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Research Article

OPTIMAL BOUNDARY CONTROL FOR A SECOND STRAIN GRADIENT THEORY-BASED BEAM MODEL

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ABSTRACT

The second strain gradient theory is a non-classical continuum theory that captures the behavior of micrometer and nanometer sized beam structures. Timoshenko and Euler-Bernoulli theories are classical beam models that neglect the effects of small size structures when compared to the second strain gradient theory-based beam model. In this study, an optimal boundary control problem is formulated for the second strain gradient theory-based beam model to control free vibrations in the system. A quadratic performance index expressing the dynamic response of the system is to be minimized while an affordable control is in use. An indirect method based on Pontryagin's maximum principle is used to derive a necessary condition analytically for optimal control. Then, the problem is transformed into a system of partial differential equations consisting of state and costate (adjoint) variables together. The solution of the control problem is carried out using the computer codes produced in MATLAB©. The effectiveness and competence of the introduced optimal boundary control are presented in numerical simulations.

Keywords: Second strain gradient theory-based beam, boundary control, maximum principle, vibration.

1. INTRODUCTION

Different beam theories such as Euler-Bernoulli, Timoshenko, and Mindlin have been developed over the last decades for the analysis of the vibrations ([1], [2]). Timoshenko beam models are generally described by the system of PDEs including the displacement, rotation angle and their derivatives with respect to space and time variables. Euler-Bernoulli beam is suitable when the cross-sectional dimension of the beam is negligible relative to its length ([3], [4]). Euler-Bernoulli and Timoshenko beam theories comparing to Mindlin's beam theory have a disadvantage for obtaining the accurate results due to ignoring the microstructural effects in the beam. Due to this observation, Mindlin presented a general theory (1964) to characterize the elastic behavior of isotropic materials taking into account of microstructural effects ([5], [6], [7]). Mindlin also considered that strains and gradient of strains are quadratically formed the potential energy density and kinetic energy function consisting of the quadratic form of both velocities and

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gradients of velocities [8]. However, using higher order gradients introduces new constants that are difficult to be determined. To overcome this difficulty Mindlin proposed three simpler versions of his theory. These new versions are known as Form I, II and III which conclude the same equation of motion. Further, Mindlin suggested a new theory, known as the second gradient elastic theory that captures the behavior of micrometer and nanometer-sized structures (1965). In his new theory, the potential energy is dependent on strains, the gradient of strains and the second gradient of str ains. The constitutive equation is a partial differential equation of sixth order ([9], [10], [11]).

In [12], a microscale Timoshenko beam model is developed based on the strain gradient elasticity theory. Ouakad et al. used a nonlocal strain gradient theory to discuss the static and dynamic response of a carbon nanotube that is electrically actuated. The aim of the authors is to examine the vibrational response of the nano-actuator with the effect of length-scale parameters. In their study, the nano-actuator is modeled as Euler–Bernoulli beam [13]. Al-shujairi et al. [14] studied the dynamic stability of a micro-beam exposed to a parametric axial excitation with different boundary conditions consisting of thermal effects. Ji et al. studied static and dynamic analyses of micro-beams by comparing the strain gradient effects for each component [15]. Oskouie et al. [16] developed Timoshenko nanobeams' strain gradient formulations. Shokravi [17] presented a forced vibration response in nanocomposite cylindrical shells - based on strain gradient beam theory. A new mesh free method for modeling strain gradient micro beams is presented by Sayyidmousavi et al [18]. Ghazavi et al. [19] investigated the nonlinear analysis of the micro-nanotube based on second strain gradient theory.

