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Research Article

HERMITE-HADAMARD AND SIMPSON TYPE INEQUALITIES FOR MULTIPLICATIVELY HARMONICALLY *P*-FUNCTIONS

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ABSTRACT

In this paper, using the Hölder-İşcan and improved power-mean integral inequalities together with an identity, we obtain new estimates on generalization of Hadamard and Simpson type inequalities for multiplicatively harmonically P-functions. The obtained results are compared with the previous ones.
Keywords: Multiplicatively P-function, multiplicatively harmonically P-functions, Hermite–Hadamard type inequalities, Simpson type inequality, Hölder-İşcan inequality, improved power-mean inequality.
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1. INTRODUCTION

Let real function f be defined on some nonempty interval I of the real line $\mathbb{R}$. The function f is said to be convex on interval I if the following inequality

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The following inequalities are well known in the literature as Hermite-Hadamard inequality [2] and Simpson inequality [1] respectively:

Theorem 1. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I and $a, b \in I$. The following double inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Theorem 2. *Let $f: [a,b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

Let $H = H(a,b) = 2ab/(a+b)$, $G = G(a,b) = \sqrt{ab}$, $L = L(a,b) = (b-a)/(\ln b - \ln a)$, $I = I(a,b) = (1/e)(b^b/a^a)^{1/(b-a)}$, $A = A(a,b) = \frac{a+b}{2}$, $A_\lambda = A_\lambda(a,b) := \lambda b + (1-\lambda)a, \lambda \in$

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$[0,1]$ and $L_p = L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}},\ p \in \mathbb{R}\backslash\{-1,0\}$, be the geometric, logarithmic, identric, arithmetic, weighted arithmetic and p-logarithmic means of a and b, respectively. Then

$$min\{a,b\} < H < G < L < I < A < max\{a,b\}.$$

In [3], S.S. Dragomir et.al. defined the following new class of functions.

Definition 1. *We recall that a function $f: I \subset \mathbb{R} \to \mathbb{R}$ is said to be P-function on I or belong to the class $P(I)$ if it is nonnegative and,*

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

for all $x, y \in I$ and $t \in [0,1]$. Note that $P(I)$ contains all nonnegative convex and quasi-convex functions [1].

Example 1. [11] *A non-negative function $f: I \to \mathbb{R}$ is called trigonometrically convex function on interval $[a,b]$, if for each $x, y \in [a,b]$ and $t \in [0,1]$,*

$$f(tx + (1-t)y) \leq \left(sin\frac{\pi t}{2}\right) f(x) + \left(cos\frac{\pi t}{2}\right) f(y).$$

Clearly, if $f(x)$ is a nonnegative function, then every trigonometric convex function is a P-function. Indeed,

$$f(tx + (1-t)y) \leq \left(sin\frac{\pi t}{2}\right) f(x) + \left(cos\frac{\pi t}{2}\right) f(y) \leq f(x) + f(y).$$

In recent years, many authors have studied errors estimations for Hermite-Hadamard and Simpson inequalities; for refinements, counterparts, generalizations see [5, 6, 14] and references therein.

In [4], İşcan gave the definition of harmonically convexity as follows:

Definition 2. *Let $I \subseteq \mathbb{R}\backslash\{0\}$ be a real interval. A function $f: I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.1) is reversed, then f is said to be harmonically concave.

The following result of the Hermite-Hadamard type inequality holds.

Theorem 3. *Let $f: I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

In [10], Kadakal gave the definition of multiplicatively P-function (or log-P-function) as follows:

Definition 3. *Let $I \neq \emptyset$ be an interval in $\mathbb{R}$. The function $f: I \to [0,\infty)$ is said to be multiplicatively P-function (or log-P-function), if the inequality*

$$f(tx + (1-t)y) \leq f(x)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

We will denote by $MP(I)$ the class of all multiplicatively P-functions on interval I. Clearly, $f: I \to (0,\infty)$ is multiplicatively P-function if and only if $\log f$ is P-function.

Remark 1. *The range of the multiplicatively P-functions is greater than or equal to* 1.

