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Research Article

A NEW ALMOST SEQUENCE SPACE OF ORDER β

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ABSTRACT

In this paper we introduce and study some properties of the new sequence space of order β which is defined using almost convergence and the modulus function. Further, some connections between strong $V_{\lambda}^{\beta}((B,f,M))$ - almost summability of sequences and $\lambda -$strong almost convergence of order β with respect to a modulus are studied.

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1. INTRODUCTION AND BACKGROUND

Let S denote the set of all real and complex sequences $x=(x_k)$. By l_∞ and c, we denote the Banach spaces of bounded and convergent sequences $x=(x_k)$ normed by $\| x \|= \sup_n |x_n|$, respectively. A linear functional γ on l_∞ is said to be a Banach limit if it has the following properties:

1) $\gamma(x) \geq 0$ if $n \geq 0$ (i.e. $x_n \geq 0$ for all n),
2) $\gamma(e)=1$ where $e=(1,1,\ldots)$,
3) $\gamma(Dx)=\gamma(x)$, where the shift operator D is defined by $D(x_n)=\{x_{n+1}\}$.

Let $\mathbf{B}$ be the set of all Banach limits on l_∞. A sequence $x \in \ell_\infty$ is said to be almost convergent if all of its Banach limits coincide. Let $\hat{c}$ denote the space of almost convergent sequences .

Lorentz [4] has shown that

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$$\hat{c}=\{x\in l_{\infty}\ :\ \lim_{m} t_{m,n}(x) \text{ exists uniformly in } n\}$$

where

$$t_{m,n}(x)=\frac{x_n+x_{n+1}+x_{n+2}+\cdots+x_{n+m}}{m+1}.$$

Maddox [5]introduced the space $[\hat{c}]$ of strongly almost convergent sequence as follows:

$$[\hat{c}]=\{x\in l_{\infty}\ :\ \lim_{m} t_{m,n}(|x-L|)=0,\ \text{uniformly in } n,\ \text{for some } L\}$$

Let $\lambda=(\lambda_i)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{i+1}\leq\lambda_i+1, \lambda_1=1.$$

The collection of such sequence λ will be denoted by Δ.

The generalized de la Valée-Poussin mean is defined by

$$T_i(x)=\frac{1}{\lambda_i}\sum_{k\in I_i} x_k$$

where $I_i=[i-\lambda_i+1, i]$. A sequence $x=(x_k)$ is said to be (V,λ)-summable to a number L, if $T_i(x)\to L \text{ as } i\to\infty$ (see [7]).

The space $[V,\lambda]$ of λ-strongly convergent sequences was introduced by Malkowsky and Savaş [7] as follows:

$$[V,\lambda]=\left\{x=(x_k):\ \lim_{i}\frac{1}{\lambda_i}\sum_{k\in I_i}|x_k-L|=0,\ \text{for some } L\right\}.$$

Note that in the special case where $\lambda_i=i$, the space $[V,\lambda]$ reduces the space w of strongly Cesàro summable sequences which is defined by

$$w=\left\{x=(x_k):\ \lim_{i}\frac{1}{i}\sum_{k=1}^{i}|x_k-L|)=0,\ \text{for some } L\right\}.$$

More results on λ-strong convergence can be seen from [8, 12, 13, 14,15].

Following Ruckle [10], a modulus function M is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $M(x)=0$ if and only if $x=0$,

(ii) $M(x+y)\leq M(x)+M(y)$ for all $x,y\geq 0$,

(iii) M increasing,

(iv) M is continuous from the right at zero.

Maddox [6] introduced and examined some properties of the sequence spaces $w_0(M)$, $w(M)$ and $w_{\infty}(M)$ defined using a modulus M, which generalized the well-known spaces w_0, w and w_{∞} of strongly summable sequences.

In 1999, E. Savas [11] defined the class of sequences, which are strongly almost Cesàro summable with respect to modulus, as follows:

$$[\hat{c}(M,p)] = \left\{ x \ : \ \lim_n \frac{1}{n} \sum_{k=1}^{n} M(| x_{k+m} - L |)^{p_k} = 0, \text{ for some L, uniformly in } m \right\}$$

and

$$[\hat{c}(M,p)]_0 = \left\{ x \ : \ \lim_n \frac{1}{n} \sum_{k=1}^{n} M(| x_{k+m} |)^{p_k} = 0, \text{ uniformly in } m \right\}.$$

where $p = (p_k)$ is a sequence of strictly positive real numbers and M be a modulus.

Waszak [16] defined the lacunary strong $(A, f)-$convergence with respect to a modulus function.

If $x = (x_k)$ is a sequence and $B = (b_{nk})$ is an infinite matrix, then Bx is the sequence whose nth term is given by $B_n(x) = \sum_{k=0}^{\infty} b_{nk} x_k$. Thus we say that x is B-summable to L if $\lim_{n\to\infty} B_n(x) = L$. Let X and Y be two sequence spaces and $B = (b_{nk})$ an infinite matrix. If for each $x \in X$ the series $B_n(x) = \sum_{k=0}^{\infty} b_{nk} x_k$ converges for each n and the sequence $Bx = B_n(x) \in Y$ we say that B maps X into Y. By (X,Y) we denote the set of all matrices which maps X into Y, and in addition if the limit is preserved then we denote the class of such matrices by $(X,Y)_{reg}$.

