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Research Article

FIXED POINT THEOREMS IN NEUTROSOPHIC METRIC SPACES

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ABSTRACT

In this paper, we introduce the neutrosophic cantractive and neutrosophic mapping. We establish some results on fixed points of a neutrosophic mapping.
Keywords: Neutrosophic Banach contraction, fixed point, complete neutrosophic metric space.

1. INTRODUCTION

Fuzzy Sets (FSs) put forward by Zadeh [1] has influenced deeply all the scientific fields since the publication of the paper. It is seen that this concept, which is very important for real-life situations, had not enough solution to some problems in time. New quests for such problems have been coming up. Atanassov [2] initiated Intuitionistic fuzzy sets (IFSs) for such cases. Neutrosophic set (NS) is a new version of the idea of the classical set which is defined by Smarandache [3]. Examples of other generalizations are FS [1] interval-valued FS [4], IFS [2], interval-valued IFS [5], the sets paraconsistent, dialetheist, paradoxist, and tautological [6], Pythagorean fuzzy sets [7].

Using the concepts Probabilistic metric space and fuzzy, fuzzy metric space (FMS) is introduced in [8]. Kaleva and Seikkala [9] have defined the FMS as a distance between two points to be a non-negative fuzzy number. In [10] some basic properties of FMS studied and the Baire Category Theorem for FMS proved. Further, some properties such as separability, countability are given and Uniform Limit Theorem is proved in [11]. Afterward, FMS has used in the applied sciences such as fixed point theory, image and signal processing, medical imaging, decision-making et al. After defined of the intuitionistic fuzzy set (IFS), it was used in all areas where FS theory was studied. Park [12] defined IF metric space (IFMS), which is a generalization of FMSs. Park used George and Veeramani's [10] idea of applying t-norm and t-conorm to the FMS meanwhile defining IFMS and studying its basic features.

Fixed point theorem for fuzzy contraction mappings is initiated by Heilpern [13]. Bose and Sahani [14] extended the Heilpern's study. Alaca et al. [15] are given fixed point theorems related to intuitionistic fuzzy metric spaces(IFMSs). Fixed point results for fuzzy metric spaces and IFMSs are studied by many researchers [16], [17], [18], [19], [20].

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Kirisci et al. [21] defined neutrosophic contractive mapping and gave a fixed point results in complete neutrosophic metric spaces. In [22], Mohamad studied fixed point aprroach in intuitionistic fuzzy metric spaces. Definitions and results of this paper are the generalizations of Mohamad's work [22] to NMSs.

2. PRELIMINARIES

Some definitions related to the fuzziness, intuitionistic fuzziness and neutrosophy are given as follows:

The fuzzy subset F of $\mathbb{R}$ is said to be a fuzzy number(FN). The FN is mapping $F:\mathbb{R} \to [0,1]$ that corresponds to each real number a to the degree of membership $F(a)$.

Let F is a FN. Then, it is known that [23]

- If $F(a_0) = 1$, for $a_0 \in \mathbb{R}$, F is said to be normal,
- If for each $\mu > 0, F^{-1}\{[0, \tau + \mu\ \}$ is open in the usual topology $\forall \tau \in [0,1)$, F is said to be upper semi continuous,
- The set $[F]^T = \{a \in\ \mathbb{R}: F(a) \geq \tau\}, \tau \in [0,1]$ is called τ −cuts of F.

Choose non-empty set F.An IFS in F is an object U defined by

$$U = \{\langle a, H_U(a), S_U(a)\rangle : a \in F\}$$

where $H_U(a): F \to [0,1]$ and $S_U(a): F \to [0,1]$ are functions for all $a \in F$ such that $0 \leq H_U(a) + S_U(a) \leq 1$ [2]. Let U be an IFN. Then,

- An IFN subset of the $\mathbb{R}$,
- If $H_U(a) = 1$ and, $S_U(a) = 0$ for $a_0 \in \mathbb{R}$, normal,
- If $H_U(\lambda a_1 + (1-\lambda)a_2) \geq \min\ \ (H_U(a_1), H_U(a_2)\ ),\ \forall\ a_1, a_2 \in \mathbb{R}$ and $\lambda \in [0,1]$, then the membership function $H_U(a)$ is called convex,
- If $S_U(\lambda a_1 + (1-\lambda)a_2) \geq \min\ \ (S_U(a_1), S_U(a_2)\ ),\ \forall\ a_1, a_2 \in \mathbb{R}$ and $\lambda \in [0,1]$, then the non-membership function $S_U(a)$ is concave,
- H_U is semi upper continuous and S_U is lower semi continuous,
- $suppU = cl(\{a \in F: S_U < 1\})$ is bounded.

An IFS $U = \{\langle a, H_U(a), S_U(a)\rangle : a \in F\}$ such that $H_U(a)$ and $1 - S_U(a)$ are FNs, where $(1 - S_U)(a) = 1 - S_U(a)$, and $H_U(a) + S_U(a) \leq 1$ is called an IFN.