Control of vibrations in structures is an ongoing research area over the last decades due to the long life span of structures. Singh et al. used piezoelectric materials to the active control of vibrations for a beam [20]. The control of vibrations with viscoelastic materials is studied by Grootenhuis [21]. Kucuk et al. [22] presented an optimal vibration control of piezolaminated smart beams by the maximum principle. In [23], Yildirim et al. presented the vibration control of the Timoshenko beam as a differential equation including the derivatives of the state variable and the fourth order time derivative by using the Pontryagin's maximum principle. Studies in the literature about vibration control and control strategies can be examined in the literature [5, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. By comparing the studies existing in the literature, the present paper has the following research results, which are important for future studies in this area. First result is to control free vibrations in the beam which is modelled using second strain gradient theory that is capable of micro and nano effects in structures. Second result is to obtain the optimal control function analytically using Pontryagin maximum principle for a beam system including sixth order derivative with respect to space variable. The present study deals with a beam model that expresses accurately the deformation behavior of beam structures at micro and nanoscales. The original contribution of the present paper to the literature is that the boundary vibration control of a second strain gradient theory based beam system, including the sixth order spatial derivative, is studied via the Pontryagin's maximum principle. Also, the mathematical model of the beam includes w_{tt} and w_{xxtt}. They represent the size effects in micro and nanoscale structures in the beam. By means of these terms, terminal conditions of the adjoint equation is re-figured by comparing the studies existing in the literature.The rest of the paper is organized as follows. The problem formulation, uniqueness, and controllability of the system are examined in Section 2. In Section 3, we discuss the optimal control problem. In Section 4, a boundary control characterization is introduced and the optimal control function is obtained via the Pontryagin's maximum principle. This approach leads to a system of partial differential equations where state and adjoint variables associated with the fixed terminal time, initial and boundary conditions. The obtained system is solved by using MATLAB. In Section 5, numerical results are presented to verify the effectiveness and competence of the introduced boundary control algorithm. The cost functional to be minimized is specified as a weighted quadratic functional of the dynamic response of the beam. The expenditure of the control energy is taken in the cost functional as a penalty term. Section 6 closes the paper by stating the findings of the present paper.

2. PROBLEM FORMULATION

The beam model under consideration is initially an undeformed rest position - Figure 1 and is mathematically formulated as follows [10]:

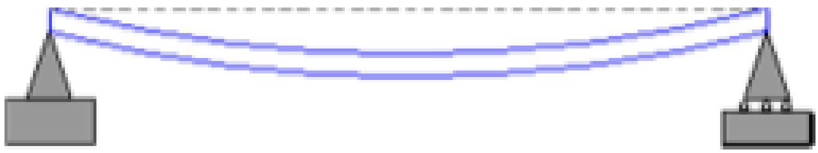

Figure 1. The scheme of the beam

$$w_{xx}-\frac{13}{12}\ell^2 w_{xxxx}+\frac{\ell^4}{72}w_{xxxxxx}-\frac{1}{c^2}w_{tt}+\frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho}w_{xxtt}=0 \tag{1}$$

where $w=w(x,t)\in S=(0,\ell)\times(0,t_f)$ is the transversal displacement at position x and time t, ℓ is the length of the beam, t_f is the fixed terminal time, $c^2=\frac{E}{\rho}$, c is a constant, E is Young's modulus, ρ' and ρ is the density of the microstructural effects. Eq. (1) is subjected to the following boundary conditions,

$$\begin{aligned} w(0,t) &= w(\ell,t) &&= 0\\ w_{xx}(0,t) &= w_{xx}(\ell,t) &&= 0\\ w_{xxxx}(0,t) &= w_{xxxx}(\ell,t) &&= p(t) \end{aligned} \tag{2}$$

where $p(t)$ is the control function to be computed optimally and the initial conditions,

$$w(x,0)=w_0(x),\quad w_t(x,0)=w_1(x) \tag{3}$$

in which $w_0(x)$ and $w_1(x)$ are known functions.
Assume that

$$\frac{\partial^j w}{\partial t^j},\frac{\partial^i w}{\partial x^i},\frac{\partial^{k+j} w}{\partial t^k\partial x^j}\in L^2(S),\ \ i=0,\ldots,6,\ \ j=0,1,2,\ \ k=0,1,2; \tag{4a}$$

$$p(t)\text{ is an analytic function, and }w_0(x)\in H^1(0,\ell), w_1(x)\in L^2(0,\ell), \tag{4b}$$

where $H(0,\ell)$=$L^2(0,\ell)$ is Hilbert space. Let the admissible control set be

$$P_{ad}=\{p:(0,t_f)\to\mathbb{R},\quad |p(t)|<m_0<\infty\},\ m_0\text{ is a constant},$$

in which the inner product of two functions q and r on S with

$$<q,r>=\int_S q(s)r(s)ds \tag{5}$$

and the norm

$$\|q\|^2=<q,q>$$

for all $q,r\in L^2(0,\ell)$. Then, the system given by Eqs. (1)-(3) has a solution [35].