The following result of the Hermite-Hadamard type inequalities holds for multiplicatively P-functions:

Theorem 4. *Let the function $f: I \to [1,\infty)$ be a multiplicatively P-function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequalities hold:*

i) $f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \leq [f(a)f(b)]^2$

ii) $f\left(\frac{a+b}{2}\right) \leq f(a)f(b)\frac{1}{b-a}\int_a^b f(x)dx \leq [f(a)f(b)]^2$

Definition 4. *A function $f: I \subseteq (0,\infty) \to \mathbb{R}$ is said to be harmonically P-function on I or belong to the class $HP(I)$ if it is nonnegative and,*

$$f\left(\frac{xy}{ty+(1-t)x}\right) \leq f(x) + f(y),$$

for all $x, y \in I$ and $t \in [0,1]$.

Hermite-Hadamard inequalities can be represented for harmonically P-function as follows.

Theorem 5. *Let $f: I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a harmonically P-function on $[a,b]$, then the following inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{2ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \leq 2[f(a)+f(b)].$$

In [9], İşcan and Olucak gave the definition of multiplicatively harmonically P-function as follows:

Definition 5. *Let $I \neq \emptyset$ be an interval in $\mathbb{R}\backslash\{0\}$. The function $f: I \to [0,\infty)$ is said to be multiplicatively harmonically P-function, if the inequality*

$$f\left(\frac{xy}{ty+(1-t)x}\right) \leq f(x)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Example 2. *The function $f: [1,\infty) \to [1,\infty), f(x) = x$ is a multiplicatively harmonically P-function. Really, for any $x, y \in [1,\infty)$ with $x < y$, we have*

$$f\left(\frac{xy}{ty+(1-t)x}\right) = \frac{xy}{ty+(1-t)x} \leq y \leq xy = f(x)f(y).$$

Example 3. *The function $f: (0,\infty) \to (0,\infty), f(x) = e^x$ is a multiplicatively harmonically P-function. Since, for any $x, y \in (0,\infty)$ with $x < y$, we have*

$$f\left(\frac{xy}{ty+(1-t)x}\right) = e^{\frac{xy}{ty+(1-t)x}} \leq e^y \leq e^x e^y = f(x)f(y).$$

We will denote by $MHP(I)$ the class of all multiplicatively harmonically P-functions on interval I.

The following result of the Hermite-Hadamard type inequalities hold for multiplicatively harmonically P-function:

Theorem 6. *Let the function $f: I \subseteq \mathbb{R}\backslash\{0\} \to [1,\infty)$ be a multiplicatively harmonically P-function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequalities hold:*

i) $f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)f([a^{-1}+b^{-1}-x^{-1}]^{-1})}{x^2}dx \leq [f(a)f(b)]^2$

ii) $f\left(\frac{2ab}{a+b}\right) \leq f(a)f(b)\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq [f(a)f(b)]^2.$

A refinement of Hölder integral inequality better approach than Hölder integral inequality can be given as follows [7]:

Theorem 7 (Hölder-İşcan Integral Inequality [7]). *Let $p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b |f(x)g(x)|dx \le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)|f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)|g(x)|^q dx\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)|f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)|g(x)|^q dx\right)^{\frac{1}{q}}\right\}$$

A refinement of power-mean integral inequality better approach than power-mean inequality as a result of the Hölder-İşcan integral inequality can be given as follows [12]:

Theorem 8 (Improved power-mean integral inequality [12]). *Let $q \ge 1$. If f and g are real functions defined on $[a,b]$ and if $|f|$, $|f||g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b |f(x)g(x)|dx \le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)|f(x)|dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)|f(x)||g(x)|^q dx\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)|f(x)|dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)|f(x)||g(x)|^q dx\right)^{\frac{1}{q}}\right\}$$

In this paper, using Hölder-İşcan integral inequality better approach than Hölder integral inequality and improved power-mean integral inequality better approach than power-mean inequality and together with an integral identity, authors obtain a generalization of Hadamard and Simpson inequalities for functions whose derivatives in absolute value at certain power are multiplicatively harmonically P-functions. In addition, we compare the results with the previous ones.

2. MAIN RESULTS

Let $f: I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I. Throughout this section we will take

$$I_f(\lambda,\mu,a,b) = (\lambda-\mu)f\left(\frac{2ab}{a+b}\right) + (1-\lambda)f(a) + \mu f(b) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du$$

where $a,b \in I$ with $a < b$ and $\lambda,\mu \in \mathbb{R}$.

In order to prove our main results we need the following identity [8].

Lemma 1. *Let $f: I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. Then for all $\lambda,\mu \in \mathbb{R}$ we have:*

$$I_f(\lambda,\mu,a,b) = ab(b-a)\left\{\int_0^{\frac{1}{2}} \frac{\mu-t}{A_t^2} f'\left(\frac{ab}{A_t}\right)dt + \int_{\frac{1}{2}}^1 \frac{\lambda-t}{A_t^2} f'\left(\frac{ab}{A_t}\right)dt\right\}, \tag{2.1}$$

where $A_t = tb + (1-t)a$.