Let's consider that F is a space of points(objects). Denote the $H_U(a)$ is a truth-MF, $M_U(a)$ is an indeterminacy-MF and $S_U(a)$ is a falsity-MF, where U is a set in F with $a \in F$. Then, if we take $I =]0^-, 1^+[$

$$H_U(a): F \to I, \qquad M_U(a): F \to I, \qquad S_U(a): F \to I.$$

There is no restriction on the sum of $H_U(a),\ M_U(a),\ S_U(a)$. Therefore,

$$0^- \leq \sup H_U(a) + \sup M_U(a) + \sup S_U(a) \leq 3^+.$$

The set U which consist of with $H_U(a),\ M_U(a),\ S_U(a)$ in F is called a neutrosophic sets(NS) and can be denoted by

$$U = \{\langle a, H_U(a),\ M_U(a), S_U(a)\rangle : a \in F, H_U(a),\ M_U(a), S_U(a) \in I\ \}. \tag{1}$$

Clearly, NS is an enhancement of [0,1] of IFSs.

An NS U is included in another NS V, $(U \subseteq V)$, if and only if,

$$\begin{array}{ll} infH_U(a) \leq infH_V(a), & supH_U(a) \leq supH_V(a) \\ infM_U(a) \geq infM_V(a), & supM_U(a) \geq supM_V(a) \\ infS_U(a) \geq infS_V(a), & supS(a) \geq supS_V(a) \end{array}$$

for any $a \in F$. However, NSs are inconvenient to practice in real problems. To cope with inconvenient situation, Wang et. al [24] customized NS's definition and single-valued NSs

(SVNSs) suggested. Ye [25], described the notion of simplified NSs, which may be characterized by three real numbers in the [0,1]. At the same time, the simplified NSs' operations may be impractical, in some cases [25]. Hence, the operations and comparison way between SNSs and the aggregation operators for simplified NSs are redefined in [26].

According to the Ye [25], a simplification of an NS U, in (1), is

$$U = \{\langle a, H_U(a),\ M_U(a), S_U(a)\rangle : a \in F\}$$

which called a simplified NS. Especially, if F has only one element $\langle\ H_U(a),\ M_U(a), S_U(a)\rangle$ is said to be an simplified NN. Expressly, we may see simplified NSs as a subclass of NSs.

An simplified NS U is comprised in another simplified NS V $(U \subseteq V)$, if and only if $H_U(a) \leq H_V(a)$, $M_U(a) \geq M_V(a)$, and $S_U(a) \geq S_V(a)$ for any $a \in F$. Then, the following operations are given by Ye[25]:

$$U + V = \langle H_U(a) + H_V(a) - H_U(a).H_V(a), M_U(a) + M_V(a) - M_U(a).M_V(a), S_U(a) + S_V(a) - S_U(a).S_V(a)\ \rangle,$$

$$U.V = \langle H_U(a).H_V(a), M_U(a).M_V(a), S_U(a).S_V(a)\rangle,$$

$$\alpha.U = \langle 1 - \left(1 - H_U(a)\right)^{\alpha},\ 1 - \left(1 - M_U(a)\right)^{\alpha},\ 1 - \left(1 - S_U(a)\right)^{\alpha}\rangle, \qquad for\ \alpha > 0,$$

$$U^{\alpha} = \langle H_U^{\alpha}(a), M_U^{\alpha}(a),\ S_U^{\alpha}(a)\rangle, \qquad for\ \alpha > 0.$$

Triangular norms (t-norms) (TN) were initiated by Menger [27]. In the problem of computing the distance between two elements in space, Menger offered using probability distributions instead of using numbers of distance. TNs are used to generalize with the probability distribution of triangle inequality in metric space conditions. Triangular conforms (t-conorms) (TC) know as dual operations of TNs. TNs and TCs are very significant for fuzzy operations (intersections and unions).

Definition 2.1. Give an operation $\odot: [0,1] \times [0,1] \to [0,1]$. If the operation $\odot$ is satisfying the following conditions, then it is called that the operation $\odot$ is continuous TN (CTN): For $s, t, u, v \in [0,1]$,

i. $s \odot 1 = s$,
ii. If $s \leq u$ and $t \leq v$, than $s \odot t \leq u \odot v$,
iii. $\odot$ is continuous,
iv. $\odot$ is commutative and associate.

Definition 2.2. Give an operation $\boxdot: [0,1] \times [0,1] \to [0,1]$. If the operation $\boxdot$ is satisfying the following conditions, then it is called that the operation $\boxdot$ is continuous TC (CTC):

i. $s \boxdot 0 = s$,
ii. If $s \leq u$ and $t \leq v$, than $s \boxdot t \leq u \boxdot v$,
iii. $\boxdot$ is continuous,
iv. $\boxdot$ is commutative and associate.

From above definitions, we note that if we choose $0 < \varepsilon_1, \varepsilon_2 < 1$ for $\varepsilon_1 > \varepsilon_2$, then there exist $0 < \varepsilon_3, \varepsilon_4 < 1$ such that $\varepsilon_1 \odot \varepsilon_3 \geq \varepsilon_2$, $\varepsilon_1 \geq \varepsilon_4 \boxdot \varepsilon_2$. Further, if we choose $\varepsilon_5 \in (0,1)$, then there exist $\varepsilon_6, \varepsilon_7 \in (0,1)$ such that $\varepsilon_6 \odot \varepsilon_6 \geq \varepsilon_5$ and $\varepsilon_7 \boxdot \varepsilon_7 \geq \varepsilon_5$.

