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Research Article

EULER-LAGRANGIAN DYNAMICAL SYSTEMS WITH RESPECT TO HORIZONTAL AND VERTICAL LIFTS ON TANGENT BUNDLE

Haşim ÇAYIR*[1]

[1]*Department of Mathematics, Giresun University, GİRESUN;* ORCID: 0000-0003-0348-8665

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ABSTRACT

The differential geometry and mahthematical physics has lots of applications. The Euler-Lagrangian mechanics are very important tools for differential geometry, classical and analytical machanics. There are many studies about Euler-Lagrangian dynamics, mechanics, formalisms, systems and equations. The classic mechanics firstly introduced by J. L. Lagrange in 1788. Because of the investigation of tensorial structures on manifolds and extension by using the lifts to the tangent or cotangent bundle, it is possible to generalize to differentiable structures on any space (resp. manifold) to extended spaces (resp. extended manifolds) [5, 6, 9]. In this study, the Euler-Lagrangian theories, which are mathematical models of mechanical systems are structured on the horizontal and the vertical lifts of an almost complex structure in tangent bundle TM. In the end, the geometrical and physical results related to Euler-Lagrangian dynamical systems are concluded.
Keywords: Euler-Lagrangian equations, dynamical systems, horizontal lift, vertical lift, tangent bundle.

1. INTRODUCTION

The differential geometry and mahthematical physics has lots of applications. The Euler-Lagrangian mechanics are very important tools for differential geometry, classical and analytical machanics.There are many studies about Euler-Lagrangian dynamics, mechanics, formalisms, systems and equations. The classic mechanics firstly introduced by J. L. Lagrange in 1788. In 1962, Klein submitted the dynamic equations for mechanical systems [4]. In 1989, M. De Leon, P.R. Rodrigues studyed the methods of differential geometry in analytical mechanics [2]. Later, Tekkoyun obtained paracomplex structure of Euler-Lagrange and Hamilton equations [7] and Kasap submitted thet the Weyl-Euler-Lagrange and Weyl-Hamilton equations on R_n^{2n} [3]. Because of the investigation of tensorial structures on manifolds and extension by using the lifts to the tangent or cotangent bundle, it is possible to generalize to differentiable structures on any space (resp. manifold) to extended spaces (resp. extended manifolds) [5, 6, 9]. In this study, the Euler-Lagrangian theories, which are mathematical models of mechanical systems are structured on the horizontal and the vertical lifts of an almost complex structure in tangent bundle. In the end, the geometrical and physical results related to Euler-Lagrangian dynamical systems are concluded.

* Corresponding Author: e-mail: hasim.cayir@giresun.edu.tr, tel: (454) 310 14 44

In this context this paper consists of two main sections. In the first section, we give some properties about the horizontal and vertical lifts of a vector field on the tangent bundle, the Euler-Lagrange (Euler's or Lagr ange's) equations and mechanical systems. In the final section, the results of the Euler-Lagrange equations with respect to horizontal and vertical lifts of an almost complex structure and the mechanical systems will be investigated on tangent bundle TM.

1.1. The Verical and Horizontal Lifts on Tangent Bundle

Let M be an $n-$dimensional Riemannian manifold with a Riemannian metric g and denote by $\pi : TM \to M$ its tangent bundle with fiber the tangent spaces to M. TM is then a $2n-$dimensional smooth manifold and some local charts induced naturally from local charts on M may be used. Namely, a system of local coordinates (U, x^i) in M induces on TM a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{i}} = y^i)$, where (x^i), $i = 1,...,n$ is a local coordinate system defined in the neighborhood U and (y^i) is the Cartesian coordinates in each tangent space $T_P M$ at an arbitrary point P in U with respect to the natural basis $\left\{ \frac{\partial}{\partial x^i} \middle| P \right\}$. Summation over repeated indices in always implied.

Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expressions in U of a vector field X on M. The vertical lift X^V and the horizontal lift X^H of X are then given respectively by [9]

$$X^V = X^i \partial_{\bar{i}}, \tag{1.1}$$

$$X^H = X^i \partial_i - y^j \Gamma^i_{jk} X^k \partial_{\bar{i}} \tag{1.2}$$

with respect to the induced coordinates, where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$ and Γ^i_{jk} are the coefficients of the Levi-Civita connection ∇ of g, and $X^V, X^H \in \Im^1_0(TM)$ of $X \in \Im^1_0(M)$.