The uniqueness of the solution to the system given by Eqs. (1)-(3) is shown next to make use of it in the uniqueness of the control.

Lemma 1. *The problem given by Eqs (1)-(3) has a unique solution.*
Proof. Suppose that w_1 and w_2 are two solutions to the system given by Eqs. (1)-(3). Then, the difference function

$$u(x,t) = w_1(x,t) - w_2(x,t)$$

will satisfy the homogeneous equation

$$u_{xx} - \frac{13}{12}\ell^2 u_{xxxx} + \frac{\ell^4}{72}u_{xxxxxx} - \frac{1}{c^2}u_{tt} + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho}u_{xxtt} = 0 \tag{6}$$

subject to homogeneous boundary conditions and initial conditions, respectively,

$$u(0,t) = u(\ell,t) = u_{xx}(0,t) = u_{xx}(\ell,t) = u_{xxxx}(0,t) = u_{xxxx}(\ell,t) = 0, \tag{7}$$

$$u(x,0) = u_t(x,0) = 0. \tag{8}$$

For the uniqueness, let the energy integral be introduced as

$$E(t) = \frac{1}{2}\int_0^\ell \{\frac{1}{c^2}u_t^2(x,t) + \frac{\ell^2\acute{\rho}}{3c^2\rho}u_{xt}^2(x,t) + u_x^2(x,t) + \frac{13\ell^2}{12}u_{xx}^2(x,t) + \frac{\ell^2}{72}u_{xxx}^2(x,t)\}dx, \tag{9}$$

and show that it is independent of t. Differentiating $E(t)$ with respect to t yields

$$\frac{dE(t)}{dt} = \int_0^\ell \{\frac{1}{c^2}u_{tt} - \frac{\ell^2\acute{\rho}}{3c^2\rho}u_{xxtt} - u_{xx} + \frac{13\ell^2}{12}u_{xxxx} - \frac{\ell^2}{72}u_{xxxxxx}\}u_t dx$$
$$+\{u_{xtt}u_t + u_x u_t + u_{xx}u_{xt} - u_{xxx}u_t + u_{xxx}u_{xxt} - u_{xxxx}u_{xt} + u_{xxxxx}u_t\}|_0^\ell.$$

By using Eq. (6) and boundary conditions given by (7), it follows that

$$\frac{dE(t)}{dt} = 0,$$

that is, $E(t) = constant$. Taking the initial conditions given by Eq. (8) into consideration, the following equality holds

$$E(t) = constant = E(0) = \frac{1}{2}\int_0^\ell \{\frac{1}{c^2}u_t^2(x,t) + \frac{\ell^2\acute{\rho}}{3c^2\rho}u_{xt}^2(x,t) + u_x^2(x,t) + \frac{13\ell^2}{12}u_{xx}^2(x,t) + \frac{\ell^2}{72}u_{xxx}^2(x,t)\}|_{t=0}dx = 0.$$

It immediately follows from Eq. (9) and the initial conditions given by Eq. (8) that $u(x,t)$ is identically equal to zero over S, i.e., $w_1 = w_2$, that completes the proof.
The uniqueness of the solution for the beam system defined by Eqs. (1)-(3) implies the uniqueness of the control function [36]. Hence, the studied system is observable and controllable by Hilbert uniqueness theorem [6], [34].

