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Research Article

MATHEMATICAL BEHAVIOR OF SOLUTIONS OF P-LAPLACIAN EQUATION WITH LOGARITHMIC SOURCE TERM

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ABSTRACT

For the p-Laplacian wave equation with logarithmic nonlinearity of initial value problem is analyzed. Focusing on the interplay between damped term and logarithmic source, we discuss the local existence of solutions.
Keywords: Existence, logarithmic nonlinearity.

1. INTRODUCTION

In this paper, we consider the following the p-Laplacian equation with logarithmic nonlinearity

$$\begin{cases} u_{tt} - div(|\nabla u|^{p-2}\nabla u) - \Delta u + u_t = kuln|u|, & x \in \Omega, \ t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = \frac{\partial}{\partial v} u(x,t) = 0, & x \in \partial\Omega, \ t > 0, \end{cases} \tag{1}$$

where $\Omega \subset R^n$ $(n \leq p)$ is a bounded domain with smooth boundary $\partial\Omega$, $p > 2$ is a costant number and k is the smallest positive constant.

Studies of logarithmic nonlinearity have a long history in physics as it occurs naturally in inflation cosmology, quantum mechanics and nuclear physics [2,3,6]. There is a lot of reference in the literature which interested in applications of logarithmic nonlinerity. The first well known working is introduced by [1] . Later, the motivated of this working a lot mathematicians studied different problem with logarithmic source term see [4,8,16,13,14,12].

Messaoudi, [11] studied the following problem

$$u_{tt} - div(|\nabla u|^{p-2}\nabla u) - \Delta u_t + |u_t|^{q-1}u_t = |u|^{p-1}u. \tag{2}$$

He studied decay of solutions of the problem (2) using the techniques combination of the perturbed energy and potential well methods. Then the problem (2) was studied by Wu and Xue [17] and Pişkin [15].

In [9], Nhan and Truong considered

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$$u_t - div(|\nabla u|^{p-2}\nabla u) - \Delta u_t = |u|^{p-2}u ln|u|, \tag{3}$$

and established the global existence, blow up and deca y of solutions for $p > 2$. The problem (3) was studied by Cao and Liu [5], they proved global boundedness and also blowing-up at infinity for $1 < p < 2$.

The present of our paper is organized as follows: Firstly, we give some notations and lemmas which will be used throughout this paper. In the last section, we established the local existence of the solutions the problem.

2. PRELIMINARIES

In this section we will give some notations and lemmas which will be used throughout this paper. For simplify notations, throughout this paper, we adopt the following abbreviations:

$$\|u\|_p = \|u\|_{L^p(\Omega)}, \|u\|_2 = \|u\| \ \ and \ \|u\|_{1,p} = \|u\|_{W_0^{1,p}\Omega)} = \left(\|u\|_p + \|\nabla u\|_p\right)^{\frac{1}{p}},$$

for $2 < p$. We denote by C and $C_i = (i = 1,2,\ldots)$ various positive constants.

(A) The constant k in (1) satisfies $0 < k < k_1$where k is the positive real number satisfying $e^{-\frac{3}{2}} = \sqrt{\frac{2\pi}{k_1}}$.

Remark 1 The function $f(s) = \sqrt{\frac{2\pi}{s}} - e^{-\frac{3}{2}}$ is a continuous and decreasing function on $(0, \infty)$, with

$$\lim_{s\to 0^+} f(s) = \infty, \ \ \lim_{s\to\infty} f(s) = -e^{-\frac{3}{2}}.$$

Then there exist a unique $k_1 > 0$ such that $f(k_1) = 0$.

Therefore;

$$e^{-\frac{3}{2}} < \sqrt{\frac{2\pi}{s}}, \forall s \in (0, k_1).$$

We define energy function as follows

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|^2 - \frac{k}{2}\int_\Omega ln|u|u^2 dx + \frac{k}{4}\|u\|^2. \tag{4}$$