Remark 2.3. [23] Take $\odot$ and $\boxdot$ are CTN and CTC, respectively. For $s, t, u, v \in [0,1]$,

a. If $s > t$, then there are u, v such that $s \odot u \geq t$ and $s \geq t \boxdot v$.
b. There are p, t such that $t \odot t \geq s$ and $s \geq p \boxdot p$.

Definition 2.4. [28] Take F be an arbitrary set, $\mathcal{E} = \{\langle a, H_U(a),\ M_U(a), S_U(a)\rangle : a \in F\}$ be a NS such that $\mathcal{E}: F \times F \times \mathbb{R}^+ \to [0,1]$. Let $\odot$ and $\boxdot$ show the CTN and CTC, respectively. The four tuple $V = (F, \mathcal{E}, \odot, \boxdot)$ is called neutrosophic metric space(NMS) when the following conditions are satisfied. $\forall a, b, c \in F$,

i. $0 \leq H(a, b, \lambda) \leq 1, \quad 0 \leq M(a, b, \lambda) \leq 1, \quad 0 \leq S(a, b, \lambda) \leq 1 \qquad \forall \lambda \in \mathbb{R}^+$,

ii. $H(a,b,\lambda) + M(a,b,\lambda) + S(a,b,\lambda) \leq 3, \quad (for\ \ \lambda \in \mathbb{R}^+)$,
iii. $H(a,b,\lambda) = 1 \quad (for\ \lambda > 0) \quad if\ and\ only\ if\ \ a = b$,
iv. $H(a,b,\lambda) = H(b,a,\lambda) \quad (for\ \lambda > 0)$,
v. $H(a,b,\lambda) \odot H(b,c,\mu) \leq H(a,c,\lambda+\mu) \quad (for\ \lambda,\mu > 0)$,
vi. $H(a,b,.)\colon [0,\infty) \to [0,1]$ is continuous,
vii. $\lim_{\lambda\to\infty} H(a,b,\lambda) = 1 \quad (\forall\lambda > 0)$,
viii. $M(a,b,\lambda) = 0 \quad (for\ \lambda > 0) \quad if\ and\ only\ if\ \ a = b$,
ix. $M(a,b,\lambda) = M(b,a,\lambda) \quad (for\ \lambda > 0)$,
x. $M(a,b,\lambda) \boxdot M(b,c,\mu) \geq M(a,c,\lambda+\mu) \quad (for\ \lambda,\mu > 0)$
xi. $M(a,b,.)\colon [0,\infty) \to [0,1]$ is continuous,
xii. $\lim_{\lambda\to\infty} M(a,b,\lambda) = 0 \quad (\forall\lambda > 0)$,
xiii. $S(a,b,\lambda) = 0 \quad (for\ \lambda > 0) \quad if\ and\ only\ if\ \ a = b$
xiv. $S(a,b,\lambda) = S(b,a,\lambda) \quad (for\ \lambda > 0)$,
xv. $S(a,b,\lambda) \boxdot S(b,c,\mu) \geq S(a,c,\lambda+\mu) \quad (for\ \lambda,\mu > 0)$,
xvi. $S(a,b,.)\colon [0,\infty) \to [0,1]$ is continuous,
xvii. $\lim_{\lambda\to\infty} S(a,b,\lambda) = 0 \quad (\forall\lambda > 0)$
xviii. If $\lambda \leq 0$, then $H(a,b,\lambda) = 0,\ M(a,b,\lambda) = 1,\ S(a,b,\lambda) = 1$.

Then $\mathcal{E} = (H,M,S)$ is called Neutrosophic metric (NM) on F.

The functions $H(a,b,\lambda),\ M(a,b,\lambda),\ S(a,b,\lambda)$ denote the degree of nearness, the degree of neutralness and the degree of non-nearness between a and b with respect to λ, respectively.

Definition 2.5. [28] Give V be a NMS, $0 < \varepsilon < 1,\ \ \lambda > 0$ and $a \in F$. The set $D(a,\varepsilon,\lambda) = \{b \in F\colon\ H(a,b,\lambda) > 1-\varepsilon,\ M(a,b,\lambda) < \varepsilon,\ \ S(a,b,\lambda) < \varepsilon\}$ is said to be the open ball (OB) (center a and radius ε with respect to λ).

Lemma 2.6. [28] Every OB $D(a,\varepsilon,\lambda)$ is an open set (OS).

Definition 2.7. Let (A_n) be a sequence in $V = (F,\mathcal{E},\odot,\boxdot)$. Then the sequence converges to a point $a \in F$ if and only if for given $\varepsilon \in (0,1)$, $\lambda > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$H(a_n,a,\lambda) > 1-\varepsilon, \quad M(a_n,a,\lambda) < \varepsilon, \quad S(a_n,a,\lambda) < \varepsilon,$$

or

$$\lim_{n\to\infty} H(a_n,a_m,\lambda) = 1, \quad \lim_{n\to\infty} M(a_n,a_m,\lambda) = 0, \quad \lim_{n\to\infty} S(a_n,a_m,\lambda) = 0 \tag{2}$$

as $\lambda \to \infty$.

