^q |f'(b)|^q}{A_t^q} dt \right)^{\frac{1}{q}} \right. \\
\left. + \left( \int_0^{1/2} t |\mu - t| dt \right)^{1-\frac{1}{q}} \left( \int_0^{1/2} t |\mu - t| \frac{|f'(a)|^q |f'(b)|^q}{A_t^q} dt \right)^{\frac{1}{q}} \right\}
\]

\[
+ 2ab(b-a) \left\{ \left( \int_1^{1/2} (1-t) |\lambda - t| dt \right)^{1-\frac{1}{q}} \left( \int_1^{1/2} (1-t) |\lambda - t| \frac{|f'(a)|^q |f'(b)|^q}{A_t^q} dt \right)^{\frac{1}{q}} \right. \\
\left. + \left( \int_1^{1/2} \left( t - \frac{1}{2} \right) |\lambda - t| dt \right)^{1-\frac{1}{q}} \left( \int_1^{1/2} \left( t - \frac{1}{2} \right) |\lambda - t| \frac{|f'(a)|^q |f'(b)|^q}{A_t^q} dt \right)^{\frac{1}{q}} \right\}
\]

\[
\leq 2ab(b-a)f'(a)|f'(b)| \left\{ \left( \int_0^{1/2} \left( \frac{t}{2} - t \right) |\mu - t| dt \right)^{1-\frac{1}{q}} \left( \int_0^{1/2} \left( \frac{t}{2} - t \right) |\mu - t| \frac{|f'(a)|^q |f'(b)|^q}{A_t^q} dt \right)^{\frac{1}{q}} \right. \\
\left. + \left( \int_0^{1/2} t |\mu - t| dt \right)^{1-\frac{1}{q}} \left( \int_0^{1/2} t |\mu - t| \frac{|f'(a)|^q |f'(b)|^q}{A_t^q} dt \right)^{\frac{1}{q}} \right\}
\]

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\[+2ab(b - a)|f'(a)||f'(b)|\left(\int_{1/2}^{1} (1 - t)|\lambda - t|\,dt\right)^{1 - \frac{1}{\mu}}\left(\int_{1/2}^{1} \frac{(1 - t)|\lambda - t|}{A_t^\mu} \,dt\right)^{\frac{1}{\mu}} \]

where

\[\int_{0}^{1/2} (1 - t)|\lambda - t|\,dt = C_1(\mu) = -\frac{\mu^2}{3} + \frac{\mu^2}{2} - \frac{\mu}{8} + \frac{1}{48}\]

\[\int_{0}^{1/2} t|\mu - t|\,dt = C_2(\mu) = \frac{\mu^3}{3} - \frac{\mu^2}{8} + \frac{1}{24}\]

\[\int_{1/2}^{1} (1 - t)|\lambda - t|\,dt = C_3(\lambda) = -\frac{\lambda^3}{3} + \lambda^2 - \frac{7\lambda}{8} + \frac{1}{16}\]

\[\int_{1/2}^{1} (t - \frac{1}{2})|\lambda - t|\,dt = C_4(\lambda) = \frac{\lambda^3}{3} - \frac{\lambda^2}{8} + \frac{1}{16}\]

which completes the proof.

In the following, Hermite-Hadamard type and Simpson type integral inequalities are obtained in special cases of \(\lambda\) and \(\mu\).

**Corollary 1.** Under the assumptions of Theorem 9 with \(\lambda = \mu = \frac{1}{2}\), the inequality (2.2) reduced to the Hermite-Hadamard type following inequality

\[\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a} \frac{f(u)}{u} \,du\right| \leq 2ab(b - a)|f'(a)||f'(b)| \times \left[C_1 \frac{1}{\mu} D_1^{\frac{1}{\mu}}\left(\frac{1}{2}, 0, a, b\right) + C_2 \frac{1}{\mu} D_2^{\frac{1}{\mu}}\left(\frac{1}{2}, q, a, b\right) + C_3 \frac{1}{\mu} D_3^{\frac{1}{\mu}}\left(\frac{1}{2}, q, a, b\right) + C_4 \frac{1}{\mu} D_4^{\frac{1}{\mu}}\left(\frac{1}{2}, q, a, b\right)\right].\]

**Corollary 2.** Under the assumptions of Theorem 9 with \(\mu = 0\) and \(\lambda = 1\), the inequality (2.2) reduced to the following Hermite-Hadamard type inequality

\[\left|f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a} \frac{f(u)}{u} \,du\right| \leq 2ab(b - a)|f'(a)||f'(b)| \times \left[C_1 \frac{1}{\mu} D_1^{\frac{1}{\mu}}(0, q, a, b) + C_2 \frac{1}{\mu} D_2^{\frac{1}{\mu}}(0, q, a, b) + C_3 \frac{1}{\mu} D_3^{\frac{1}{\mu}}(1, q, a, b) + C_4 \frac{1}{\mu} D_4^{\frac{1}{\mu}}(1, q, a, b)\right].\]