3. OPTIMAL CONTROL PROBLEM

It is desired to determine an optimal control function $p(t)$ placed on the boundary to damp out undesired vibrations. To this end, the cost functional (performance index) that is to be minimized over the time interval $0 \le t \le t_f$ is defined in two parts: The first part measures the dynamical response of the system at the terminal time t_f and the second term is the penalty function that minimizes the expenditure of the control force used over $[0, t_f]$. The performance index is defined as

$$\mathcal{J}(p(t)) = \int_0^\ell [\mu_1 w^2(x,t_f) + \mu_2 w_t^2(x,t_f)]dx + \mu_3\int_0^{t_f} p^2(t)dt \tag{10}$$

where $p(t) \in L^2(0,t_f)$ is to be determined optimally and μ_1, μ_2 and μ_3 weighting coefficients satisfying $\mu_1, \mu_2 \ge 0$, $\mu_1 + \mu_2 \ne 0$ and $\mu_3 > 0$. Hence, the optimal boundary control problem of our main interest is expressed in the following manner:

$$\mathcal{J}(p^\circ(t)) = \min_{p(t)\in P_{ad}} \mathcal{J}(p(t)) \tag{11}$$

subject to Eqs. (1)-(3).

4. BOUNDARY CONTROL CHARACTERIZATION

The Pontryagin's maximum principle provides an optimal control function analytically and is used to derive a necessary condition for the optimal control in terms of Hamiltonian. Since the performance index function satisfies convexity, optimality conditions obtained from results of maximum principle is also sufficient condition [30]. The Pontryagin's maximum principle also enables us to obtain a relationship between state and control variables implicitly. For this purpose, let us introduce a Hamiltonian and an adjoint variable v. The adjoint system related to Eqs. (1)-(3) is

$$v_{xx}-\frac{13}{12}\ell^2 v_{xxxx}+\frac{\ell^4}{72}v_{xxxxxx}-\frac{1}{c^2}v_{tt}+\frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho}v_{xxtt}=0. \tag{12}$$

The boundary conditions are

$$\begin{aligned} v(0,t)=v(\ell,t) &= 0,\\ v_{xx}(0,t)=v_{xx}(\ell,t) &= 0,\\ v_{xxxx}(0,t)=v_{xxxx}(\ell,t) &= 0, \end{aligned} \tag{13}$$

and terminal conditions are

$$\begin{aligned} \frac{\ell^2\acute{\rho}}{3c^2\rho}v_{xxt}(x,t_f)-\frac{1}{c^2}v_t(x,t_f) &= 2\mu_1 w(x,t_f),\\ \frac{\ell^2\acute{\rho}}{3c^2\rho}v_{xx}(x,t_f)-\frac{1}{c^2}v(x,t_f) &= -2\mu_2 w_t(x,t_f). \end{aligned} \tag{14}$$

For the problem given by Eq. (1)-(3), the maximum principle is expressed as follows:

Theorem 1. (**Maximum Principle)** *If $p^\circ(t)\in P_{ad}$ is an optimal control then it satisfies the maximum principle:*

$$max_{p(t)\in P_{ad}}\ \mathcal{H}(t;v,p)=\mathcal{H}(t;v^\circ,p^\circ) \tag{15}$$

where the Hamiltonian is

$$\mathcal{H}(t;v,p)=-p(t)R(t)-\mu_3 p^2(t) \tag{16}$$

in which

$$R(t)=\frac{\ell^4}{72}\big(v_x(0,t)-v_x(\ell,t)\big). \tag{17}$$

Proof. First, let us introduce an operator,

$$\Psi(w)=w_{xx}-\frac{13}{12}\ell^2 w_{xxxx}+\frac{\ell^4}{72}w_{xxxxxx}-\frac{1}{c^2}w_{tt}+\frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho}w_{xxtt}, \tag{19}$$

and deviations are given by

$$\Delta w=w(x,t)-w^\circ(x,t),\quad \Delta p=p(x,t)-p^\circ(x,t). \tag{20}$$

The operator defined by Eq. (19) satisfies

$$\Psi(\Delta w)=0,$$

and subjected to the following boundary conditions and initial conditions, respectively,