Theorem 9. *Let $f: I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a < b$. If $|f'|^q$ is multiplicatively harmonically P-function on the interval $[a,b]$ for some fixed $q \ge 1$ and $0 \le \mu \le \frac{1}{2} \le \lambda \le 1$, then the following inequality holds* for $q > 1$

$$I_f(\lambda,\mu,a,b) \le 2ab(b-a)|f'(a)||f'(b)|\left[C_1^{1-\frac{1}{q}}(\mu)D_1^{\frac{1}{q}}(\mu,q,a,b) + C_2^{1-\frac{1}{q}}(\mu)D_2^{\frac{1}{q}}(\mu,q,a,b) + C_3^{1-\frac{1}{q}}(\lambda)D_3^{\frac{1}{q}}(\lambda,q,a,b) + C_4^{1-\frac{1}{q}}(\lambda)D_4^{\frac{1}{q}}(\lambda,q,a,b)\right], \tag{2.2}$$

where

$$\int_0^{1/2}\left(\frac{1}{2}-t\right)|\mu-t|dt = C_1(\mu) = -\frac{\mu^3}{3} + \frac{\mu^2}{2} - \frac{\mu}{8} + \frac{1}{48}, \tag{2.3}$$

$$\int_0^{1/2} t|\mu-t|dt = C_2(\mu) = \frac{\mu^3}{3} - \frac{\mu}{8} + \frac{1}{24}$$

$$\int_{1/2}^{1}(1-t)|\lambda-t|dt = C_3(\lambda) = -\frac{\lambda^3}{3}+\lambda^2-\frac{7\lambda}{8}+\frac{1}{4},$$
$$\int_{1/2}^{1}\left(t-\frac{1}{2}\right)|\lambda-t|dt = C_4(\lambda) = \frac{\lambda^3}{3}-\frac{\lambda^2}{2}+\frac{\lambda}{8}+\frac{1}{16}$$

$$D_1(\mu,q,a,b) = \begin{cases} \frac{1}{2(b-a)^2}\left[-L_{-2q+2}^{-2q+2}(A,a)+(A+a)L_{-2q+1}^{-2q+1}(A,a)-aAL_{-2q}^{-2q}(A,a)\right], \\ \qquad \mu=0 \\ \left\{\frac{\mu}{(b-a)^2}\left[AA_\mu L_{-2q}^{-2q}(A_\mu,a)-(A+A_\mu)L_{-2q+1}^{-2q+1}(A_\mu,a)+L_{-2q+2}^{-2q+2}(A_\mu,a)\right]\right. \\ \left.+\frac{(1-2\mu)}{2(b-a)^2}\left[-AA_\mu L_{-2q}^{-2q}(A,A_\mu)+(A+A_\mu)L_{-2q+1}^{-2q+1}(A,A_\mu)-L_{-2q+2}^{-2q+2}(A,A_\mu)\right]\right\} \\ \qquad 0<\mu<\frac{1}{2} \\ \frac{1}{2(b-a)^2}\left[A^2L_{-2q}^{-2q}(A,a)-2AL_{-2q+1}^{-2q+1}(A,a)+L_{-2q+2}^{-2q+2}(A,a)\right], \\ \qquad \mu=\frac{1}{2} \end{cases}$$

$$D_2(\mu,q,a,b) = \begin{cases} \frac{1}{2(b-a)^2}\left[-L_{-2q+2}^{-2q+2}(A,a)-2aL_{-2q+1}^{-2q+1}(A,a)+a^2L_{-2q}^{-2q}(A,a)\right], \\ \qquad \mu=0 \\ \left\{\frac{\mu}{(b-a)^2}\left[-aA_\mu L_{-2q}^{-2q}(A_\mu,a)+(a+A_\mu)L_{-2q+1}^{-2q+1}(A_\mu,a)-L_{-2q+2}^{-2q+2}(A_\mu,a)\right]\right. \\ \left.+\frac{(1-2\mu)}{2(b-a)^2}\left[aA_\mu L_{-2q}^{-2q}(A,A_\mu)-(a+A_\mu)L_{-2q+1}^{-2q+1}(A,A_\mu)+L_{-2q+2}^{-2q+2}(A,A_\mu)\right]\right\} \\ \qquad 0<\mu<\frac{1}{2} \\ \frac{1}{2(b-a)^2}\left[-aAL_{-2q}^{-2q}(A,a)+(a+A)L_{-2q+1}^{-2q+1}(A,a)-L_{-2q+2}^{-2q+2}(A,a)\right], \\ \qquad \mu=\frac{1}{2} \end{cases}$$