**Corollary 3.** Under the assumptions of Theorem 9 with \(\mu = \frac{1}{6}\) and \(\lambda = \frac{5}{6}\), the inequality (2.2) reduced to the following Simpson type inequality

\[\left|\frac{1}{3} \left[f(a)+f(b)\right] + 2f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a} \frac{f(u)}{u} \,du\right| \leq 2ab(b - a)|f'(a)||f'(b)| \times \left[C_1 \frac{1}{\mu} D_1^{\frac{1}{\mu}}\left(\frac{1}{6}, q, a, b\right) + C_2 \frac{1}{\mu} D_2^{\frac{1}{\mu}}\left(\frac{1}{6}, q, a, b\right) + C_3 \frac{1}{\mu} D_3^{\frac{1}{\mu}}\left(\frac{5}{6}, q, a, b\right) + C_4 \frac{1}{\mu} D_4^{\frac{1}{\mu}}\left(\frac{5}{6}, q, a, b\right)\right].\]
Corollary 4. Under the assumptions of Theorem 9, if we take \( q = 1 \) in the inequality (2.2), then we have the following inequality:

\[
I_f(\lambda, \mu, a, b) \leq 2ab(b - a)|f'(a)||f''(b)|
\times \left[ D_1(\mu, 1, a, b) + D_2(\mu, 1, a, b) + D_3(\lambda, 1, a, b) + D_4(\lambda, 1, a, b) \right].
\]

Theorem 10. Let \( f: I \subset (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^* \) such that \( f' \in L[a, b] \), where \( a, b \in I^* \) with \( a < b \). If the function \(|f'|^q\) is multiplicatively harmonically \( P \)-function on \([a, b]\) for some fixed \( q > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( 0 \leq \mu \leq \frac{1}{2} \leq \lambda \leq 1 \), then the following inequality holds

\[
I_f(\lambda, \mu, a, b) \leq 2ab(b - a)|f'(a)||f''(b)|
\times \left\{ C_5^p(\mu, p)D_5^p(q, a, b) + C_6^p(\mu, p)D_5^p(q, a, b) + C_7^p(\mu, p)D_5^p(q, a, b) + C_8^p(\lambda, p)D_5^p(q, a, b) \right\} \quad (2.4)
\]

where

\[
C_5(\mu, p) = \int_0^{1/2} \left( \frac{1}{2} - t \right) |\mu - t|^p dt = \left( \frac{1}{2} - \mu \right) \left[ \frac{\mu^{p+2} - \left( \frac{1}{2} - \mu \right)^{p+2}}{p+2} \right],
\]

\[
C_6(\mu, p) = \int_0^{1/2} t |\mu - t|^p dt = \mu \left[ \frac{\mu^{p+2} - \left( \frac{1}{2} - \mu \right)^{p+2}}{p+2} \right],
\]

\[
C_7(\lambda, p) = \int_{1/2}^1 (1 - t) |\lambda - t|^p dt = (1 - \lambda) \left[ \frac{(\lambda - \frac{1}{2})^{p+1} + (1 - \lambda)^{p+1}}{p+1} \right] - \left[ \frac{(\lambda - \frac{1}{2})^{p+2} + (1 - \lambda)^{p+2}}{p+2} \right],
\]

\[
C_8(\lambda, p) = \int_{1/2}^1 \left( \frac{1}{2} - t \right) |\lambda - t|^p dt = \left( \frac{1}{2} - \lambda \right) \left[ \frac{(\lambda - \frac{1}{2})^{p+1} - (\frac{1}{2} - \lambda)^{p+1}}{p+1} \right] - \left[ \frac{(\lambda - \frac{1}{2})^{p+2} + \frac{1}{2} - \lambda)^{p+2}}{p+2} \right].
\]

Proof. Since the function \(|f'|^q\) is multiplicatively harmonically \( P \)-function on interval \([a, b]\) and using Lemma 1 and Hölder-İşcan integral inequality, we have

\[
I_f(\lambda, \mu, a, b) \leq 2ab(b - a)|f'(a)||f''(b)| \left\{ \left( \int_0^{1/2} \left( \frac{1}{2} - t \right) |\mu - t|^p dt \right)^{\frac{1}{p}} \left( \int_0^{1/2} \left( \frac{1}{2} - t \right) \frac{|f'(a)|^q |f''(b)|^q}{A_t^{2q}} dt \right)^{\frac{1}{q}} \right\} ^{\frac{1}{p}}
\]

\[
+ \left( \int_0^{1/2} t |\mu - t|^p dt \right)^{\frac{1}{p}} \left( \int_0^{1/2} t \frac{|f'(a)|^q |f''(b)|^q}{A_t^{2q}} dt \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_{1/2}^1 (1 - t) |\lambda - t|^p dt \right)^{\frac{1}{p}} \left( \int_{1/2}^1 (1 - t) \frac{|f'(a)|^q |f''(b)|^q}{A_t^{2q}} dt \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_{1/2}^1 \left( \frac{1}{2} - t \right) |\lambda - t|^p dt \right)^{\frac{1}{p}} \left( \int_{1/2}^1 \left( \frac{1}{2} - t \right) \frac{|f'(a)|^q |f''(b)|^q}{A_t^{2q}} dt \right)^{\frac{1}{q}}
\]