Lemma 2 $E(t)$ is a nonincreasing function of $t \geq 0$

$$E'(t) = -\|u_t\|^2 \leq 0. \tag{5}$$

Proof. We show that $E'(t) = -\|u_t\|^2 \leq 0$. Multiplying the equation (1) by u_t and integrating on Ω we have

$$\begin{gathered}\int_\Omega u_{tt}u_t dx - \int_\Omega div(|\nabla u|^{p-2}\nabla u)u_t dxc \\ +\int_\Omega \nabla u \nabla u_t dx + \int_\Omega u_t u_t dx \\ = \int_\Omega k u ln|u| u_t dx,\end{gathered}$$

$$\begin{gathered}\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |u_t|^2 dx\right) + \frac{d}{dt}\left(\frac{1}{p}\int_\Omega |\nabla u|^p dx\right) + \frac{d}{dt}\left(\frac{1}{2}\int_\Omega |\nabla u|^2 dx\right) \\ + \frac{d}{dt}\left(-\frac{k}{2}\int_\Omega ln|u|u^2 dx + \frac{k}{4}\|u\|^2\right) \\ = -\|u_t\|^2,\end{gathered}$$

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|^2 - \frac{k}{2}\int_\Omega ln|u|u^2 dx + \frac{k}{4}\|u\|^2\right] = -\|u_t\|^2,$$

$$E'(t) = -\|u_t\|^2.$$

Lemma 3 [7] (Logarithmic Sob olev Inequality). Let u be any function $u \in {H_0}^1(\Omega)$ and $\alpha > 0$ be any number

$$\int_\Omega ln|u|u^2 dx < \frac{1}{2}\|u\|^2 ln\|u\|^2 + \frac{\alpha^2}{2\pi}\|\nabla u\|^2 - (1 + ln\alpha)\|u\|^2.$$

Lemma 4 [4] (Logarithmic Gronwall Inequality) Let $c > 0, \gamma \in L^1(0, T, R^+)$ and assume that the function $w: [0, T] \to [1, \infty]$ satisfies

$$w(t) \le c\left(1 + \int_0^t \gamma(s)w(s)\ln w(s)\,ds\right), 0 \le t \le T,$$

where

$$w(t) \le ce^{\int_0^t c\gamma(s)ds}, 0 \le t \le T.$$

3. LOCAL EXISTENCE

In this section we state and prove the local existence result for the problem (1). The proof is based Faedo-Galerkin method.

Definition 5 A function u defined on $[0, T]$ is called a weak solution of (1) if

$$u \in C\left([0, T): W_0^{1,p}(\Omega)\right), \quad u_t \in C\big([0, T): L^2(\Omega)\big)$$

and u satisfies

$$\begin{cases} \int_\Omega u_{tt}(x,t)w(x)dx + \int_\Omega \nabla u(x,t)\nabla w(x)dx \\ + \int_\Omega |\nabla u|^{p-2}\nabla u(x,t)\nabla w(x)dx + \int_\Omega u_t(x,t)w(x)dx \\ = k\int_\Omega u(x,t)ln|u(x,t)|w(x)dx, \end{cases}$$

for $w \in H_0^1(\Omega)$.

Theorem 6 Let $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$. Then the problem (1) has a global weak solution on $[0, T]$.

Proof. We will use the Faedo-Galerkin method to construct approximate solutions. Let $\{w_j\}_{j=1}^{\infty}$ be an orthogonal basis of the "separable" space $W_0^{1,p}(\Omega)$ which is orthonormal in $L^2(\Omega)$. Let

$$V_m = span\{w_1, w_2, \ldots, w_m\}$$

and let the projections of the initial data on the finite dimensional subspace V_m be given by

$$u_0^m(x) = \sum_{j=1}^m a_j w_j \to u_0 \; in \; W_0^{1,p}(\Omega)$$
$$u_1^m(x) = \sum_{j=1}^m b_j w_j \to u_1 \; in \; L^2(\Omega)$$

for $j = 1,2, \ldots, m$.

