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Research Article

THE EXISTENCE OF POSITIVE SOLUTIONS AND A LYAPUNOV TYPE INEQUALITY FOR BOUNDARY VALUE PROBLEMS OF THE FRACTIONAL CAPUTO-FABRIZIO DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, a Lyapunov-type inequality and the existence of the positive solutions for boundary value problems of the nonlinear fractional Caputo-Fabrizio differential equation have been presented. By using the Guo Krasnoselskii's fixed point theorem on cone and the properties of the associated Green`s function, we prove the existence of the positive solution. Finally, we gave some numerical examples to validate the theoretical findings.
Keywords: Lyapunov-type inequality, fractional Caputo-Fabrizio derivative, Green's function, positive solution, fixed point.

1. INTRODUCTION

In recent years, the fractional differential calculus has been appeared frequently and has been studied by many researchers so that it has strong mathematical background and many papers are attributed to the development of it. For more details, the readers can refer to [1, 2, 3, 4]. Physical phenomena such as control systems, mechanics, viscoelasticity can be modelled by fractional calculus [5],[6][7]. More frequently used fractional derivatives are the Riemann-Lioville and Caputo derivative [1]. Some other commonly used fractional derivatives in the literature include conformable fractional derivative [8-11]. In the recent years, a new definition of fractional order derivative has been defined by Caputo and Fabrizio [12] with a regular kernel. Some existence results on this new fractional calculus have been studied in [13],[14]. This new definition can describe better heterogeneousness and systems with different scales with memory effects [13]. In this paper, we fisrt derive a Lyapunov-type inequlaity for the fractional boundary value problem of the fractionla Caputo-Fabrizio differential equation of order $\sigma \in (2,3]$ and then we prove the existence of positive solutions of the fractional boundary value problem of the fractional Caputo-Fabrizio differential equation of order $\sigma \in (1,2]$. The well-known Lyapunov inequality provides that if there is a non-zero solution of the differential equatiıon

$$u''(x) + r(x)u(x) = 0,\ a < x < b,$$
$$u(a) = u(b) = 0,$$

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where r is a continuous function, then one has

$$\int_a^b |r(t)|dt > \frac{4}{b-a}.$$

The constant 4 can not be changed any bigger number. There are number of papers and studies that provide generalizations and extensions of (2), e.g., see [15-16]. The studies for Lyapunov-type inequalities for the fractional boundary value problems have recently been investigated in a number of papers, see e.g. [17-21]. To the best of our knowledge, there is a few result on Lyapunov-type ineqalities for the fractional Caputo-Fabrizio differential equation, see [20]. To fill this gap, we provide a Lyapunov-type inequality for the boundary value problem of the fractional Caputo-Fabrizio differential equaiton. To provide a Lyapunov-type inequality, one usually converts to the fractional differental equation to an integral equaiton with the corresponding Green's function. Then, one finds the maximum value of the Green's function to obtain the desired inequality. However, finding the maximum value of the Green function is not easy task. Secondly, we prove the existence of the positive solution of the fractional boundary value problem of the fractional Caputo-Fabrizio differential equaiton. The existence of (positive) solutions has been investigated by many researchers [22-27]. Although many papers are devoted to the existence of positive solutions of fractional differential equations in the sense of Riemann-Lioville or Caputo derivative, there is no result, to the best of our knowledge, on the existence of positive solution of fractional boundary value problem involving the Caputo-Fabrizio derivative. This is the first paper on the existence of positive solution for the fractional Caputo-Fabrizio differential equaiton of order $\alpha + 1 \in (1,2)$.

The rest of the paper is organized as follows. In Section 2, some definitions and related studies have been introduced. A Lyapunov-type inequality have been presented in Section 3. The existence of positive solution of fractional boundary value problem is provided in Section 4. In Section 5, some numerical examples are examined.

2. DEFINITIONS AND PREVIOUS RESULTS

We recall the frequently used fractional derivatives in the literature and related results of the new fractional Caputo-Fabrizio derivative that are needed in the sequel.