\[
\leq 2ab(b - a)|f'(a)||f''(b)| \left\{ \left( \int_0^{1/2} \left( \frac{1}{2} - t \right) |\mu - t|^p dt \right)^{\frac{1}{p}} \left( \int_0^{1/2} \left( \frac{1}{2} - t \right) A_t^{-2q} dt \right)^{\frac{1}{q}} \right\} ^{\frac{1}{p}}
\]

\[
+ \left( \int_0^{1/2} t |\mu - t|^p dt \right)^{\frac{1}{p}} \left( \int_0^{1/2} t A_t^{-2q} dt \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_{1/2}^1 (1 - t) |\lambda - t|^p dt \right)^{\frac{1}{p}} \left( \int_{1/2}^1 (1 - t) A_t^{-2q} dt \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_{1/2}^1 \left( \frac{1}{2} - t \right) |\lambda - t|^p dt \right)^{\frac{1}{p}} \left( \int_{1/2}^1 \left( \frac{1}{2} - t \right) A_t^{-2q} dt \right)^{\frac{1}{q}}
\]

here it is seen by simple computation that

\[
D_5(q, a, b) = \int_0^{1/2} \left( \frac{1}{2} - t \right) A_t^{-2q} dt = \frac{1}{2(b - a)} \left[ AL_{-2q}^{-2q}(A, a) - L_{-2q+1}^{-2q+1}(A, a) \right],
\]

\[
D_6(q, a, b) = \int_0^{1/2} t A_t^{-2q} dt = \frac{1}{2(b - a)} \left[ L_{-2q+1}^{-2q+1}(A, a) - AL_{-2q}^{-2q}(A, a) \right],
\]

\[
D_7(q, a, b) = \int_{1/2}^1 (1 - t) A_t^{-2q} dt = \frac{1}{2(b - a)} \left[ bL_{-2q}^{-2q}(b, A) - L_{-2q+1}^{-2q+1}(b, A) \right],
\]

\[
D_8(q, a, b) = \int_{1/2}^1 \left( \frac{1}{2} - t \right) A_t^{-2q} dt.
\]
\[ D_\theta(q, a, b) = \int_{q/2}^{1} \left( t - \frac{1}{2} \right) A^2 dt = \frac{1}{2(b-a)} \left[ L^{-2q+1}(b, A) - AL^{-2q}(b, A) \right]. \]

Therefore, the proof is completed.

In the following, Hermite-Hadamard type and Simpson type integral inequalities are obtained in special cases of \( \mu \) and \( \lambda \).

**Corollary 5.** Under the assumptions of Theorem 10 with \( \lambda = \mu = \frac{1}{2} \), the inequality (2.4) reduced to the following Hermite-Hadamard type inequality

\[
\left| f(\frac{a+f(b)}{2}) - \frac{ab}{b-a} \frac{f(u)}{u^2} du \right| \leq 2ab(b-a)|f'(a)||f'(b)| \times \left\{ \frac{1}{5} \left( \frac{1}{5}, p \right) D^2_5(q, a, b) + \frac{1}{6} \left( \frac{1}{6}, p \right) D^3_6(q, a, b) + \frac{1}{7} \left( \frac{1}{7}, p \right) D^4_7(q, a, b) \right\}.
\]

**Corollary 6.** Under the assumptions of Theorem 10 with \( \mu = 0 \) and \( \lambda = 1 \), the inequality (2.4) reduced to the following Hermite-Hadamard type inequality

\[
\left| f(\frac{2ab}{a+b}) - \frac{ab}{b-a} \frac{f(u)}{u^2} du \right| \leq 2ab(b-a)|f'(a)||f'(b)| \times \left\{ \frac{1}{6} \left( 0, p \right) D^2_6(0, a, b) + \frac{1}{6} \left( 1, p \right) D^2_6(q, a, b) + \frac{1}{6} \left( 2, p \right) D^2_6(q, a, b) \right\}.
\]

**Corollary 7.** Under the assumptions of Theorem 10 with \( \mu = \frac{1}{6} \) and \( \lambda = \frac{5}{6} \), the inequality (2.4) reduced to the following Simpson type inequality

\[
\left| \frac{1}{3} \left[ f(\frac{a+f(b)}{2}) \right] - \frac{ab}{b-a} \frac{f(u)}{u^2} du \right| \leq 2ab(b-a)|f'(a)||f'(b)| \times \left\{ \frac{1}{5} \left( \frac{1}{5}, p \right) D^2_5(q, a, b) + \frac{1}{6} \left( \frac{1}{6}, p \right) D^3_6(q, a, b) + \frac{1}{7} \left( \frac{1}{7}, p \right) D^4_7(q, a, b) \right\}.
\]

**Remark 2.** Since \( |f'(x)| = x^q \), \( x > 0 \) and \( |f'(x)| = e^x \) are multiplicatively \( P \)-functions, by considering \( |f'(x)| = x^q \) \( \mu = 1 \) and \( |f'(x)| = e^x \) \( \mu = 0 \) applications can be made for the above two theorems and the results of these theorems.

**REFERENCES**