$$\Delta w(x,t)=\Delta w_{xx}(x,t)=0,\quad \Delta w_{xxxx}(x,t)=\Delta p(t),\quad x=0 \text{ and } x=\ell \tag{21}$$

$$\Delta w(x,0)=\Delta w_t(x,0)=0. \tag{22}$$

Consider

$$\int_0^\ell \int_0^{t_f} (\Delta w \Psi(v) - v\Psi(\Delta w))dtdx = \int_0^\ell \int_0^{t_f} \{ \Delta w(v_{xx} - \frac{13}{12}\ell^2 v_{xxxx} + \frac{\ell^4}{72} v_{xxxxxx} - \frac{1}{c^2} v_{tt} + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho} v_{xxtt})$$
$$-v(\Delta w_{xx} - \frac{13}{12}\ell^2 \Delta w_{xxxx} + \frac{\ell^4}{72}\Delta w_{xxxxxx} - \frac{1}{c^2}\Delta w_{tt} + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho}\Delta w_{xxtt})\} dtdx$$
$$= \underbrace{\int_0^\ell \int_0^{t_f} [\Delta w v_{xx} - v\Delta w_{xx}]dtdx}_{I} + \underbrace{\int_0^\ell \int_0^{t_f} [\frac{13}{12}\ell^2 \Delta w_{xxxx} v - \frac{13}{12}\ell^2 v_{xxxx}\Delta w]dtdx}_{II}$$
$$+ \underbrace{\int_0^\ell \int_0^{t_f} [\frac{\ell^4}{72}\Delta w v_{xxxxxx} - \frac{\ell^4}{72} v\Delta w_{xxxxxx}]dtdx}_{III} +$$
$$\underbrace{\int_0^\ell \int_0^{t_f} [\frac{1}{c^2}\Delta w_{tt} v - \frac{1}{c^2} v_{tt}\Delta w]dtdx}_{IV}$$
$$+ \underbrace{\int_0^\ell \int_0^{t_f} [\frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho}\Delta w v_{xxtt} - \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho} v\Delta w_{xxtt}]dtdx}_{V} = 0. \quad (23)$$

In view of Eqs. (21)-(22), the integration by parts is applied to the terms I, II, III, IV, V Eq. (23). It is observed that calculating I

$I = \int_0^\ell \int_0^{t_f} [\Delta w v_{xx} - v\Delta w_{xx}]dtdx = 0$ (Integration by parts used twice).

Similarly using integration by parts four times in II yields

$$II = \int_0^\ell \int_0^{t_f} [\frac{13}{12}\ell^2 \Delta w_{xxxx} v - \frac{13}{12}\ell^2 v_{xxxx}\Delta w]dtdx = 0.$$

By applying integration by parts six times to III, the following relation is obtained

$$III = \int_0^\ell \int_0^{t_f} [\frac{\ell^4}{72}\Delta w v_{xxxxxx} - \frac{\ell^4}{72} v\Delta w_{xxxxxx}]dtdx =$$
$$\int_0^{t_f} \frac{\ell^4}{72}(v_x(\ell,t)\Delta w_{xxxx}(\ell,t) - v_x(0,t)\Delta w_{xxxx}(0,t))dt.$$

Similarly, the following equalities are observed:

$$IV = \int_0^\ell \int_0^{t_f} [\frac{1}{c^2}\Delta w_{tt} v - \frac{1}{c^2} v_{tt}\Delta w]dtdx = \int_0^\ell \frac{1}{c^2}(\Delta w_t(x,t_f)v(x,t_f) - \Delta w(x,t_f)v_t(x,t_f))dx,$$
$$V = \int_0^\ell \int_0^{t_f} [\frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho}\Delta w v_{xxtt} - \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho} v\Delta w_{xxtt}]dtdx = \int_0^\ell \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho}(\Delta w(x,t_f)v_{xxt}(x,t_f) - \Delta w_t(x,t_f)v_{xx}(x,t_f))dx.$$