Definition 1 Let $\alpha \in (n, n+1]$, $n \in N$ and $f \in C^{n+1}[a,b], a < b$. The fractional Caputo derivative of the function f of order α defined as

$$D_C^\alpha f(x) = \frac{1}{\Gamma(n+1-\alpha)} \int_a^x (x-t)^{n-\alpha} f^{(n+1)}(t)dt.$$

Definition 2 Let $\alpha \in (n, n+1]$, $n \in N$ and $f \in C^{n+1}[a,b], a < b$. The fractional Caputo-Fabrizio derivative of the function f of order α defined as

$$D_C^\alpha f(x) = \frac{1}{\mathrm{n}+1-\alpha} \int_a^x \exp\left(-\frac{\alpha-n}{n+1-\alpha}(x-t)\right) f^{(n+1)}(t)dt.$$

Definition 3 Let $\alpha \in (n, n+1]$, $n \in N$ and $f \in C^{n+1}[a,b], a < b$. The fractional Caputo-Fabrizio integral operator of order α defined as

$$I_a^\alpha f(x) = (1+n-\alpha) I_a^n f(x) + (\alpha - n) I_a^{n+1} f(x), \quad x \geq a,$$

where $I_a^n f(x)$ is the iterated Cauchy integral given by

$$I_a^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt, \text{ with the adaption } I_a^0 f(x) = f(x).$$

Lemma 4 [15] Let $\alpha \in (n, n+1]$. Then we have the following relation

$$I_a^\alpha D_{CF}^\alpha f(x) = f(x) - \sum_{k=0}^n \frac{(x-a)^k}{k!} f^{(k)}(a).$$

We first prove a Lyapunov-type inequality for the following boundary value problem of the fractional Caputo-Fabrizio differential equation in the next section.

$$D_{CF}^{\sigma}u(x) + f(x, u(x)) = 0, \quad \sigma \in (2,3], \quad a \le x \le b,$$
$$u(a) = u'(a) = 0, \quad u(b) = 0. \tag{1}$$

3. LYAPUNOV-TYPE INEQUALITY FOR FRACTIONAL BOUNDARY VALUE PROBLEMS

We present a Lyapunov-type inequality for the fractional boundary value problem (1) when $f(x, u(x)) = p(x)u(x)$ for $p \in C([a, b], \mathbb{R})$ in this section. As applications of this inequality, we obtain a Lyapunov-type inequality for the linear third order ordinary differential equation and a lower bound for the eigenvalues of the fractional boundary value problems.

Lemma 5 Let $h \in C([a, b], \mathbb{R})$. Then the following fractional boundary value problem of the fractional Caputo-Fabrizio differential equation

$$D_{CF}^{\sigma}u(x) + h(x) = 0, \quad \sigma \in (2,3], \quad a \le x \le b,$$
$$u(a) = u'(a) = 0, \quad u(b) = 0. \tag{2}$$

has the solution $u(x)$ that is also solution of the following integral equation

$$u(x) = \int_a^b H(x,t)h(t)dt,$$

where the Green's function $H(x, t)$ is given by

$$\mathrm{H(x,t)} = \begin{cases} h_1(x,t), & a \le t \le x \le b, \\ h_2(x,t), & a \le t \le x \le b, \end{cases}$$

with

$$h_1(x,t) = h_2(x,t) - \frac{2(3-\sigma)(x-t)(b-a)^2 + (\sigma-2)(x-t)^2(b-a)^2}{2(b-a)^2},$$

and

$$h_2(x,t) = \frac{2(3-\sigma)(x-a)^2(b-t) + (\sigma-2)(x-a)^2(b-t)^2}{2(b-a)^2}.$$

Proof. By Lemma 4, we get

$$I_a^{\alpha} D_{CF}^{\alpha} u(x) = u(x) - u(a) - (x-a)u'(a) - \frac{(x-a)^2}{2}u''(a).$$