Definition 2.8. [28] Take V to be a NMS. A sequence (a_n) in F is called Cauchy if for each $\varepsilon > 0$ and each $\lambda > 0$, there exist $n_0 \in \mathbb{N}$ such that

$$H(a_n,a_m,\lambda) > 1-\varepsilon, \quad M(a_n,a_m,\lambda) < \varepsilon, \quad S(a_n,a_m,\lambda) < \varepsilon,$$

for all $n,m \geq n_0$. V is called complete if every Cauchy sequence is convergent with respect to $\tau_{\mathcal{E}}$.

Remark 2.9. We can define the topology induced by NMS with the using Definition 2.5 and Lemma 2.6. The set

$$\tau_{\mathcal{E}} = \{A \subset F\colon\ for\ each\ \ a \in A,\ \ there\ exist\ \lambda > 0\ \ and\ \ \varepsilon \in (0,1)\ \ such\ that\ \ D(a,\varepsilon,\lambda) \subset A\}$$

is topology on F. It can be easily seen that every NM $\mathcal{E}$ on F generates a topology $\tau_{\mathcal{E}}$ on F which has as a base the family of Oss of the form $\{D(a,\varepsilon,\lambda)\colon\ a \in F,\ \ \varepsilon \in (0,1),\ \ \lambda > 0\}$. Let $\left\{D\left(a,\frac{1}{n},\frac{1}{n}\right)\colon\ n \in \mathbb{N}\right\}$ be a local base at a. Then, $\tau_{\mathcal{E}}$ is the first countable and for $D_n = \left\{(a,b)\colon\ H\left(a,b,\frac{1}{n}\right) > 1-\frac{1}{n}\ ,\ \ M\left(a,b,\frac{1}{n}\right) < \frac{1}{n},\ \ S\left(a,b,\frac{1}{n}\right) < \frac{1}{n}\right\}$, $\quad \mathcal{D} = \{D_n\colon\ n \in \mathbb{N}\}$ is a countable uniformly on F whose induced topology coincides with the topology $\tau_{\mathcal{E}}$. Therefore, $(F, \tau_{\mathcal{E}})$ is metrizable.

3. NEUTROSOPHIC CONTRACTIVE MAPPING

The following definitions and results are given in [21]:

Proposition 3.1. Let V be the NMS. For any $\varepsilon \in (0,1]$, define $h: F \times F \times \mathbb{R}^+$ as follows:

$$h_\varepsilon(a,b) = inf\{\lambda > 0: \ H(a,b,\lambda) > 1-\varepsilon, M(a,b,\lambda) < \varepsilon, S(a,b,\lambda) < \varepsilon\}$$

Then,

i. $(F, h_\varepsilon : \varepsilon \in (0,1])$ is generating space of quasi-metric family.

ii. The topology $\tau_{\mathcal{E}}$ on $(F, h_\varepsilon : \varepsilon \in (0,1])$ coincides with the $\mathcal{E}$ −topology on V, that is, h_ε is a compatible symmetric for $\tau_{\mathcal{E}}$.

Definition 3.2. Let V be a NMS. The mapping $f: F \to F$ is called neutrosophic contraction (NC) if there exists $k \in (0,1)$ such that

$$\frac{1}{H(f(a),f(b),\lambda)} - 1 \le k\left(\frac{1}{H(a,b,\lambda)-1}\right),$$
$$M(f(a),f(b),\lambda) \le kM(a,b,\lambda),$$
$$S(f(a),f(b),\lambda) \le kS(a,b,\lambda),$$

for each $a, b, \in F$ and $\lambda > 0$.

Definition 3.3. Let V be a NMS and let $f: F \to F$ be a NC mapping. Then there exists $c \in F$ such that $c = f(c)$. That is, c is called neutrosophic fixed point (NFP) of f.

Proposition 3.4. Suppose that f is a NC. Then, f^n is also a NC. Furthermore, if k is the constant for f, then k^n is the constant for f^n.

Proposition 3.5. Let f be a NC and $a \in F$. $f[D(a,\varepsilon,\lambda)] \subset D(a,\varepsilon,\lambda)$ for large enough values of ε.

Proposition 3.6. The inclusion $f^n[D(a,\varepsilon,\lambda)] \subset D(f^n(a),\varepsilon^*,\lambda)$ is hold for all n, where $\varepsilon^* = k^n \times \varepsilon$.

Now, we will give new definitions and results:

Definition 3.7. Choose two NMSs $(F, \mathcal{E}_1, \odot, \boxdot)$ and $(G, \mathcal{E}_2, \odot, \boxdot)$. Let $\mathcal{D}_i$ the uniformly generated by $\mathcal{E}_i$ $(i = 1,2)$. A mapping $f: F \to G$ is uniformly continuous with respect to $\mathcal{D}_1$ and $\mathcal{D}_2$ if and only if for a given $\varepsilon_2 \in (0,1)$ and $\lambda_2 > 0$, there exists $\varepsilon_1 \in (0,1)$ and $\lambda_1 > 0$ such that $H_1(a,b,\lambda_1) \ge 1-\varepsilon_1$, $M_1(a,b,\lambda_1) \le \varepsilon_1$, $S_1(a,b,\lambda_1) \le \varepsilon_1$ implies $H_2(f(a),f(b),\lambda_2) \ge 1-\varepsilon_2$, $M_2(f(a),f(b),\lambda_2) \le \varepsilon_2$, $M_2(f(a),f(b),\lambda_2) \le \varepsilon_2$ for each $a, b \in F$.