Substituting these equalities into Eq. (23) and using terminal conditions given by Eq. (14) give

$$\int_0^\ell \{ 2\mu_1 \Delta w(x,t_f)w(x,t_f) + 2\mu_2 \Delta w_t(x,t_f)w_t(x,t_f)\} dx = \int_0^{t_f} \{ \frac{\ell^4}{72}(v_x(0,t) - v_x(\ell,t))\} \Delta p(t)dt.$$

To analyze the deviation in the performance index functional, the following equality is observed

$$\Delta\mathcal{J}(p) = \mathcal{J}(p) - \mathcal{J}(p^\circ) = \int_0^\ell \{ \mu_1[w^2(x,t_f) - w^{\circ^2}(x,t_f)] + \mu_2[w_t^2(x,t_f) - w_t^{\circ^2}(x,t_f)]\} dx$$
$$+\mu_3 \int_0^{t_f} [p^2(t) - p^{\circ^2}(t)]dt \quad (24)$$

Using Taylor series for $w^2(x,t_f)$ and $w_t^2(x,t_f)$ about $w^\circ(x,t_f)$ and $w_t^\circ(x,t_f)$, respectively, leads to the following relation

$$w^2(x,t_f) - w^{\circ^2}(x,t_f) = 2w^\circ(x,t_f)\Delta w(x,t_f) + r_1, \quad (25a)$$

$$w_t^2(x,t_f) - w_t^{o^2}(x,t_f) = 2w_t^\circ(x,t_f)\Delta w_t(x,t_f) + r_2, \quad (25b)$$

where remainders r_1 and r_2 are

$$r_1 = 2(\Delta w)^2 + \ldots > 0 \text{ and } \quad r_2 = 2(\Delta w_t)^2 + \ldots > 0.$$

Substituting Eq. (25) into Eq. (24) results in

$$\Delta\mathcal{J}(p)=\int_0^{\ell}\{2\mu_1[w^o(x,t_f)\Delta w^\circ(x,t_f)+r_1]+2\mu_2[w_t^\circ(x,t_f)\Delta w_t^\circ(x,t_f)+r_2]\}dx+\mu_3\int_0^{t_f}[p^2(t)-p^{\circ 2}(t)]dt\geq 0.$$

Since $2\mu_1 r_1+2\mu_2 r_2\geq 0$, the following inequality is obtained

$$\Delta\mathcal{J}(p)\geq\int_0^{t_f}\{\frac{\ell^4}{72}(v_x(0,t)-v_x(\ell,t))\}f(t)dt+\mu_3\int_0^{t_f}[p^2(t)-p^{\circ 2}(t)]dt\geq 0$$

and

$$\Delta\mathcal{J}(p)\geq\int_0^{t_f}\{\frac{\ell^4}{72}p(t)R(t)+\mu_3 p^2(t)-(\frac{\ell^4}{72}p^\circ(t)R(t)+\mu_3 p^{\circ 2}(t))\}dt\geq 0.$$

Hence,

$$\max\mathcal{H}(t;v,p)=\mathcal{H}(t;v^\circ,p^\circ),\forall\, p\in P_{ad}$$

and

$$\mathcal{J}(p)\geq\mathcal{J}(p^\circ),\forall\, p\in P_{ad}.$$

The first variation of Hamiltonian $\mathcal{H}(t;v,p)$ vanishes at p°; therefore, the optimal control function indicates clearly as follows;

$$p^\circ(t)=\frac{\ell^4}{144}\frac{\{v_x(\ell,t)-v_x(0,t)\}}{\mu_3}. \quad (26)$$

5. NUMERICAL SIMULATIONS

In this section, numerical simulations of the developed theory in the previous sections are presented to show the effectiveness of the technique by using MATLAB. In this process, firstly homogeneous boundary conditions are obtained by defining a new variable. Then, the adjoint system given by Eqs. (12)-(14) is solved using the eigenfunction expansion method in terms of the *sine* Fourier series and optimal control function Eq. (26) is computed. Finally, the obtained distributed parameter system with homogeneous boundary conditions is solved by using the *N*th-terms of *sine* Fourier series, similarly by finding undetermined constants.
In the numerical simulations, the following parameters are taken [37]:

- The size of the unit cell is taken as ℓ =1 *m*,
- The density of the microstructural effects are given by $\rho'=\rho=6.10^4 kg/m^3$,
- $c^2=\frac{E}{\rho}$, $E=2.10^7 N/m^2$, $t_f=3$,
- The displacement and velocity of the beam are given at the midpoint of the beam, that is, $x=0.5$,
- Weighting coefficients are taken into account as $\mu_1=\mu_2=1$ and $\mu_3=10^{-3}$ for controlled case.