Thus, by applying the high order fractional Caputo-Fabrizio integral operator to the equation (2), we find that

$$u(x) - u(a) - (x-a)u'(a) - \frac{(x-a)^2}{2}u''(a) = -I_a^{\sigma} h(x).$$

The boundary conditions $u(a) = u'(a) = 0$ imply that

$$u(x) = \frac{(x-a)^2}{2}u''(a) - I_a^{\sigma} h(x).$$

Using the boundary condition $u(b) = 0$, we obtain

$$u(x) = \frac{(3-\sigma)}{(b-a)^2}\int_{\mathrm{a}}^{\mathrm{b}}(x-a)^2(b-t)h(t)dt + \frac{\sigma-2}{2(b-a)^2}\int_{\mathrm{a}}^{\mathrm{b}}(x-a)^2(b-t)^2h(t)dt$$
$$-(3-\sigma)\int_a^x (x-t)h(t)dt - \frac{\sigma-2}{2}\int_a^x (x-t)^2 h(t)dt,$$

or, equivalently

$$\mathrm{u(x)} = \int_{\mathrm{a}}^{\mathrm{x}} h_1(x,t)h(t)dt + \int_x^b h_1(x,t)h(t)dt$$
$$= \int_{\mathrm{a}}^{\mathrm{b}} H(x,t)h(t)dt.$$

The proof is completed.

Lemma 6 The Green's function $H(x, t)$ has the following bound:

$$H(x,t) \leq \frac{2(3-\sigma)(b-a)+(\sigma-2)(b-a)^2}{2}, \quad \sigma \in (2,3], \ x,t \in [a,b].$$

Proof. Since $x - a \leq b - a$, we find that

$$\frac{2(3-\sigma)(x-a)^2(b-t)+(\sigma-2)(x-a)^2(b-t)^2}{2(b-a)^2}$$

$$-\frac{2(3-\sigma)(x-t)(b-a)^2+(\sigma-2)(x-t)^2(b-a)^2}{2(b-a)^2} \leq \frac{2(3-\sigma)(b-t)+(\sigma-2)(b-t)^2-2(3-\sigma)(x-t)-(\sigma-2)(x-t)^2}{2}.$$

Note that for $a \leq t \leq x \leq b$, we have

$$2(3-\sigma)((b-t)-(x-t)) = 2(3-\sigma)(b-x) \leq 2(3-\sigma)(b-a),$$

and

$$(\sigma-2)((b-t)^2-(x-t)^2) \leq (\sigma-2)(b-t)^2 \leq (\sigma-2)(b-a)^2.$$

We now see that for $a \leq t \leq x \leq b$,

$$H(x,t) \leq \frac{2(3-\sigma)(b-a)+(\sigma-2)(b-a)^2}{2}.$$

For the case when $a \leq x \leq t \leq b$, we have $x - a \leq t - a \leq b - a$, and we can write

$$\frac{2(3-\sigma)(x-a)(x-a)(b-t)+(\sigma-2)(x-a)(x-a)(b-t)(b-t)}{2(b-a)^2}$$

$$\leq \frac{2(3-\sigma)(t-a)(t-a)(b-t)+(\sigma-2)(t-a)(b-t)(t-a)(b-t)}{2(b-a)^2}.$$

Now, using the inequalities $(t-a)(b-t) \leq \big((t-a)+(b-t)\big)^2$ and $(t-a)(b-t) \leq (b-a)^2$, we have

$$\frac{2(3-\sigma)(t-a)(t-a)(b-t)+(\sigma-2)(t-a)(b-t)(t-a)(b-t)}{2(b-a)^2}$$

$$\leq \frac{2(3-\sigma)(b-a)+(\sigma-2)(b-a)^2}{2}.$$

This shows that

$$H(x,t) \leq \frac{2(3-\sigma)(b-a)+(\sigma-2)(b-a)^2}{2} \text{ for } a \leq x \leq t \leq b.$$

Thus, the proof is completed.

Next, we establish a Lyapunov-type inequality for the fractional boundary value problem of the fractional Caputo-Fabrizio differential equation of order $\sigma \in (2,3]$ in the next theorem.