Definition 3.8. Take a NMS V. The mapping $f: F \to F$ is called λ −uniformly continuous if for each η, $(\eta \in (0,1))$, there exists $0 < \varepsilon < 1$ such that $H(a,b,\lambda) \ge 1-\eta$, $M(a,b,\lambda) \le \eta$, $S(a,b,\lambda) \le \eta$ implies $H(f(a),f(b),\lambda) \ge 1-\eta$, $M(f(a),f(b),\lambda) \le \eta$, $S(f(a),f(b),\lambda) \le \eta$, for each $a, b \in F$ and $\lambda > 0$.

Definition 3.9. Let V be a NMS. A mapping $f: F \to F$ is NC, if there exists $k \in (0,1)$ such that

$$\frac{1}{H(f(a),f(b),\lambda)} - 1 \le k\left(\frac{1}{H(a,b,\lambda)} - 1\right),$$
$$\frac{1}{M(f(a),f(b),\lambda)} - 1 \ge \frac{1}{k}\left(\frac{1}{M(a,b,\lambda)} - 1\right),$$
$$\frac{1}{S(f(a),f(b),\lambda)} - 1 \ge \frac{1}{k}\left(\frac{1}{S(a,b,\lambda)} - 1\right)$$

for each $a, b \in F$ and $\lambda > 0$.

In this definition, k is said to the contractive constant of f.

Proposition 3.10. Let V be a NMS and $f: F \to F$ a mapping. Then, f is λ −uniformly continuous if and only if for each $\delta > 0$ there exists $\eta > 0$ such that

$$\frac{1}{H(a,b,\lambda)} - 1 \leq \eta, \qquad \frac{M(a,b,\lambda)}{1-M(a,b,\lambda)} \leq \eta, \qquad \frac{S(a,b,\lambda)}{1-S(a,b,\lambda)} \leq \eta$$

implies

$$\frac{1}{H(f(a),f(b),\lambda)} - 1 \leq \delta, \qquad \frac{M(f(a),f(b),\lambda)}{1-M(f(a),f(b),\lambda)} \leq \delta, \qquad \frac{S(f(a),f(b),\lambda)}{1-S(f(a),f(b),\lambda)} \leq \delta$$

for each $a, b \in F$ and $\lambda > 0$.

Proposition 3.11. Let (F, d) be a metric space. Then the mapping $f: F \to F$ is metric contractive on the metric space (F, d) with contractive constant k if and only if f is NC, with contractive constant k, on the standard NMS $(F, \mathcal{E}, \odot, \boxdot)$ induced by d.

Remark 3.12. It is known that the sequence (a_n) in a metric space (F, d) is called contractive, if there exists $k \in (0,1)$ such that $d(a_{n+1}, a_{n+2}) \leq k. d(a_n, a_{n+1})$ for all $n \in \mathbb{N}$.

Proposition 3.13. Let V be a NMS induced by the metric d on F. The sequence (a_n) in F is contractive in (F, d) if and only if (a_n) is NC in V.

The above propositions can be easily proved. Now, we will give Banach fixed point theorem for NC mappings as follows:

Theorem 3.14. Let V be a complete NMS with (2) in which NC sequences are Cauchy. Let $T: F \to F$ be a NC mapping. Then, T has a unique fixed point.

Proof. Let $a \in F$ and $a_n = T^n(a),\ n \in \mathbb{N}$. For each $\lambda > 0$,

$$\frac{1}{H(T(a),T^2(a),\lambda)} - 1 \leq k\left(\frac{1}{H(a,a_1,\lambda)} - 1\right),$$
$$\frac{1}{M(T(a),T^2(a),\lambda)} - 1 \geq \frac{1}{k}\left(\frac{1}{M(a,a_1,\lambda)} - 1\right),$$
$$\frac{1}{S(T(a),T^2(a),\lambda)} - 1 \geq \frac{1}{k}\left(\frac{1}{S(a,a_1,\lambda)} - 1\right).$$

By induction we have, for $n \in \mathbb{N}$,

$$\frac{1}{H(a_{n+1},a_{n+2},\lambda)} - 1 \leq k\left(\frac{1}{H(a_n,a_{n+1},\lambda)} - 1\right),$$
$$\frac{1}{M(a_{n+1},a_{n+2},\lambda)} - 1 \geq \frac{1}{k}\left(\frac{1}{M(a_n,a_{n+1},\lambda)} - 1\right),$$
$$\frac{1}{S(a_{n+1},a_{n+2},\lambda)} - 1 \geq \frac{1}{k}\left(\frac{1}{S(a_n,a_{n+1},\lambda)} - 1\right).$$