$$D_3(\lambda,q,a,b) = \begin{cases} \frac{1}{2(b-a)^2}\left[-bAL_{-2q}^{-2q}(A,b)+(b+A)L_{-2q+1}^{-2q+1}(A,b)-L_{-2q+2}^{-2q+2}(A,b)\right], \\ \qquad \lambda=\frac{1}{2} \\ \left\{\frac{(1-\lambda)}{(b-a)^2}\left[-bA_\lambda L_{-2q}^{-2q}(A_\lambda,b)+(b+A_\lambda)L_{-2q+1}^{-2q+1}(A_\lambda,b)-L_{-2q+2}^{-2q+2}(A_\lambda,b)\right]\right. \\ \left.+\frac{(2\lambda-1)}{2(b-a)^2}\left[-bA_\lambda L_{-2q}^{-2q}(A,A_\lambda)-(b+A_\lambda)L_{-2q+1}^{-2q+1}(A,A_\lambda)+L_{-2q+2}^{-2q+2}(A_\lambda,A)\right]\right\} \\ \qquad \frac{1}{2}<\lambda<1 \\ \frac{1}{2(b-a)^2}\left[b^2L_{-2q}^{-2q}(b,A)-2bL_{-2q+1}^{-2q+1}(b,A)+L_{-2q+2}^{-2q+2}(b,A)\right], \\ \qquad \lambda=1 \end{cases}$$

$$D_4(\lambda,q,a,b) = \begin{cases} \frac{1}{2(b-a)^2}\left[A^2L_{-2q}^{-2q}(A,b)-2AL_{-2q+1}^{-2q+1}(A,b)+L_{-2q+2}^{-2q+2}(A,b)\right], \\ \qquad \lambda=\frac{1}{2} \\ \left\{\frac{(1-\lambda)}{(b-a)^2}\left[AA_\lambda L_{-2q}^{-2q}(A_\lambda,b)-(A+A_\lambda)L_{-2q+1}^{-2q+1}(A_\lambda,b)+L_{-2q+2}^{-2q+2}(A_\lambda,b)\right]\right. \\ \left.+\frac{(2\lambda-1)}{2(b-a)^2}\left[-AA_\lambda L_{-2q}^{-2q}(A,A_\lambda)+(A+A_\lambda)L_{-2q+1}^{-2q+1}(A,A_\lambda)-L_{-2q+2}^{-2q+2}(A_\lambda,A)\right]\right\} \\ \qquad \frac{1}{2}<\lambda<1 \\ \frac{1}{2(b-a)^2}\left[-AbL_{-2q}^{-2q}(A,b)+(A+b)L_{-2q+1}^{-2q+1}(A,b)-L_{-2q+2}^{-2q+2}(A,b)\right], \\ \qquad \lambda=1 \end{cases}$$

$$D_1(\mu,1,a,b)=\begin{cases}\frac{1}{2(b-a)^2}\left[(A+a)L^{-1}(A,a)-2\right], & \mu=0\\ \left\{\frac{\mu}{(b-a)^2}\left[\frac{A+a}{a}-\left(A+A_\mu\right)L^{-1}\left(A_\mu,a\right)\right]\right.\\ \left.+\frac{(1-2\mu)}{2(b-a)^2}\left[\left(A+A_\mu\right)L^{-1}\left(A,A_\mu\right)-2\right]\right\} & 0<\mu<\frac{1}{2},\\ \frac{1}{2(b-a)^2}\left[\frac{A+a}{2}-2AL^{-1}(A,a)\right], & \mu=\frac{1}{2}\end{cases}$$

$$D_2(\mu,1,a,b)=\begin{cases}\frac{1}{2(b-a)^2}\left[\frac{A+a}{2}-2aL^{-1}(A,a)\right], & \mu=0\\ \left\{\frac{\mu}{(b-a)^2}\left[\left(a+A_\mu\right)L^{-1}\left(A_\mu,a\right)-2\right]\right.\\ \left.+\frac{(1-2\mu)}{2(b-a)^2}\left[\frac{A+a}{A}-\left(a+A_\mu\right)L^{-1}\left(A,A_\mu\right)\right]\right\} & 0<\mu<\frac{1}{2},\\ \frac{1}{2(b-a)^2}\left[(A+a)L^{-1}(A,a)-2\right], & \mu=\frac{1}{2}\end{cases}$$