Definition 1.1.1 Let M be a manifold with an affine connection ∇ and a tensor field $\tilde{F}$ of type $(1,1)$ in TM by [9]

$$\tilde{F} X^H = X^V \text{ and } \tilde{F} X^V = -X^H \tag{1.3}$$

for any $X \in \Im^1_0(M)$. Then we obtain

$$\tilde{F}^2 = -I$$

So, $\tilde{F}$ is an almost complex structure in tangent bundle. In fact, we have by virtue of (1.3)

$\tilde{F}^2 X^H = \tilde{F}(\tilde{F}X^H) = \tilde{F}X^V = -X^H$ and $\tilde{F}^2 X^V = \tilde{F}(\tilde{F}X^V) = \tilde{F}(-X^H) = -X^V$ for any $X \in \Im_0^1(M)$.

The projection $\pi : TM \to M$, $\pi(u) = x$, a po int $u \in TM$ will be denoted by (x, y), its local coordinates being (x^i, y^i). There are the natural basis $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i})$ and dual basis (dx^i, dy^i) of the tangent space TM and the cotangent space $T^*(M)$ at the point $u \in TM$, respectively. In addition, any vector field $X \in \chi(TM)$ can be uniquely written as follows (see [2] p. 197)

$$X = X^H + X^V, \tag{1.4}$$

where $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ is a local basis adapted to the horizontal distribution and the vertical disribution. Then $(dx^i, \delta y^i)$ is dual basis of $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ basis. Moreover, from the sources (see [8] p. 76) and (see [9] p. 88), we can write

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma_j^i(x, y)\frac{\partial}{\partial y^i} \text{ and } \delta y^i = dy^i + \Gamma_j^i(x, y)dx^j, \tag{1.5}$$

where Γ_j^i are local coefficients of nonlinear connectin Γ on TM, $\tilde{F}$ is an almost complex structure on TM .

Thus, from Definition 1 and (1.5) we get

$$\tilde{F}(\frac{\delta}{\delta x^i}) = \frac{\partial}{\partial y^i}; \tilde{F}(\frac{\partial}{\partial y^i}) = -\frac{\delta}{\delta x^i}. \tag{1.6}$$

1.2. Euler-Lagrangian Dynam cs System and Equations on Tangent Bundle

The Langrangian mechanics is a reformulation of classical mechanics. Lagrangian mechanics is widely used to solve mechanical problems in physics and engineering when Newton's formulation of classical mechanics is not convenient. Lagrangian mechanics applies to the dynamics of particles, fields are described using a Lagrangian density. Lagrange's equations are also used in optimization problems of dynamic systems. A Euler-Lagrange and Hamilton space has been certified as an excellent model for some important problems in relativity, gauge theory and electromagnetism. Euler-Lagrangian gives a model for both the gravitational and electromagnetic field in a very natural blending of the geometrical structures of the space with the characteristic properties of these physical fields. Klein (1962) submitted the dynamic equations for mechanical systems that his description is as follows [4].

Let M be an $n-$dimensional manifold and TM its tangent bundle with canonical projection $\tau_M : TM \to M$. TM is called the phase space of velocities of the base manifold M . Let $L : TM \to R$ be a differentiable function on TM and is called the Langrangian function. We consider closed $2-$form on TM and $\Phi_L = -d(d_J L)$ and i_X is $2-$form reduction function that reduces the $1-$form. Consider the equation

$$i_X \Phi_L = dE_L, \tag{1.7}$$

where the semispray X is a vector field. A vector field arises in a situation where, for some reason, there is a direction and magnitude assigned to each point of the space or of a surface, typically examples are fluid dynamics, wheather prediction,... A classical example would be to represent the velocity of the wind with a vector that it does not depend on the time.