Theorem 7 For $\sigma \in (2,3]$ and $p \in C([a,b],\mathbb{R})$, if the following fractional boundary value problem

$$\begin{aligned} &D_{CF}^{\sigma}u(x) + p(x)u(x) = 0, \quad \sigma \in (2,3], \quad a \leq x \leq b, \\ &u(a) = u'(a) = 0, \quad u(b) = 0. \end{aligned} \tag{3}$$

has a non zero solution, then the function p obeys the following integral inequality

$$\int_a^b |p(\eta)| d\eta \geq \frac{2}{2(3-\sigma)(b-a)+(\sigma-2)(b-a)^2}.$$

Proof. Let $C[a,b]$C[a,b] be the Banach space with maximum norm, that is,

$$\| u \| = \max_{x\in[a,b]} |u(x)|, \quad u \in C[a,b].$$

By Lemma 5, the solution of the boundary value problem (3) has the form

$$u(x) = \int_a^b H(x,t)p(t)u(t)dt \quad x,t \in [a,b].$$

Taking the maximum norm of the both side of the equation yields

$$\| u \| \leq \max_{x\in[a,b]} \int_a^b |H(x,t)p(t)| \, \| u \| \, dt$$

which gives that

$$\max_{x\in[a,b]} \int_a^b |H(x,t)p(t)|dt \geq 1.$$

The bound on the Green's function H leads to have

$$1 \leq \frac{2(3-\sigma)(b-a)+(\sigma-2)(b-a)^2}{2} \int_a^b |p(t)|dt,$$

or

$$\int_a^b |p(t)|dt \geq \frac{2}{2(3-\sigma)(b-a)+(\sigma-2)(b-a)^2}.$$

This completes the proof.

We now give some consequences of Theorem 7. First, we obtain a Lyapunov-type inequality for the linear third order ordinary differential equation.

Corollary 8 If $p \in C([a,b],\mathbb{R})$ and the following third order ordinary differential equation

$$u'''(x) + p(x)u(x) = 0, \quad a \leq x \leq b,$$
$$u(a) = u'(a) = 0, \quad u(b) = 0,$$

has a non zero solution, then one has

$$\int_a^b |p(\eta)|d\eta \geq \frac{2}{(b-a)^2}.$$

Secondly, we present a bound for the eigenvalues of fractional boundary value problems of the fractional Caputo-Fabrizio differential equation.

Corollary 9 If λ is an eigenvalue of the following fractional boundary value problem

$$D_{CF}^{\sigma}u(x) + \lambda u(x) = 0, \quad \sigma \in (2,3], \quad a \leq x \leq b,$$
$$u(a) = u'(a) = 0, \quad u(b) = 0,$$

then

$$|\lambda| \geq \frac{2}{2(3-\sigma)(b-a)^2+(\sigma-2)(b-a)^3}.$$

Proof. Since λ is an eigenvalue of the fractional boundary value problem, there exists a non zero solution. Thus, appealing to Theorem 7 when p replaced by λ, we get

$$\int_a^b |\lambda|\, ds \geq \frac{2}{2(3-\sigma)(b-a)+(\sigma-2)(b-a)^2},$$

which leads to

$$|\lambda| \geq \frac{2}{2(3-\sigma)(b-a)^2+(\sigma-2)(b-a)^3},$$

that completes the proof.

4. POSITIVE SOLUTIONS FOR NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEMS

We further study the existence of positive solutions of the following nonlinear fractional differential equation

$$\begin{aligned} &\mathrm{D}_{\mathrm{CF}}^{\sigma}u(x) = h(x,u(x)), \quad \sigma \in (1,2], \quad 0 \leq x \leq 1, \\ &u(0) = u(1) = 0. \end{aligned} \tag{4}$$

where D_{CF}^{σ} is the Caputo-Fabrizio fractional derivative and $h \in C([0,1]\times[0,\infty),[0,\infty))$. We first convert the fractional differential equation to the integral equation in the next lemma.