$$D_3(\lambda,1,a,b)=\begin{cases}\frac{1}{2(b-a)^2}\left[(b+A)L^{-1}(A,b)-2\right], & \lambda=\frac{1}{2}\\ \left\{\frac{(1-\lambda)}{(b-a)^2}\left[(b+A_\lambda)L^{-1}(A_\lambda,b)-2\right]\right.\\ \left.+\frac{(2\lambda-1)}{2(b-a)^2}\left[\frac{A+b}{A}-(b+A_\lambda)L^{-1}(A,A_\lambda)\right]\right\} & \frac{1}{2}<\lambda<1,\\ \frac{1}{2(b-a)^2}\left[\frac{b+A}{2}-2bL^{-1}(b,A)\right], & \lambda=1\end{cases}$$

$$D_4(\lambda,1,a,b)=\begin{cases}\frac{1}{2(b-a)^2}\left[\frac{b+A}{b}-2AL^{-1}(A,b)\right], & \lambda=\frac{1}{2}\\ \left\{\frac{(1-\lambda)}{(b-a)^2}\left[\frac{b+A}{b}+(A+A_\lambda)L^{-1}(A_\lambda,b)\right]\right.\\ \left.+\frac{(2\lambda-1)}{2(b-a)^2}\left[(A+A_\lambda)L^{-1}(A,A_\lambda)-2\right]\right\} & \frac{1}{2}<\lambda<1,\\ \frac{1}{2(b-a)^2}\left[(A+b)L^{-1}(A,b)-2\right], & \lambda=1.\end{cases}$$

Proof. Since $|f'|^q$ is multiplicatively harmonically P-function on interval $[a,b]$, $\left|f'\left(\frac{ab}{A_t}\right)\right|^q\leq |f'(a)|^q|f'(b)|^q$ for all $t\in[0,1]$. Hence, using Lemma 1 and improved power-mean integral inequality we obtain

$$\begin{aligned}I_f(\lambda,\mu,a,b)\leq\ & 2ab(b-a)\left\{\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)|\mu-t|dt\right)^{1-\frac{1}{q}}\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)|\mu-t|\frac{|f'(a)|^q|f'(b)|^q}{A_t^{2q}}dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^{1/2}t|\mu-t|dt\right)^{1-\frac{1}{q}}\left(\int_0^{1/2}t|\mu-t|\frac{|f'(a)|^q|f'(b)|^q}{A_t^{2q}}dt\right)^{\frac{1}{q}}\right\}\\ &+2ab(b-a)\left\{\left(\int_{1/2}^1(1-t)|\lambda-t|dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^1(1-t)|\lambda-t|\frac{|f'(a)|^q|f'(b)|^q}{A_t^{2q}}dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{1/2}^1\left(t-\frac{1}{2}\right)|\lambda-t|dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^1\left(t-\frac{1}{2}\right)|\lambda-t|\frac{|f'(a)|^q|f'(b)|^q}{A_t^{2q}}dt\right)^{\frac{1}{q}}\right\}\\ &\leq 2ab(b-a)|f'(a)||f'(b)|\left\{\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)|\mu-t|dt\right)^{1-\frac{1}{q}}\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)\frac{|\mu-t|}{A_t^{2q}}dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^{1/2}t|\mu-t|dt\right)^{1-\frac{1}{q}}\left(\int_0^{1/2}t|\mu-t|\frac{1}{A_t^{2q}}dt\right)^{\frac{1}{q}}\right\}\end{aligned}$$

$$+2ab(b-a)|f'(a)||f'(b)|\left\{\left(\int_{1/2}^{1}(1-t)|\lambda-t|dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^{1}\frac{(1-t)|\lambda-t|}{A_t^{2q}}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{1/2}^{1}\left(t-\frac{1}{2}\right)|\lambda-t|dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\frac{|\lambda-t|}{A_t^{2q}}dt\right)^{\frac{1}{q}}\right\},$$