We shall see that for motion in a potential, $E_L = V(L) - L$ is an energy function and $V = J(X)$ a Liouville vector field. Here dE_L denotes the differential of E. We shall see that (1.7) under a certain condition on X is the intrinsical expression of the Euler-Lagrange equations of motion. This equation is named as Euler-Lagrange dynamic equation. The triple (TM, Φ_L, X) is known as Euler-Langrangian system on the tangent bundle TM. The operations run on (1.7) for any coordinate system $(q^i(t), p_i(t))$. Infinite dimension Euler-Lagrangian's equation is obtained the form below [1, 2]:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}^i}\right) - \frac{\partial}{\partial q^i} = 0, \frac{dq^i}{dt} = \dot{q}^i, i = 1,..., n. \tag{1.8}$$

In this paper all geometrical object fields and all mappings are considered of the class C^∞, expressed by the words " differentiate" or " smooth" . The indices i, j ...run over set $\{1,..., n\}$ and Einstein convention of summarizing is adopted over all this paper. R, $\mathfrak{F}(TM)$, $\chi(TM)$, $\chi(T^*M)$ denote the set of real numbers, the set of real functions on TM, the set of vector fields on TM and the set of $1-$forms on T^*M.

2. EULER-LAGRANGİAN DYNAMİCAL SYSTEMS ON TANGENT BUNDLE

In this section, the Euler-Lagrange equations for classical mechanics structured by means of almost complex structure $\tilde{F}$ defined by (1.3) under the consideration of the basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ on distributions horizontal and vertical of the tangent bundle TM of manifold M. Let (x^i, y^i) be its local coordinates. For the vector field $X \in \chi(TM)$ given by (1.4), we have

$$X = X^i \frac{\delta}{\delta x^i} + X^i \frac{\partial}{\partial y^i}, \tag{2.1}$$

where the dot indicates the derivative with respect to time t. The Liouville vector field on the bundle denoted by $V = \tilde{F}(X)$ and expressed by

$$V = \tilde{F}(X) = \tilde{F}(X^i \frac{\delta}{\delta x^i} + X^i \frac{\partial}{\partial y^i}) \tag{2.2}$$
$$= X^i \frac{\partial}{\partial y^i} - X^i \frac{\delta}{\delta x^i}$$

The maps given by T, $P : TM \to R$ such that $T = \frac{1}{2} m_i (x^i)^2$, $P = m_i gh$ are called the kinetic energy and potential energy of the mechanical systems, respectively. Here m_i, g and h stand for mass of a mechanical systems having m particles, the gravity acceleration and distance to the origin of a mechanical system on the tangent bundle TM, respectively. Then $L : TM \to R$ is a map that satisfies the conditions;

$i)$ $L = T - P$ is a Lagrangian function,

$ii)$ the function given by $E_L = V(L) - L$ is a Lagrangian energy.

The operator $i_{\tilde{F}}$ induced by $\tilde{F}$ and shown by

$$i_{\tilde{F}} \omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., \tilde{F}(X_i), ..., X_r) \tag{2.3}$$

is said to be vertical derivation, where $\omega \in \wedge^r TM$, $X_i \in \chi(TM)$. The vertical differentiation $d_{\tilde{F}}$ is defined by

$$d_{\tilde{F}} = [i_{\tilde{F}}, d] = i_{\tilde{F}} d - d i_{\tilde{F}} \tag{2.4}$$

where d is the usual exterior derivation. For an almost complex structure $\tilde{F}$, the closed fundemental form is the closed $2-$form by $\Phi_L = -dd_{\tilde{F}} L$ such that

$$d_{\tilde{F}} : \hat{o}(TM) \to T^* M \tag{2.5}$$

In addition, for $d = \frac{\delta}{\delta x^j} dx^j + \frac{\partial}{\partial y^j} \delta y^j$ (see [8] p. 77), we obtain

$$d_{\tilde{F}} = \tilde{F}(d) = \frac{\partial}{\partial y^i} dx^i - \frac{\delta}{\delta x^i} \delta y^i \tag{2.6}$$

and

$$d_{\tilde{F}} L = \tilde{F}(d) = \frac{\partial L}{\partial y^i} dx^i - \frac{\delta L}{\delta x^i} \delta y^i, \tag{2.7}$$

where L is a Lagrangian function.