Then (a_n) is a NC sequence. Therefore, it is a Cauchy sequence. Hence (a_n) converges to b, for some $b \in F$. Now, we must show that b is a fixed point for T. Recall that

$$\frac{M(T(b),T(a_n),\lambda)}{1-M(T(b),T(a_n),\lambda)} \leq k\left(\frac{M(b,a_n,\lambda)}{1-M(b,a_n,\lambda)}\right)$$

and

$$\frac{S(T(b),T(a_n),\lambda)}{1-S(T(b),T(a_n),\lambda)} \leq k\left(\frac{S(b,a_n,\lambda)}{1-S(b,a_n,\lambda)}\right).$$

Therefore, by completeness, for $n \to \infty$,

$$\frac{1}{H(T(b),T(a_n),\lambda)} - 1 \leq k\left(\frac{1}{H(b,a_n,\lambda)} - 1\right) \to 0$$

and

$$\frac{M(T(b),T(a_n),\lambda)}{1-M(T(b),T(a_n),\lambda)} \leq k\left(\frac{M(b,a_n,\lambda)}{1-M(b,a_n,\lambda)}\right) \to 0,$$
$$\frac{S(T(b),T(a_n),\lambda)}{1-S(T(b),T(a_n),\lambda)} \leq k\left(\frac{S(b,a_n,\lambda)}{1-S(b,a_n,\lambda)}\right) \to 0.$$

Then $\lim_n H(T(b), T(a_n), \lambda) = 1$ and $\lim_n M(T(b), T(a_n), \lambda) = 0$, $\lim_n S(T(b), T(a_n), \lambda) = 0$ for every $\lambda > 0$. Therefore, $\lim_n T(a_n) = T(b)$, that is, $\lim_n a_{n+1} = T(b)$ and then $T(b) = b$.

To show uniqueness, assume $T(c) = c$ for some $c \in F$. Then, for $\lambda > 0$, we have, for $n \to \infty$,

$$\frac{1}{H(b,c,\lambda)} - 1 = \frac{1}{H(T(b),T(c),\lambda)} - 1 \le k\left(\frac{1}{H(b,c,\lambda)} - 1\right) = k\left(\frac{1}{H(T(b),T(c),\lambda)} - 1\right)$$
$$\le k^2\left(\frac{1}{H(b,c,\lambda)} - 1\right) \le \cdots \le k^n\left(\frac{1}{H(b,c,\lambda)} - 1\right) \to 0.$$

As well as for $n \to \infty$,

$$\frac{M(b,c,\lambda)}{1-M(b,c,\lambda)} = \frac{M(T(b),T(c),\lambda)}{1-M(T(b),T(c),\lambda)} \le k\left(\frac{M(b,c,\lambda)}{1-M(b,c,\lambda)}\right) = k\left(\frac{M(T(b),T(c),\lambda)}{1-M(T(b),T(c),\lambda)}\right)$$
$$\le k^2\left(\frac{M(b,c,\lambda)}{1-M(b,c,\lambda)}\right) \le \cdots \le k^n\left(\frac{M(b,c,\lambda)}{1-M(b,c,\lambda)}\right) \to 0,$$
$$\frac{S(b,c,\lambda)}{1-S(b,c,\lambda)} = \frac{S(T(b),T(c),\lambda)}{1-S(T(b),T(c),\lambda)} \le k\left(\frac{S(b,c,\lambda)}{1-S(b,c,\lambda)}\right) = k\left(\frac{S(T(b),T(c),\lambda)}{1-S(T(b),T(c),\lambda)}\right)$$
$$\le k^2\left(\frac{S(b,c,\lambda)}{1-S(b,c,\lambda)}\right) \le \cdots \le k^n\left(\frac{S(b,c,\lambda)}{1-S(b,c,\lambda)}\right) \to 0.$$

Thus, $H(b,c,\lambda) = 1, M(b,c,\lambda) = 0, S(b,c,\lambda) = 0$ and hence $b = c$.

From Proposition 3.13 and Theorem 3.14, we can give the following corollary:

Corollary 3.15. Let V be a complete NMS and $T: F \to F$ an NC mapping. Then T has a unique fixed point.

Definition 3.16. A sequence (λ_n) is called $s-$increasing sequence if there exists $m_0 \in \mathbb{N}$ such that $\lambda_m + 1 \le \lambda_{m+1}$, for all $m \ge m_0$.

Now, we define the infinite "product" with $\odot$ in V, as follows:

$$\prod_{i=1}^{\infty} H(a,b,\lambda_i) = H(a,b,\lambda_1) \odot H(a,b,\lambda_2) \odot \cdots \odot H(a,b,\lambda_n) \odot \cdots,$$
$$\prod_{i=1}^{\infty} M(a,b,\lambda_i) = M(a,b,\lambda_1) \odot M(a,b,\lambda_2) \odot \cdots \odot M(a,b,\lambda_n) \odot \cdots,$$
$$\prod_{i=1}^{\infty} S(a,b,\lambda_i) = S(a,b,\lambda_1) \odot S(a,b,\lambda_2) \odot \cdots \odot S(a,b,\lambda_n) \odot \cdots.$$

The following property holds in the classical infinite product of real numbers:

$\prod_{i=1}^{\infty} d_i$ is convergent, if the sequence of the successive products $e_n = \prod_{i=1}^{\infty} d_i$ is convergent, i.e., (e_n) converges to a nonzero real number as $n \to \infty$.