where

$$\int_0^{1/2}\left(\frac{1}{2}-t\right)|\mu-t|dt = C_1(\mu) = -\frac{\mu^3}{3}+\frac{\mu^2}{2}-\frac{\mu}{8}+\frac{1}{48},$$
$$\int_0^{1/2} t|\mu-t|dt = C_2(\mu) = \frac{\mu^3}{3}-\frac{\mu}{8}+\frac{1}{24}$$
$$\int_{1/2}^{1}(1-t)|\lambda-t|dt = C_3(\lambda) = -\frac{\lambda^3}{3}+\lambda^2-\frac{7\lambda}{8}+\frac{1}{4},$$
$$\int_{1/2}^{1}\left(t-\frac{1}{2}\right)|\lambda-t|dt = C_4(\lambda) = \frac{\lambda^3}{3}-\frac{\lambda^2}{2}+\frac{\lambda}{8}+\frac{1}{16}$$
$$\int_0^{1/2}\left(\frac{1}{2}-t\right)|\mu-t|\frac{1}{A_t^{2q}}dt = D_1(\mu,q,a,b), \qquad \int_0^{1/2} t|\mu-t|\frac{1}{A_t^{2q}}dt = D_2(\mu,q,a,b)$$
$$\int_{1/2}^{1}(1-t)|\lambda-t|\frac{1}{A_t^{2q}}dt = D_3(\lambda,q,a,b), \quad \int_{1/2}^{1}\left(t-\frac{1}{2}\right)|\lambda-t|\frac{1}{A_t^{2q}}dt = D_4(\lambda,q,a,b)$$

which completes the proof.

In the following, Hermite-Hadamard type and Simpson type integral inequalities are obtained in special cases of λ and μ.

Corollary 1. *Under the assumptions of Theorem 9 with $\lambda=\mu=\frac{1}{2}$, the inequality (2.2) reduced to the Hermite-Hadamard type following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right| \le 2ab(b-a)|f'(a)||f'(b)|$$
$$\times$$
$$\left[C_1^{1-\frac{1}{q}}\left(\frac{1}{2}\right)D_1^{\frac{1}{q}}\left(\frac{1}{2},q,a,b\right)+C_2^{1-\frac{1}{q}}\left(\frac{1}{2}\right)D_2^{\frac{1}{q}}\left(\frac{1}{2},q,a,b\right)\ C_3^{1-\frac{1}{q}}\left(\frac{1}{2}\right)D_3^{\frac{1}{q}}\left(\frac{1}{2},q,a,b\right)+\right.$$
$$\left.C_4^{1-\frac{1}{q}}\left(\frac{1}{2}\right)D_4^{\frac{1}{q}}\left(\frac{1}{2},q,a,b\right)\right].$$

Corollary 2. *Under the assumptions of Theorem 9 with $\mu=0$ and $\lambda=1$, the inequality (2.2) reduced to the following Hermite-Hadamard type inequality*

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right| \le 2ab(b-a)|f'(a)||f'(b)|$$
$$\times$$
$$\left[C_1^{1-\frac{1}{q}}(0)D_1^{\frac{1}{q}}(0,q,a,b)+C_2^{1-\frac{1}{q}}(0)D_2^{\frac{1}{q}}(0,q,a,b)+C_3^{1-\frac{1}{q}}(1)D_3^{\frac{1}{q}}(1,q,a,b)+C_4^{1-\frac{1}{q}}(1)D_4^{\frac{1}{q}}(1,q,a,b)\right].$$

Corollary 3. *Under the assumptions of Theorem 9 with $\mu=\frac{1}{6}$ and $\lambda=\frac{5}{6}$, the inequality (2.2) reduced to the following Simpson type inequality*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{2ab}{a+b}\right)\right]-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right| \le 2ab(b-a)|f'(a)||f'(b)|$$
$$\times$$
$$\left[C_1^{1-\frac{1}{q}}\left(\frac{1}{6}\right)D_1^{\frac{1}{q}}\left(\frac{1}{6},q,a,b\right)+C_2^{1-\frac{1}{q}}\left(\frac{1}{6}\right)D_2^{\frac{1}{q}}\left(\frac{1}{6},q,a,b\right)+C_3^{1-\frac{1}{q}}\left(\frac{5}{6}\right)D_3^{\frac{1}{q}}\left(\frac{5}{6},q,a,b\right)+\right.$$
$$\left.C_4^{1-\frac{1}{q}}\left(\frac{5}{6}\right)D_4^{\frac{1}{q}}\left(\frac{5}{6},q,a,b\right)\right].$$