We consider closed $2-$form on TM and $\Phi_L = -d(d_J L)$ and i_X is $2-$form reduction function that reduces the $1-$form, we obtain

$$\Phi_L = -(\frac{\delta}{\delta x^j}dx^j + \frac{\partial}{\partial y^j}\delta y^j)(\frac{\partial L}{\partial y^i}dx^i - \frac{\delta L}{\delta x^i}\delta y^i) \tag{2.8}$$

$$= -\frac{\delta(\partial L)}{\delta x^j \partial y^i}dx^j \wedge dx^i + \frac{\delta^2 L}{\delta x^j \delta x^i}dx^j \wedge \delta y^i$$

$$-\frac{\partial^2 L}{\partial y^j \partial y^i}\delta y^j \wedge dx^i + \frac{\partial(\delta L)}{\partial y^j \delta x^i}\delta y^j \wedge \delta y^i$$

Let X be the second order differential equation (semispray) determined by (1.7) and i_X is $2-$form reduction function that reduces the $1-$form

$$i_X \Phi_L = \Phi_L(X) = -X^i \frac{\delta(\partial L)}{\delta x^j \partial y^j}\delta_i^j dx^i + X^i \frac{\delta(\partial L)}{\delta x^j \partial y^i}\delta_i^i dx^j \tag{2.9}$$

$$+ X^i \frac{\delta^2 L}{\delta x^j \delta x^i}\delta_i^j \delta y^i - X^i \frac{\delta^2 L}{\delta x^j \delta x^i}\delta_i^i dx^j - X^i \frac{\partial^2 L}{\partial y^j \partial y^i}\delta_i^j dx^i$$

$$+ X^i \frac{\partial^2 L}{\partial y^j \partial y^i}\delta_i^i \delta y^j + X^i \frac{\partial(\delta L)}{\partial y^j \delta x^i}\delta_i^j \delta y^i - X^i \frac{\partial(\delta L)}{\partial y^j \delta x^i}\delta_i^i \delta y^j$$

$$= X^i \frac{\delta(\partial L)}{\delta x^j \partial y^i}dx^j - X^i \frac{\delta^2 L}{\delta x^j \delta x^i}dx^j + X^i \frac{\partial^2 L}{\partial y^j \partial y^i}\delta y^j - X^i \frac{\partial(\delta L)}{\partial y^j \delta x^i}\delta y^j,$$

where $(f \wedge g)(v) = f(v)g - g(v)f$, δ is the Kronecker delta, $(dx^i \wedge dx^j)(\frac{\partial}{\partial x_k}) = dx^i \frac{\partial}{\partial x_k}dx^j - dx^i dx^j \frac{\partial}{\partial x_k} = \delta_{x_k}^i dx^j - dx^i \delta_{x_k}^j$ and $\delta_{x_k}^i = 0, \delta_i^i = 1.$

Since closed $2-$form Φ_L on TM is in the symplectic structure, it is found

$$E_L = V(L) - L = \widetilde{F}(X)(L) - L = X^i \frac{\partial L}{\partial y^i} - X^i \frac{\delta L}{\delta x^i} - L. \tag{2.10}$$

Hence, for

$$d = \frac{\delta}{\delta x^j}dx^j + \frac{\partial}{\partial y^j}\delta y^j,$$

we get

$$dE_L = (\frac{\delta}{\delta x^j}dx^j + \frac{\partial}{\partial y^j}\delta y^j)(X^i\frac{\partial L}{\partial y^i} - X^i\frac{\delta L}{\delta x^i} - L) \tag{2.11}$$
$$= X^i\frac{\delta(\partial L)}{\delta x^j \partial y^i}dx^j - X^i\frac{\delta^2 L}{\delta x^j \delta x^i}dx^j - \frac{\delta L}{\delta x^j}dx^j$$
$$+ X^i\frac{\partial^2 L}{\partial y^j \partial y^i}\delta y^j - X^i\frac{\partial(\delta L)}{\partial y^j \delta x^i}\delta y^j - \frac{\partial L}{\partial y^j}\delta y^j$$