Lemma 10 [15] If $u(x)$ is the solution of the fractional boundary value problem (4), then $u(x)$ also solves the following integral equation

$$u(x) = \int_0^1 G(x,t)h(t,u(t)dt,$$

where

$$\mathrm{G(x,t)} = \begin{cases} (1-x)\big((\sigma-1)t-2+\sigma\big), & 0 \le t \le x \le 1, \\ x\big((\sigma-1)(1-t)+2-\sigma\big), & 0 \le t \le x \le 1. \end{cases} \tag{5}$$

Consider the Banach space $C[0,1]$ equipped with maximum norm $\| y \| = \max_{t\in[0,1]} |y(t)|$. We define the cone $Q = \{y \in C[0,1]: y(x) \ge 0, x \in [0,1]\}$. Let us define the operator $\mathcal{K}: Q \to Q$ by

$$\mathcal{K}u(x) = \int_0^1 G\,(x,t)h\big(t,y(t)\big)dt, \quad x \in [0,1], \tag{6}$$

where the Green's function $G(x,t)$ is defined by (5). We show the existence of solutions for the fractional boundary value problem (4) by proving the operator $\mathcal{K}$ has a fixed point in Q.

Lemma 11 The operator $\mathcal{K}: Q \to Q$ is completely continuous.

Proof. We first show that the operator $\mathcal{K}$ is continuous. For $u_1(x), u_2(x) \in [0,\infty)$ and $\epsilon > 0$, there exists $\delta > 0$ with $|u_1(x) - u_2(t)| < \delta$ so that $|h(x,u_1) - h(x,u_2)| < \frac{\epsilon}{3-\sigma}$ since $h \in C([0,1]\times[0,\infty),[0,\infty))$.

$$\begin{aligned} |\mathcal{K}u_1(x) - \mathcal{K}u_2(x)| &\le \int_0^x (1-x)((\sigma-1)t-2+\sigma)|h(t,u_1(t)) - h(t,u_2(t))|\,dt \\ &\quad + \int_x^1 x\,\big((\sigma-1)(1-t)+2-\sigma\big)\big|h\big(t,u_1(t)\big) - h\big(t,u_2(t)\big)\big|dt \\ &\le \left((\sigma-1)\frac{x^2(1-x)}{2} + (2-\sigma)x(1-x)\right)\frac{\epsilon}{3-\sigma} \\ &\quad + x((\sigma-1)(\frac{1}{2} - x + \frac{x^2}{2}) + (2-\sigma)x(1-x))\frac{\epsilon}{3-\sigma} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

showing that the operator $\mathcal{K}$ is continuous. Let $B \subset Q$ be a bounded subset. We show that $\mathcal{K}(B)$ is bounded and equicontinuous. Let $M = \max_{x\in[0,1],u\in B}|h(x,u(x))|$. We have for $u \in B$

$$\begin{aligned} |\mathcal{K}u(x)| &\le \int_0^{\mathrm{x}} (1-x)((\sigma-1)t-2+\sigma)|h(t,u(t))|\,dt + \int_{\mathrm{x}}^1 x\,\big((\sigma-1)(1-t)+2-\sigma\big) \\ &\le M(\frac{\sigma-1}{2} + 2 - \sigma) + M(\frac{\sigma-1}{2} + 2 - \sigma) = (3-\sigma)M \end{aligned}$$

which shows that $\mathcal{K}(B)$ is bounded. Now, let $x_1, x_2 \in [0,1], x_1 < x_2$ and $u \in B$, then