Corollary 4. *Under the assumptions of Theorem 9, If we take $q=1$ in the inequality (2.2), then we have the following inequality:*

$$I_f(\lambda,\mu,a,b)\le 2ab(b-a)|f'(a)||f'(b)|$$
$$\times\left[D_1(\mu,1,a,b)+D_2(\mu,1,a,b)+D_3(\lambda,1,a,b)+D_4(\lambda,1,a,b)\right].$$

Theorem 10. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If the function $|f'|^q$ is multiplicatively harmonically P-function on $[a,b]$ for some fixed $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $0\le\mu\le\frac{1}{2}\le\lambda\le 1$, then the following inequality holds*

$$I_f(\lambda,\mu,a,b)\le 2ab(b-a)|f'(a)||f'(b)|$$
$$\times\left\{C_5^{\frac{1}{p}}(\mu,p)D_5^{\frac{1}{q}}(q,a,b)+C_6^{\frac{1}{p}}(\mu,p)D_6^{\frac{1}{q}}(q,a,b)+C_7^{\frac{1}{p}}(\lambda,p)D_7^{\frac{1}{q}}(q,a,b)+C_8^{\frac{1}{p}}(\lambda,p)D_8^{\frac{1}{q}}(q,a,b)\right\} \quad (2.4)$$

where

$$C_5(\mu,p)=\int_0^{1/2}\left(\frac{1}{2}-t\right)|\mu-t|^p dt=\left(\frac{1}{2}-\mu\right)\left[\frac{\mu^{p+1}+\left(\frac{1}{2}-\mu\right)^{p+1}}{p+1}\right]+\left[\frac{\mu^{p+2}-\left(\frac{1}{2}-\mu\right)^{p+2}}{p+2}\right],$$
$$C_6(\mu,p)=\int_0^{1/2}t|\mu-t|^p dt=\mu\left[\frac{\mu^{p+1}+\left(\frac{1}{2}-\mu\right)^{p+1}}{p+1}\right]+\left[\frac{\left(\frac{1}{2}-\mu\right)^{p+2}-\mu^{p+2}}{p+2}\right],$$
$$C_7(\lambda,p)=\int_{1/2}^1(1-t)|\lambda-t|^p dt=(1-\lambda)\left[\frac{\left(\lambda-\frac{1}{2}\right)^{p+1}+(1-\lambda)^{p+1}}{p+1}\right]+\left[\frac{\left(\lambda-\frac{1}{2}\right)^{p+2}-(1-\lambda)^{p+2}}{p+2}\right],$$
$$C_8(\lambda,p)=\int_{1/2}^1\left(t-\frac{1}{2}\right)|\lambda-t|^p dt=\left(\lambda-\frac{1}{2}\right)\left[\frac{\left(\lambda-\frac{1}{2}\right)^{p+1}-\left(\frac{1}{2}-\lambda\right)^{p+1}}{p+1}\right]-\left[\frac{\left(\lambda-\frac{1}{2}\right)^{p+2}+\left(\frac{1}{2}-\lambda\right)^{p+2}}{p+2}\right]$$

Proof. Since the function $|f'|^q$ is multiplicatively harmonically P-function on interval $[a,b]$ and using Lemma 1 and Hölder-İşcan integral inequality, we have

$$I_f(\lambda,\mu,a,b)\le 2ab(b-a)\left\{\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)|\mu-t|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)\frac{|f'(a)|^q|f'(b)|^q}{A_t^{2q}}dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^{1/2}t|\mu-t|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1/2}t\frac{|f'(a)|^q|f'(b)|^q}{A_t^{2q}}dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{1/2}^1(1-t)|\lambda-t|^p dt\right)^{\frac{1}{p}}\left(\int_{1/2}^1(1-t)\frac{|f'(a)|^q|f'(b)|^q}{A_t^{2q}}dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{1/2}^1\left(t-\frac{1}{2}\right)|\lambda-t|^p dt\right)^{\frac{1}{p}}\left(\int_{1/2}^1\left(t-\frac{1}{2}\right)\frac{|f'(a)|^q|f'(b)|^q}{A_t^{2q}}dt\right)^{\frac{1}{q}}\right\}$$
$$\le 2ab(b-a)|f'(a)||f'(b)|\left\{\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)|\mu-t|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)A_t^{-2q}dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^{1/2}t|\mu-t|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1/2}tA_t^{-2q}dt\right)^{\frac{1}{q}}+\left(\int_{1/2}^1(1-t)|\lambda-t|^p dt\right)^{\frac{1}{p}}\left(\int_{1/2}^1(1-t)A_t^{-2q}dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{1/2}^1\left(t-\frac{1}{2}\right)|\lambda-t|^p dt\right)^{\frac{1}{p}}\left(\int_{1/2}^1\left(t-\frac{1}{2}\right)A_t^{-2q}dt\right)^{\frac{1}{q}}\right\},$$