Theorem 3.17. Let V be a complete NMS with (2) such that for each $\eta > 0$ and an $s-$increasing sequence (λ_n) there exists $n_0 \in \mathbb{N}$ such that $\prod_{n \ge n_0} H(a,b,\lambda_n) > 1 - \eta$, $\prod_{n \ge n_0} M(a,b,\lambda_n) < \eta$, $\prod_{n \ge n_0} S(a,b,\lambda_n) < \eta$. Choose $k \in (0,1)$. Let $T: F \to F$ be a mapping satisfying $H(T(a),T(b),k\lambda) \ge H(a,b,\lambda)$, $M(T(a),T(b),k\lambda) \ge M(a,b,\lambda)$ and $S(T(a),T(b),k\lambda) \ge S(a,b,\lambda)$ for all $a, b \in F$. Then, T has a unique fixed point.

Proof. Let $a \in F$ and $a_n = T^n(a), n \in \mathbb{N}$. We have,

$$H(a_1,a_2,\lambda) = H(T(a),T^2(a),\lambda) \ge H\left(a,T(a),\frac{\lambda}{k}\right) = H\left(a,a_1,\frac{\lambda}{k}\right),$$
$$M(a_1,a_2,\lambda) = M(T(a),T^2(a),\lambda) \le M\left(a,T(a),\frac{\lambda}{k}\right) = M\left(a,a_1,\frac{\lambda}{k}\right),$$
$$S(a_1,a_2,\lambda) = S(T(a),T^2(a),\lambda) \le S\left(a,T(a),\frac{\lambda}{k}\right) = S\left(a,a_1,\frac{\lambda}{k}\right).$$

By induction, for $n \in \mathbb{N}$,

$$H(a_n,a_{n+1},\lambda) > H\left(a,a_1,\frac{\lambda}{k^n}\right), \quad M(a_n,a_{n+1},\lambda) < M\left(a,a_1,\frac{\lambda}{k^n}\right), \quad S(a_n,a_{n+1},\lambda) < S\left(a,a_1,\frac{\lambda}{k^n}\right).$$

Let $\lambda, \eta > 0$. For $m, n \in \mathbb{N}$, we suppose $n < m$, if we take $s_i > 0, (i = n, \cdots, m-1)$, satisfying $s_n + \cdots + s_{m-1} \le 1$, then

$$H(a_n, a_m, \lambda) > H(a_n, a_{n+1}, s_n\lambda) \odot \quad \cdots \odot H(a_{m-1}, a_m, s_{m-1}\lambda) \geq$$
$$H\left(a, a_1, \frac{s_n\lambda}{k^n}\right) \odot \cdots \odot H\left(a, a_1, \frac{s_{m-1}\lambda}{k^{m-1}}\right),$$
$$M(a_n, a_m, \lambda) < M(a_n, a_{n+1}, s_n\lambda) \odot \cdots \odot M(a_{m-1}, a_m, s_{m-1}\lambda) \leq$$
$$M\left(a, a_1, \frac{s_n\lambda}{k^n}\right) \odot \cdots \odot M\left(a, a_1, \frac{s_{m-1}\lambda}{k^{m-1}}\right),$$
$$S(a_n, a_m, \lambda) < S(a_n, a_{n+1}, s_n\lambda) \odot \cdots \odot S(a_{m-1}, a_m, s_{m-1}\lambda) \leq$$
$$S\left(a, a_1, \frac{s_n\lambda}{k^n}\right) \odot \cdots \odot S\left(a, a_1, \frac{s_{m-1}\lambda}{k^{m-1}}\right).$$

In particular case, since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$, we can take $s_i = \frac{1}{i(i+1)}, (i = n, \cdots m-1)$, and then

$$H(a_n, a_m, b) \geq H\left(a, a_1, \frac{\lambda}{n(n+1)k^n}\right) \odot \cdots \odot H\left(a, a_1, \frac{\lambda}{m(m-1)k^{m-1}}\right) \geq \prod_{n=1}^{\infty} H\left(a, a_1, \frac{\lambda}{n(n+1)k^n}\right),$$
$$M(a_n, a_m, b) \leq M\left(a, a_1, \frac{\lambda}{n(n+1)k^n}\right) \odot \cdots \odot M\left(a, a_1, \frac{\lambda}{m(m-1)k^{m-1}}\right) \leq \prod_{n=1}^{\infty} M\left(a, a_1, \frac{\lambda}{n(n+1)k^n}\right),$$
$$S(a_n, a_m, b) \leq S\left(a, a_1, \frac{\lambda}{n(n+1)k^n}\right) \odot \cdots \odot S\left(a, a_1, \frac{\lambda}{m(m-1)k^{m-1}}\right) \leq \prod_{n=1}^{\infty} S\left(a, a_1, \frac{\lambda}{n(n+1)k^n}\right).$$