$$\begin{aligned} |\mathcal{K}u(x_1) - \mathcal{K}u(x_2)| &\le \int_0^{\mathrm{x}_1} [\,(1-x_1)\big((\sigma-1)t-2+\sigma\big) - (1-x_2)((\sigma-1)t-2+ \\ &\qquad \sigma)]|h(t,u(t))|\,dt \\ &\quad + \int_{\mathrm{x}_1}^{\mathrm{x}_2} (\ 1-x_2)((\sigma-1)t-2+\sigma)|h(t,u(t))|\,dt \\ &\quad + \int_{\mathrm{x}_2}^1 [x_1\big((\sigma-1)(1-t)+2-\sigma\big) - x_2((\sigma-1)(1-t)+2-\sigma)]|h(t,u(t))|\,dt \\ &\quad -(1-x_2)((\sigma-1)t-2+\sigma)]|h(t,u(t))|\,dt \\ &\quad + \int_{\mathrm{x}_1}^{\mathrm{x}_2} x_1\,\big((\sigma-1)(1-t)+2-\sigma\big)\big|h\big(t,u(t)\big)\big|dt \\ &\le M[(x_2-x_1)((\sigma-1)x_1^2 - (2-\sigma)x_1) + (1-x_2)((\sigma-1)(\frac{x_2^2}{2} - \frac{x_1^2}{2}) \\ &\quad -\big(2-\sigma(x_2-x_1)\big) + (x_1-x_2)\left((\sigma-1)\left(\frac{1}{2} - x_2 + \frac{x_2^2}{2}\right) + (2-\sigma)(x_2-x_1)\right) \\ &\quad + x_1(9\sigma-1)(x_1 - x_2 + \frac{x_2^2}{2} - \frac{x_1^2}{2}) + (2-\sigma)(x_2-x_1)]. \end{aligned}$$

We see that $|\mathcal{K}u(x_1) - \mathcal{K}u(x_2)| \to 0$ as $x_2 \to x_1$ implying that $\mathcal{K}(B)$ is equicontinuous. By the well known theorem of Arzela- Ascolli, we conclude that the operator $\mathcal{K}$ is completely continuous.

Next, we show the existence of positive solution for the fractional differential equation (4) by the help of Guo-Krasnoselskii fixed point theorem.

Theorem 12 Let $\sigma \in (1,2]$ and $h \in C\big([0,1] \times [0,\infty), [0,\infty)\big)$. Assume that there exist positive constants $r_1 < r_2$ so that

(A) $h(x,t) \geq \frac{8r_1}{\sigma-1}$ $for\ (x,t) \in [0,1] \times [0, r_1)$,

(B) $h(x,t) \leq \frac{r_2}{3-\sigma}$ $for\ (x,t) \in [0,1] \times [0, r_2)$.

Then, there exists at least one postive solution of the fractional boundary value problem (4).

Proof. Let us define two subsets of the convex set Q: $\omega_1 = \{u \in Q : \| u \| \leq r_1\}$ and $\omega_2 = \{u \in Q : \| u \| \leq r_2\}$. If $u \in Q \cap \partial\omega_1$, then we have $0 \leq u(x) \leq r_1$ for $x \in [0,1]$. We then obtain by using the assumption (A)

$$\| \mathcal{K}u \| = \max_{0\leq x\leq 1} |\mathcal{K}u(x)| = \max_{0\leq x\leq 1} \int_0^1 G\,(x,t) h(t, u(t))\, dt \geq \frac{8r_1}{\sigma-1} \max_{0\leq x\leq 1} \int_0^1 G\,(x,t)\, dt.$$

We compute

$$\begin{aligned}\max_{0\leq x\leq 1} \int_0^1 G\,(x,t) d &= \max_{0\leq x\leq 1} \int_0^x (1-x)\,[(\sigma-1)t - 2 + \sigma] dt \\ &\quad + \int_{\mathrm{x}}^1 x\,[(\sigma-1)(1-t) + 2 - \sigma]\, dt) \\ &= \max_{0\leq x\leq 1} ((\sigma-1)\frac{(1-x)x^2}{2} - (2-\sigma)x(1-x) + (\sigma-1)x(\frac{1}{2} - x + \frac{x^2}{2}) \\ &\quad + (2-\sigma)x(1-x)) = \max_{0\leq x\leq 1} \frac{\sigma-1}{2}(x - x^2) = \frac{\sigma-1}{8}.\end{aligned}$$