Also, in the simulations, it is assumed that beam under consideration subject to the following initial conditions;

$$w(x,0)=\sqrt{2}\sin(\pi x),\quad w_t(x,0)=\sqrt{2}\cos(\pi x).$$

Let us define the dynamical response of the system at the terminal time t_f and control force spend in $[0,t_f]$ as follows, respectively,

$$\mathcal{J}(w)=\int_0^1[w^2(x,t_f)+w_t^2(x,t_f)]dx,\quad \mathcal{J}(p)=\int_0^{t_f}p^2(t)dt.$$

The results obtained for $\mathcal{J}(w)$ and $\mathcal{J}(p)$ by using the values $\mu_{1,2} = 1$ and different values of μ_3 are summarized in Table 1. By observing Table 1, it is concluded that, as the weighted coefficient μ_3 on the control function decreases, the dynamic response of the beam decreases corresponding to an increase in the control force. Similar observations can be done by examining Figure 2 and Figure 3. Displacement and velocity of the beam are plotted for the controlled and uncontrolled cases in Figure 2 and Figure 3, respectively. These figures show that introduced boundary control damps out the free vibrations such that the amplitude of the vibrations is close to zero at the terminal time t_f. By observing the Figure 2, the difference between controlled and uncontrolled displacements of the beam means that introduced boundary control algorithm is very effective and it reaches the objective of the control problem. Figure 4 shows the optimal control solutions for the case with $t_f = 3$ and different weighted coefficients μ_3 .

Table 1. The results obtained for $\boldsymbol{\mathcal{J}(w)}$ and $\boldsymbol{\mathcal{J}(p)}$ by using the values $\boldsymbol{\mu_{1,2} = 1}$ and different values of $\boldsymbol{\mu_3}$

μ_3	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
$\mathcal{J}(w)$	0.6583e-4	0.5986e-2	0.5766	46.6916	1166.54	3238.67	3717.51
$\mathcal{J}(p)$	8151.19	8132.44	7952.54	6440.64	1609.13	44.67	0.51

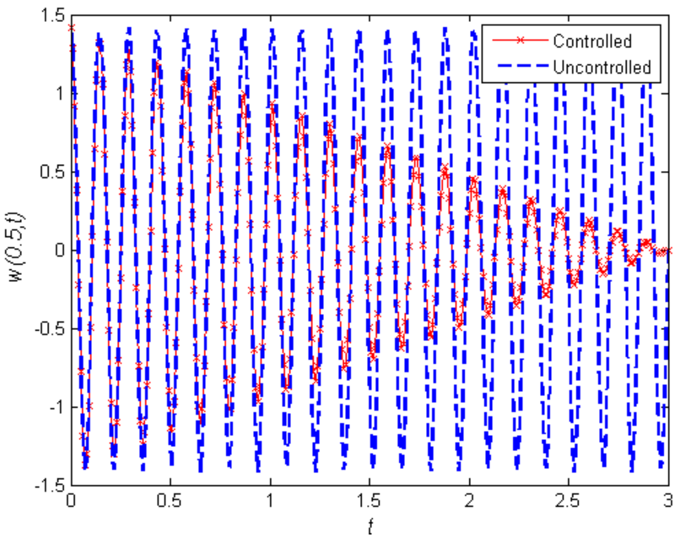

Figure 2. The displacement in the controlled/ uncontrolled beam.