here it is seen by simple computation that

$$D_5(q,a,b)=\int_0^{1/2}\left(\frac{1}{2}-t\right)A_t^{-2q}dt=\frac{1}{2(b-a)}\left[AL_{-2q}^{-2q}(A,a)-L_{-2q+1}^{-2q+1}(A,a)\right],$$
$$D_6(q,a,b)=\int_0^{1/2}tA_t^{-2q}dt=\frac{1}{2(b-a)}\left[L_{-2q+1}^{-2q+1}(A,a)-aL_{-2q}^{-2q}(A,a)\right],$$
$$D_7(q,a,b)=\int_{1/2}^1(1-t)A_t^{-2q}dt=\frac{1}{2(b-a)}\left[bL_{-2q}^{-2q}(b,A)-L_{-2q+1}^{-2q+1}(b,A)\right],$$

$$D_8(q,a,b)=\int_{1/2}^{1}\left(t-\frac{1}{2}\right)A_t^{-2q}dt=\frac{1}{2(b-a)}\left[L_{-2q+1}^{-2q+1}(b,A)-AL_{-2q}^{-2q}(b,A)\right].$$

Therefore, the proof is completed.

In the following, Hermite-Hadamard type and Simpson type integral inequalities are obtained in special cases of λ and μ.

Corollary 5. *Under the assumptions of Theorem 10 with $\lambda=\mu=\frac{1}{2}$, the inequality (2.4) reduced to the following Hermite-Hadamard type inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right|\leq 2ab(b-a)|f'(a)||f'(b)|$$
$$\times\left\{C_5^{\frac{1}{p}}\left(\frac{1}{2},p\right)D_5^{\frac{1}{q}}(q,a,b)+C_6^{\frac{1}{p}}\left(\frac{1}{2},p\right)D_6^{\frac{1}{q}}(q,a,b)+C_7^{\frac{1}{p}}\left(\frac{1}{2},p\right)D_7^{\frac{1}{q}}(q,a,b)+C_8^{\frac{1}{p}}\left(\frac{1}{2},p\right)D_8^{\frac{1}{q}}(q,a,b)\right\}.$$

Corollary 6. *Under the assumptions of Theorem 10 with $\mu=0$ and $\lambda=1$, the inequality (2.4) reduced to the following Hermite-Hadamard type inequality*

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right|\leq 2ab(b-a)|f'(a)||f'(b)|$$
$$\times\left\{C_5^{\frac{1}{p}}(0,p)D_5^{\frac{1}{q}}(q,a,b)+C_6^{\frac{1}{p}}(0,p)D_6^{\frac{1}{q}}(q,a,b)+C_7^{\frac{1}{p}}(1,p)D_7^{\frac{1}{q}}(q,a,b)+C_8^{\frac{1}{p}}(1,p)D_8^{\frac{1}{q}}(q,a,b)\right\}.$$

Corollary 7. *Under the assumptions of Theorem 10 with $\mu=\frac{1}{6}$ and $\lambda=\frac{5}{6}$, the inequality (2.4) reduced to the following Simpson type inequality*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{2ab}{a+b}\right)\right]-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right|\leq 2ab(b-a)|f'(a)||f'(b)|$$
$$\times\left\{C_5^{\frac{1}{p}}\left(\frac{1}{6},p\right)D_5^{\frac{1}{q}}(q,a,b)+C_6^{\frac{1}{p}}\left(\frac{1}{6},p\right)D_6^{\frac{1}{q}}(q,a,b)+C_7^{\frac{1}{p}}\left(\frac{5}{6},p\right)D_7^{\frac{1}{q}}(q,a,b)+C_8^{\frac{1}{p}}\left(\frac{5}{6},p\right)D_8^{\frac{1}{q}}(q,a,b)\right\}.$$

Remark 2. *Since $|f'(x)|^q=x$ and $|f'(x)|^q=e^x$ are multiplicatively P-functions, by considering $|f'(x)|=x^{\frac{1}{q}}, x>0$ and $|f'(x)|=e^x$ functions, applications can be made for the above two theorems and the results of these theorems.*

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