Thus we find that $\| \mathcal{K}u \| \geq r_1 \geq \| u \|$ for $u \in Q \cap \partial\omega_1$. If $u \in Q \cap \partial\omega_2$, then we have $0 \leq u(x) \leq r_2$ for $x \in [0,1]$. We then obtain by using the assumption (B),

$$\begin{aligned}\| \mathcal{K}u \| &= \max_{0\leq x\leq 1} |\mathcal{K}u(x)| = \max_{0\leq x\leq 1} \int_0^1 G\,(x,t) h\big(t, u(t)\big) dt \leq \frac{r_2}{3-\sigma} \max_{0\leq x\leq 1} \int_0^1 G\,(x,t) dt \\ &\leq \frac{r_2}{3-\sigma} \max_{0\leq x\leq 1} ((\sigma-1)\frac{(1-x)x^2}{2} + (2-\sigma)x(1-x) \\ &\quad + (\sigma-1)x(\frac{1}{2} - x + \frac{x^2}{2}) + (2-\sigma)x(1-x))\end{aligned}$$

Since $(1-x)x^2 \leq (1-x)x \leq 1$ and $x(\frac{1}{2} - x + \frac{x^2}{2}) \leq \frac{1}{2}$ for $x \in [0,1]$, we get

$\| \mathcal{K}u \| \leq \frac{r_2}{3-\sigma}(\frac{3-\sigma}{2} + \frac{3-\sigma}{2}) = r_2$, so we have $\| \mathcal{K}u \| \leq \| u \|$ for $u \in Q \cap \partial\omega_2$. Therefore, by Guo Krasnoselskii's fixed point theorem, the existence of at least one positive solution is proved. This completes the proof.

5. NUMERICAL EXAMPLES

In this section, we give some numerical examples to show the existence of positve solutions of the fractional differential equations.

Example 13 Consider the following fractional boundary value problem

$$\begin{aligned}&\mathrm{D}_{\mathrm{CF}}^{\frac{3}{2}} u(x) = \exp\big(-u(x)\big) + 1, \qquad 0 \leq x \leq 1, \\ &u(0) = u(1) = 0.\end{aligned} \tag{7}$$

We select $r_1 = \frac{1}{17}$ and $r_2 = 4$. We then have

$$h\big(x, u(x)\big) = \exp\big(-u(x)\big) + 1 \leq 2 \leq \frac{4}{\frac{3}{2}} = \frac{8}{3}\ \ for\ (x,u) \in [0,1] \times [0,4],$$

$$h\big(x, u(x)\big) = \exp\big(-u(x)\big) + 1 \geq 1 \geq \frac{8\left(\frac{1}{17}\right)}{\frac{1}{2}} = \frac{16}{17}\ for\ (x,u) \in [0,1] \times \left[0, \frac{1}{17}\right].$$

By Theorem 12, there exists at least one positive solution of (7) satisfying $\frac{1}{17} \leq ||u|| \leq 4$.

Example 14 Consider the following fractional boundary value problem

$$\begin{aligned} &\mathrm{D}_{\mathrm{CF}}^{\frac{3}{2}} u(x) = \frac{\mathrm{u(x)+1}}{1+\mathrm{x}^2}, \qquad 0 \leq x \leq 1, \\ &u(0) = u(1) = 0. \end{aligned} \tag{8}$$

We select $r_1 = \frac{1}{32}$ and $r_2 = 3$. We then have

$$h\big(x, u(x)\big) = \frac{\mathrm{u(x)+1}}{\mathrm{x}^2+1} \leq \frac{2}{x^2+1} \leq 2 \leq \frac{3}{\frac{3}{2}} = 2 \;\; for\ (x,u) \in [0,1] \times [0,3],$$

$$h\big(x, u(x)\big) = \frac{u(x)+1}{x^2+1} \geq \frac{1}{x^2+1} \geq \frac{1}{2} \geq \frac{8\left(\frac{1}{32}\right)}{\frac{1}{2}} = \frac{1}{2} \;\; for\ (x,u) \in [0,1] \times [0, \tfrac{1}{32}].$$

By Theorem 12, there exists at least one positive solution of (8) satisfying $\frac{1}{32} \leq ||u|| \leq 3$.

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