We look for the approximate solutions

$$u^m(x,t) = \sum_{j=1}^m h_j^m(t) w_j(x),$$

of the approximate problem in V_m

$$\begin{cases} \int_\Omega (u_{tt}^m w + |\nabla u^m|^{p-2}\nabla u^m \nabla w + \nabla u^m \nabla w + u_t^m w)dx = k\int_\Omega u^m ln|u^m| w dx, \\ u^m(0) = u_0^m = \sum_{j=1}^m (u_0, w_j)\, w_j, \\ u_t^m(0) = u_1^m = \sum_{j=1}^m (u_1, w_j)\, w_j. \end{cases} \tag{6}$$

This leads to a system of ordinary differantial equations for unknown functions $h_j^m(t)$. Based on standard existence theory for ordinary differantial equation, one can obtain functions

$$h_j: [0, t_m] \to R,\ j = 1,2,\ldots,\mathrm{m} \tag{7}$$

which satisfy (6) in a maximal interval $[0, t_m)$, $0 < t_m \leq T$. Next, we show that $t_m = T$ and that the local solution is uniformly bounded independent of m and t. For this purpose, let us replace w by u_t^m in (6) and integrate by parts we obtain

$$\frac{d}{dt}E^m(t) = -\|u_t^m\|^2 \leq 0, \tag{8}$$

where

$$E^m(t) = \frac{1}{2}\|u_t^m\|^2 + \frac{1}{p}\|\nabla u^m\|_p^p + \frac{1}{2}\|\nabla u^m\|^2$$
$$-\frac{k}{2}\int_\Omega |u^m|^2 ln|u^m| dx + \frac{k}{4}\|u^m\|^2. \tag{9}$$

Integrating (8) with respect to t from 0 to t, we obtain

$$E^m(t) \leq E^m(0). \tag{10}$$

By the Logarithmic Sobolev inequality leads to

$$E^m(t) = \frac{1}{2}\|u_t^m\|^2 + \frac{1}{p}\|\nabla u^m\|_p^p + \frac{1}{2}\|\nabla u^m\|^2$$
$$-\frac{k}{2}\int_\Omega |u^m|^2 ln|u^m| dx + \frac{k}{4}\|u^m\|^2,$$
$$\geq \frac{1}{2}\|u_t^m\|^2 + \frac{1}{p}\|\nabla u^m\|_p^p + \frac{1}{2}\|\nabla u^m\|^2 + \frac{k}{4}\|u^m\|^2$$
$$-\frac{k}{2}\left[\frac{1}{2}\|u^m\|^2 ln\|u^m\|^2 + \frac{\alpha^2}{2\pi}\|\nabla u^m\|^2 - (1 + ln\alpha)\|u^m\|^2\right],$$
$$= \frac{1}{2}\|u_t^m\|^2 + \frac{1}{p}\|\nabla u^m\|_p^p + \left(1 - \frac{k\alpha^2}{2\pi}\right)\|\nabla u^m\|^2$$
$$+\frac{1}{2}\left[\frac{k}{2}(1 - ln\|u^m\|^2) + k(1 + ln\alpha)\right]\|u^m\|^2 \tag{11}$$

Then, using of (10) and taking $C = 2E^m(0)$ we get

$$\|u_t^m\|^2 + \left(1 - \frac{k\alpha^2}{2\pi}\right)\|\nabla u^m\|^2$$
$$\frac{2}{p}\|\nabla u^m\|_p^p + + \left(\frac{3k}{2} + kln\alpha\right)\|u^m\|^2$$
$$\leq C + \frac{k}{2}\|u^m\|^2 ln\|u^m\|^2. \tag{12}$$

Now, choosing

$$e^{-\frac{3}{2}} < \alpha < \sqrt{\frac{2\pi}{k}} \tag{13}$$

will make

$$\frac{3k}{2} + kln\alpha > 0 \ \ and\ 1 - \frac{k\alpha^2}{2\pi} > 0$$

This selection is possible thanks to (A). So, we have

$$\|u_t^m\|^2 + \|\nabla u^m\|_p^p + \|\nabla u^m\|^2 + \|u^m\|^2 < c(1 + \|u^m\|^2 ln\|u^m\|^2). \tag{14}$$

We know that

$$u^m(.,t) = u^m(.,0) + \int_0^t \frac{\partial u^m}{\partial \tau}(.,\tau)d\tau.$$

Then, using Cauchy-Schwarz inequality $(a + b)^2 \leq 2(a^2 + b^2)$, we obtain

$$\|u^m(t)\|^2 = \left\|u^m(.,0) + \int_0^t \frac{\partial u^m}{\partial \tau}(.,\tau)d\tau\right\|^2$$
$$\leq 2\|u^m(0)\|^2 + 2\left\|\int_0^t \frac{\partial u^m}{\partial \tau}(.,\tau)d\tau\right\|^2$$
$$\leq 2\|u^m(0)\|^2 + 2T\int_0^t \|u_t^m(\tau)\|^2\, d\tau. \qquad (15)$$