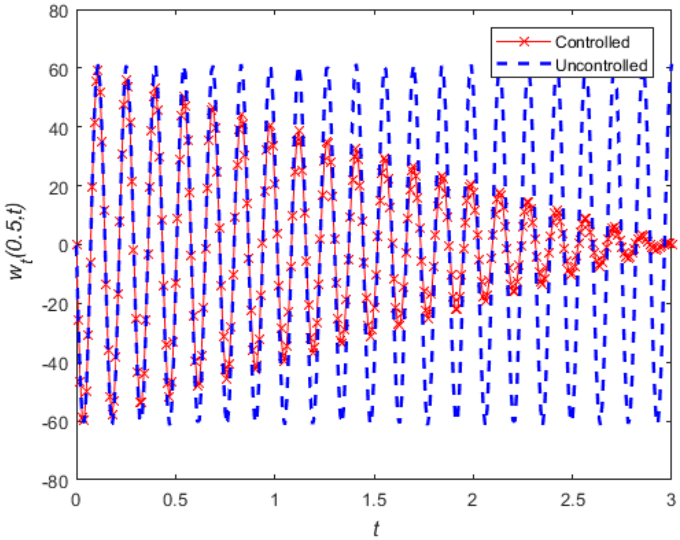

Figure 3. The velocity in the controlled/ uncontrolled beam.

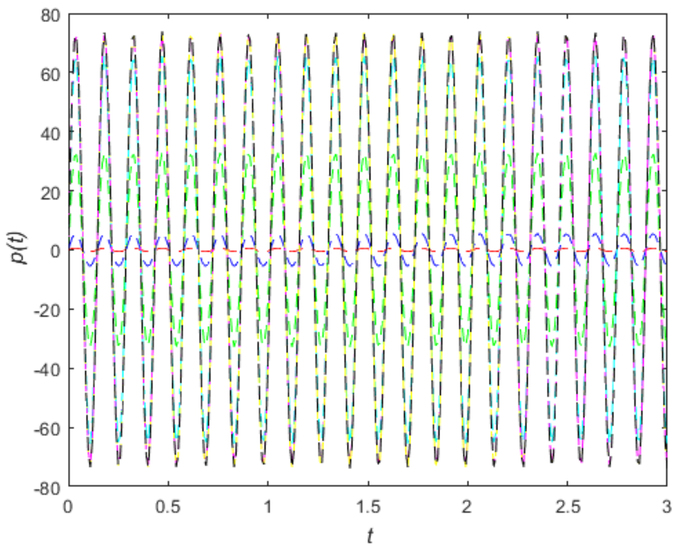

Figure 4. Optimal control solution for the case with $\boldsymbol{t_f = 3}$ and different weight coefficients $\boldsymbol{\mu}_3$

6. CONCLUSION

Analyzing micro and nanoscale structures has recently become one of the most researched topics in nonclassical continuum theories. Mindlin's second strain gradient theory (1965) is an extremely important nonclassical theory when it comes to the correct expression of small-scale effects. Timoshenko and Euler Bernoulli beam theories are other different types of beam theories, but they do not meet the expectations about microscopic effects of the structures when compared to the second strain gradient theory. In this paper, the optimal vibration control of a beam model described by a linear higher-order distributed parameter system is presented. The beam model to be controlled is the second strain gradient theory based beam model that captures the behavior of micrometer and nanometer-sized structures. The boundary control is implemented to the beam model via the Pontryagin's maximum principle. The uniqueness of the solution and controllability of the system are discussed. A cost functional to be minimized in the control duration is chosen as a sum of performance measure and penalty function. The performance measure is expressed as a dynamic response of the beam and penalty function assures the minimum expenditure of control forces. The Pontryagin's maximum principle is used to compute the optimal control function analytically that leads to a partial differential equations system including state and adjoint variables, which are related by the terminal, initial and boundary conditions. Numerical results are presented in graphical and table forms by using MATLAB©. It is observed that the dynamic response decreases corresponding to an increase in the control. Comparisons of the displacement and velocity profiles for the controlled and uncontrolled beam are presented. In addition, optimal control solutions for different weight coefficients are obtained. It is shown that the introduced boundary control is effective for the second strain gradient theory-based beam.

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