If we write $\lambda_n = \frac{\lambda}{n(n+1)k^n}$, it is easy to prove that $(\lambda_{n+1} - \lambda_n) \to \infty$, as $n \to \infty$, so (λ_n) is an s −increasing sequence, and then there exists $n_0 \in \mathbb{N}$ such that

$$\prod_{n=n_0}^{\infty} H\left(a, a_1, \frac{\lambda}{n(n+1)k^n}\right) > 1 - \eta, \quad \prod_{n=n_0}^{\infty} M\left(a, a_1, \frac{\lambda}{n(n+1)k^n}\right) < \eta,$$
$$\prod_{n=n_0}^{\infty} S\left(a, a_1, \frac{\lambda}{n(n+1)k^n}\right) < \eta.$$

Therefore, $H(a_n, a_m, \lambda) > 1 - \eta$, $M(a_n, a_m, \lambda) < \eta$, $S(a_n, a_m, \lambda) < \eta$, for $m, n \geq n_0$. Hence (a_n) is a Cauchy sequence. Since F is complete, there is $b \in F$ such that $\lim_n a_n = b$. We must show that b is a fixed point for T. We have, for $n \to \infty$,

$$H(T(b), b, \lambda) \geq H\left(T(b), T(a_n), \frac{\lambda}{2}\right) \odot H\left(a_{n+1}, b, \frac{\lambda}{2}\right) \geq H\left(b, a_n, \frac{\lambda}{2k}\right) \odot H\left(a_{n+1}, b, \frac{\lambda}{2}\right) \to 1 \odot 1,$$
$$M(T(b), b, \lambda) \leq M\left(T(b), T(a_n), \frac{\lambda}{2}\right) \boxdot M\left(a_{n+1}, b, \frac{\lambda}{2}\right) \leq M\left(b, a_n, \frac{\lambda}{2k}\right) \boxdot M\left(a_{n+1}, b, \frac{\lambda}{2}\right) \to 0 \boxdot 0,$$
$$S(T(b), b, \lambda) \leq S\left(T(b), T(a_n), \frac{\lambda}{2}\right) \boxdot S\left(a_{n+1}, b, \frac{\lambda}{2}\right) \leq S\left(b, a_n, \frac{\lambda}{2k}\right) \boxdot S\left(a_{n+1}, b, \frac{\lambda}{2}\right) \to 0 \boxdot 0$$

by definition of convergence sequence and by the continuity of $\odot$ and $\boxdot$.

So $H(T(b), b, \lambda) = 1$, $M(T(b), b, \lambda) = 0$, $S(T(b), b, \lambda) = 0$ and then we get $T(b) = b$. Now, let's show the uniqueness. Assume $T(c) = c$ for some $c \in F$. Then,

$$1 \geq H(c, b, \lambda) = H(T(c), T(b), \lambda) \geq H\left(c, b, \frac{\lambda}{k}\right) = H\left(T(c), T(b), \frac{\lambda}{k}\right) \geq H\left(c, b, \frac{\lambda}{k^2}\right) \geq \cdots \geq$$
$$H\left(c, b, \frac{\lambda}{k^n}\right),$$
$$0 \leq M(c, b, \lambda) = M(T(c), T(b), \lambda) \leq M\left(c, b, \frac{\lambda}{k}\right) = M\left(T(c), T(b), \frac{\lambda}{k}\right) \leq M\left(c, b, \frac{\lambda}{k^2}\right) \leq \cdots \leq$$
$$M\left(c, b, \frac{\lambda}{k^n}\right),$$
$$0 \leq S(c, b, \lambda) = S(T(c), T(b), \lambda) \leq S\left(c, b, \frac{\lambda}{k}\right) = S\left(T(c), T(b), \frac{\lambda}{k}\right) \leq S\left(c, b, \frac{\lambda}{k^2}\right) \leq$$
$$\cdots S\left(c, b, \frac{\lambda}{k^n}\right).$$

Now, it is easy to verify that $\left(\frac{t}{k^n}\right)$ is an s −increasing sequence, then by assumption, for a given $\eta \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $\prod_{n \geq n_0} H\left(c, b, \frac{\lambda}{k^n}\right) \geq 1 - \eta$, $\prod_{n \geq n_0} M\left(c, b, \frac{\lambda}{k^n}\right) \leq \eta$, $\prod_{n \geq n_0} S\left(c, b, \frac{\lambda}{k^n}\right) \leq \eta$ and clearly $\lim_n H\left(c, b, \frac{\lambda}{k^n}\right) = 1$, $\lim_n M\left(c, b, \frac{\lambda}{k^n}\right) = 0$, $\lim_n S\left(c, b, \frac{\lambda}{k^n}\right) = 0$. Hence, $H(c, b, \lambda) = 1$, $M(c, b, \lambda) = 0$, $S(c, b, \lambda) = 0$ and so $c = b$.

4. CONCLUSION

Neutrosophic metric space with CTN and CTC is defined by Kirisci and Simsek[28]. Kirisci et al [21] argued fixed point results for NMS. In this paper, fixed point results in NMS were discussed. New infinite products are defined by CTN. The Banach Contraction Theorem for NMS is proved by the new defined infinite products.

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