So, using of inequality (14) and (15) we get

$$\|u^m(t)\|^2 \leq 2\|u^m(0)\|^2 + 2Tc(1 + \|u^m\|^2 ln\|u^m\|^2). \qquad (16)$$

If we put $C_1 = max\{2\|u^m(0)\|^2, 2Tc\}$, (16) leads to

$$\|u^m\|^2 \leq 2C_1\left(1 + \int_0^t \|u^m\|^2 ln\|u^m\|^2\, d\tau\right).$$

Without loss of generality, we take $C_1 \geq 1$, we have

$$\|u^m\|^2 \leq 2C_1\left(1 + \int_0^t (C_1 + \|u^m\|^2) ln\big((C_1 + \|u^m\|^2)\big)\, d\tau\right).$$

Thanks to Logarithmic Gronwall inequality, we obtain

$$\|u^m\|^2 \leq 2C_1 e^{2C_1 t} = C_2.$$

Therefore, from inequality (14), it follows that

$$\|u_t^m\|^2 + \|\nabla u^m\|_p^p + \|\nabla u^m\|^2 + \|u^m\|^2 \leq C_3 = C(1 + C_2 \ln C_2),$$

where C_3 is a positive constant independent of m and t. If these operations (14) are applied to each term of inequality, this implies

$$\max_{t\in(0,t_m)}\|u_t^m\|^2 + \max_{t\in(0,t_m)}\|\nabla u^m\|_p^p + \max_{t\in(0,t_m)}\|\nabla u^m\|^2 + \max_{t\in(0,t_m)}\|u^m\|^2 \leq 4C_3 \qquad (17)$$

So, the approximate solution is uniformly bounded independent of m and t. Therefore, we can extend t_m to T. Moreover, we obtain

$$\begin{cases} u^m, is\ uniformly\ bounded\ in\ L^\infty\left(0,T; W_0^{(1,p)}(\Omega)\right), \\ u_t^m, is\ uniformly\ bounded\ in\ L^\infty\big(0,T; L^2(\Omega)\big). \end{cases} \qquad (18)$$

Hence we can infer from (17) and (18) that there exists a subsequence of (u^m) (still denoted by (u^m), such that

$$\begin{cases} u^m \to u,\ weakly^*\ in\ L^\infty\left(0,T; W_0^{(1,p)}(\Omega)\right), \\ u_t^m \to u_t,\ weakly^*\ in\ L^\infty\big(0,T; L^2(\Omega)\big), \\ u^m \to u, weakly\ in\ L^2\big(0,T; H_0^1(\Omega)\big), \\ u_t^m \to u_t, weakly\ \ in\ L^2\big(0,T; L^2(\Omega)\big). \end{cases} \qquad (19)$$

Then using (19) and Aubin-Lions lemma, we have

$$u^m \to u, strongly\ in\ L^2\big(0,T; L^2(\Omega)\big)$$

which implies

$$u^m \to u,\ \Omega \times (0,T).$$

Since the map $s \to sln|s|^k$ is continuous, we have the convergence

$$u^m ln|u^m|^k \to uln|u|^k,\ \Omega \times (0,T). \qquad (20)$$

By the Sobolev embedding theorem $(H_0^1(\Omega) \hookrightarrow L^\infty(\Omega))$, it is clear that $\left|u^m ln|u^m|^k - uln|u|^k\right|$ is bounded in $L^\infty(\Omega \times (0,T))$. Next, taking into account the Lebesgue bounded convergence theorem, we have

$u^m ln|u^m|^k \to u ln|u|^k \ strongly \ in \ L^2\big(0,T;L^2(\Omega)\big)$ (21)

We integrate (6) over $(0,t)$ to obtain, $\forall w \in V_m$

$$k\int_0^t \int u^m ln|u^m| w dx ds = \int_\Omega u_t^m w dx - \int_\Omega u_1^m w dx + \int_0^t \int_\Omega |\nabla u^m|^{p-2}\nabla u^m \nabla w dx ds + \int_0^t \int_\Omega \nabla u^m \nabla w dx ds + \int_0^t \int_\Omega u_t^m w dx ds.$$