Using the equation $i_X\Phi_L = dE_L$ defined by (1.7) and considering (2.9) and (2.11), we get

$$X^i\frac{\delta(\partial L)}{\delta x^j \partial y^i}dx^j - X^i\frac{\delta^2 L}{\delta x^j \delta x^i}dx^j + X^i\frac{\partial^2 L}{\partial y^j \partial y^i}\delta y^j - X^i\frac{\partial(\delta L)}{\partial y^j \delta x^i}\delta y^j \tag{2.12}$$

$$= X^i\frac{\delta(\partial L)}{\delta x^j \partial y^i}dx^j - X^i\frac{\delta^2 L}{\delta x^j \delta x^i}dx^j - \frac{\delta L}{\delta x^j}dx^j + X^i\frac{\partial^2 L}{\partial y^j \partial y^i}\delta y^j$$
$$- X^i\frac{\partial(\delta L)}{\partial y^j \delta x^i}\delta y^j - \frac{\partial L}{\partial y^j}\delta y^j.$$

Thus, we obtain the following equation

$$-X^i\frac{\delta^2 L}{\delta x^i \delta x^j}dx^j - X^i\frac{\delta(\partial L)}{\partial y^i \delta x^j}dx^j + \frac{\delta L}{\delta x^j}dx^j + X^i\frac{\partial(\delta L)}{\delta x^i \partial y^j}\delta y^j + X^i\frac{\partial^2 L}{\partial y^i \partial y^j}\delta y^j + \frac{\partial L}{\partial y^j}\delta y^j = 0. \tag{2.13}$$

If a curvature denoted by $\alpha : R \to TM$ is considered to be an integral curve of $X \in \chi(TM)$ defined by (2.1), i.e. $X(\alpha(t)) = \frac{d\alpha(t)}{dt}$, then we obtain

$$-\frac{d}{dt}(\frac{\delta L}{\delta x^j}) + \frac{\delta L}{\delta x^j} + \frac{d}{dt}(\frac{\partial L}{\partial y^j}) + \frac{\partial L}{\partial y^j} = 0. \tag{2.14}$$

or

$$\frac{d}{dt}(\frac{\delta L}{\delta x^j}) - \frac{\delta L}{\delta x^j} = 0 \tag{2.15}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial y^j}) + \frac{\partial L}{\partial y^j} = 0. \tag{2.16}$$

Thus, the equations given by (2.15) and (2.16) are seen to be a Euler-Lagrange equations according to the horizontal and vertical distribution on TM. The triple (TM, Φ_L, X) is seen to be a mechanical system with taking into account the basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ with respect to the horizontal and vertical distributions, respectively.

3. CONCLUSION

In this paper, the Euler-Lagrangian theories, which are mathematical models of mechanical systems are structured on the horizontal and the vertical lifts of an almost complex structure in tangent bundle TM. In the end, the geometrical and physical results related to Euler-Lagrangian dynamical systems are concluded.

REFERENCES

[1] R. Abraham, J.E. Marsden, T. Ratiu, Manifolds Tensor Analysis and Applications, Springer, 2001.

[2] M. de Leon, P.R. Rodrigues, Methods of Differential Geometry in Analytical Mechanics, Elsevier Sc. Pub. Com. Inc., 1989.

[3] Z. Kasap, Weyl-mechanical systems on tangent manifolds of constant W-sectional curvature, IJGMMP, 10 (2013), no.10, 1-13.

[4] J. Klein, Escapes Varialionnels et Mécanique, Ann. Inst. Fourier, Gronoble, 12 (1962).

[5] A. A. Salimov, Tensor Operators and Their applications, Nova Science Publ., New York, 2013.

[6] S. Sasaki, On the diferential geometry of tangent bundles of Riemannian manifolds, Tohoku Math J, 10 (1958), 338-358.

[7] M. Tekkoyun, On para-Euler-Lagrange and para-Hamilton equations, Phys. Lett. A, 340 (2005), 7-11.

[8] M. Tekkoyun, Mechanical Systems on Manifolds, Geometry Balkan Press, Bucharest, Romania, 2014.

[9] K. Yano, S .Ishihara, Tangent and Cotangent Bundles Differential geometry. Pure and Applied Mathematics. Mercel Dekker, Inc, New York, 1973.