Convergences (19), (21) are sufficient to pass to the limit in (22)

$$\int_\Omega u_t w dx = \int_\Omega u_1 w dx - \int_0^t \int_\Omega |\nabla u|^{p-2}\nabla u \nabla w dx ds - \int_0^t \int_\Omega \nabla u \nabla w dx ds - \int_0^t \int_\Omega u_t w dx ds \int + k\int_0^t \int_\Omega u ln|u| w dx ds. \quad (22)$$

which implies that (22) is valid $\forall w \in H_0^1(\Omega)$.Using the fact that the terms in the right-hand side of (23) are absolutely continuous since they are functions of t defined by integrals over $(0,t)$, hence it is differentiable for a.e. $t \in R^+$. Thus, differentiating (23), we obtain, for a.e. $t \in (0,T)$ and any $\forall w \in H_0^1(\Omega)$,

$$\int_\Omega u(x,t) ln|u(x,t)|^k w(x,t) dx = \int_\Omega u_{tt}(x,t) w(x) dx + \int_\Omega |\nabla u|^{p-2}\nabla u(x,t) \nabla w(x) + \int_\Omega \nabla u(x,t) \nabla w(x) dx + \int_\Omega u_t(x,t) w(x) dx, \quad (23)$$

If we take initial data, we note that

$$u^m \to u, weakly \ in \ L^2\left(0,T;W_0^{(1,p)}(\Omega)\right),$$
$$u_t^m \to u_t, weakly \ in \ L^2\big(0,T;L^2(\Omega)\big).$$

Thus, using Lion's Lemma [10], we have

$$u^m \to u, in \ C\big([0,T];L^2(\Omega)\big).$$

Therefore, $u^m(x,0)$ makes sense and

$$u^m(x,0) \to u(x,0), in \ L^2(\Omega)$$

We have

$$u^m(x,0) \to u_0(x,0), in \left(H_0^1(\Omega) \cap L^p(\Omega)\right)$$

Hence

$$u(x) = u_0(x)$$

Now, multiply (6) by $\varphi \in C_0^\infty(0,T)$ and integrate over $(0,T)$, we obtain for $\forall w \in V_m$, and because of

$$(u_t^m\}\varphi(t))' = u_{tt}^m \varphi(t) + u^m \varphi'(t)$$

we get

$$-\int_0^t \int_\Omega u_t^m w \varphi'(t) dx = \int_0^t \int_\Omega |\nabla u^m|^{p-2}\nabla u^m \nabla w \varphi(t) dx dt - \int_0^t \int_\Omega \nabla u^m \nabla w \varphi(t) dx dt - \int_0^t \int_\Omega u_t^m w \varphi(t) dx dt + k\int_0^t \int_\Omega u^m ln|u^m| w \varphi(t) dx dt.$$

As $\to \infty$, we have for $\forall w \in H_0^1(\Omega)$ and $\varphi \in C_0^\infty(0,T)$

$$-\int_0^t \int_\Omega u_t w\varphi'(t)dx = \int_0^t \int_\Omega |\nabla u|^{p-2}\nabla u \nabla w\varphi(t)dxdt$$
$$-\int_0^t \int_\Omega \nabla u\nabla w\varphi(t)dxdt - \int_0^t \int_\Omega u_t w\varphi(t)dxdt$$
$$+k\int_0^t \int_\Omega u ln|u| w\varphi(t)dxdt.$$

This means

$$u_{tt} \in L^2[0,T), H^{-2}(\Omega),$$

on the other hand, because of

$$u_{tt} \in L^2[0,T), L^2(\Omega),$$

we obtain

$$u_{tt} \in C[0,T), H^{-2}(\Omega).$$

So that

$$u_t^m(x,0) \to u_t(x,0), H^{-2}(\Omega),$$

but

$$u_t^m(x,0) = u_1^m(x) \to u^1(x), L^2(\Omega).$$

Hence

$$u_t(x,0) = u_1(x).$$

This finished the proof of the theorem.

Conflict of interest The authors declare that they have no conflict of interest.

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