Let $B = (b_{nk})$ be a nonnegative regular matrix summability method. Connor [3] further extended Maddox's results by giving the following definition:

Definition 1.1. *Let M be a modulus and B be a nonnegative regular summability method. We let*

$$w(B,M) = \left\{ x \ : \ \lim_n \sum_{k=1}^{\infty} b_{nk} M(| x_k - L |) = 0 \right\}$$

and

$$w(B,M)_0 = \left\{ x \ : \ \lim_n \sum_{k=1}^{\infty} b_{nk} M(| x_k |) = 0 \right\}.$$

Later on Nuray and Savas [9] extended Connor's results by using almost convergence as follows:

Definition 1.2. *Let M be a modulus and B be a nonnegative regular summability method. We let*

$$\hat{w}(B,M) = \left\{ x \ : \ \lim_n \sum_{k=1}^{\infty} b_{nk} M(| x_{k+m} - L |) = 0, \text{for some L, uniformly in } m \right\}$$

and

$$\hat{w}(B,M)_0 = \left\{ x \ : \ \lim_n \sum_{k=1}^{\infty} b_{nk} M(| x_{k+m} |) = 0, \text{ uniformly in } m \right\}.$$

By a f -function we understand a continuous non-decreasing function $f(u)$ defined for $u \geq 0$ and such that $f(0)=0,\ f(u)>0$,for $u>0$ and $f(u)\to\infty$ as $u\to\infty$, (see, [16]).

A function f is said to satisfy (Δ_2) -condition,(for all large u) if there exists constant $K>1$ such that $f(2u)\leq Kf(u)$.

In the this paper, we introduce and study some properties of the almost convergence sequence space of order β which is establish using the modulus and infinite matrix and hence as special cases, some known results are also obtained .

2. MAIN RESULTS

Let $\Delta=(\lambda_j)$ be same as above, f be given f -function and M be given modulus function, respectively. Moreover, let $B=(b_{nk})$ be the real matrix and $0<\beta\leq 1$ be given. Then we write,

$$\hat{V}_\lambda^\beta(B,f,M,p)_0=\left\{x=(x_k):\ \lim_j \frac{1}{\lambda_j^\beta}\sum_{n\in I_j} M\left(\left|\sum_{k=1}^{\infty} a_{nk}\, f(|\, x_{k+m}\, |)\right|\right)=0,\ \text{uniformly in } m\right\}.$$

If $x\in\hat{V}_\lambda^\beta(B,f,M)_0$, the sequence x is said to be λ -strong $(B,\ f)$ -almost convergent of order β to zero with respect to a modulus M .

If $\lambda_j = j,$ we have

$$\hat{V}_\lambda^\beta(B,f,M)_0=\left\{x=(x_k):\ \lim_j \frac{1}{j^\beta}\sum_{n=1}^{j} M\left(\left|\sum_{k=1}^{\infty} b_{nk}\, f(|\, x_{k+m}\, |)\right|\right)=0,\ \text{uniformly in } m\right\}.$$

If we take $f(x)=x$ for all x , we write

$$\hat{V}_\lambda^\beta(B,f,M,p)_0=\left\{x=(x_k):\ \lim_j \frac{1}{\lambda_j^\beta}\sum_{n\in I_j} M\left(\left|\sum_{k=1}^{\infty} a_{nk}\, (|\, x_{k+m}\, |)\right|\right)=0,\ \text{uniformly in } m\right\}.$$

If $M(x)=x$, we write

$$\hat{V}_\lambda^\beta(B,f)_0=\left\{x=(x_k):\ \lim_j \frac{1}{\lambda_j^\beta}\sum_{n\in I_j} \left(\left|\sum_{k=1}^{\infty} b_{nk}\, f(|\, x_{k+m}\, |)\right|\right)=0,\ \text{uniformly in } m\right\}.$$

If we take $B=I$ and $f(x)=x$ respectively, then we have

$$\hat{V}_\lambda^\beta(I,M)_0=\left\{x=(x_k):\ \lim_j \frac{1}{\lambda_j^\beta}\sum_{k\in I_j} M\left(\left|x_{k+m}\right|\right)=0,\ \text{uniformly in } m\right\}.$$

If we take $B=I$, $f(x)=x$ and $M(x)=x$ respectively, then we have

$$\hat{V}_{\lambda}^{\beta}(I)=\left\{x=(x_k):\ \lim_j \frac{1}{\lambda_j^{\beta}}\sum_{k\in I_j}|x_{k+m}|=0,\text{uniformly in } m\right\}$$

which was defined and studied by Savaş and Savaş [11].

If we define the matrix $B=(b_{nk})$ as follows:

$$b_{nk}:=\begin{cases}\frac{1}{n}, & \text{if} \quad n\geq k,\\ 0, & \text{otherwise.}\end{cases}$$

then we have,

$$\hat{V}_{\lambda}^{\beta}(\mathbf{C},f,M)_0=\left\{x=(x_k):\ \lim_j \frac{1}{\lambda_j^{\beta}}\sum_{n\in I_j}M\left(\left|\frac{1}{n}\sum_{k=1}^{n}f(|x_{k+m}|)\right|\right)=0,\right\}.$$

We now have

Theorem 2.1. *Let the* f *-function* $f(u)$ *satisfies the condition* (Δ_2) *and let the matrix has the property*

$$b_{n1}+b_{n2}+...\leq K$$

for $n=1,2,...$ *Then the following conditions are true.*

(a) If $x=(x_k)\in\hat{V}_{\lambda}^{\beta}(B,f,M,p)$ *and* α *is an arbitrary number, then* $\alpha x\in\hat{V}_{\lambda}^{\beta}(B,f,M)$.

(b) If $x,y\in\hat{V}_{\lambda}^{\beta}(B,f,M)$ *where* $x=(x_k)$, $y=(y_k)$ *and* α,η *are given numbers, then* $\alpha x+\eta y\in\hat{V}_{\lambda}^{\beta}(B,f,M)$.

Proof. (a) Let $x\in\hat{V}_{\lambda}^{\beta}(B,f,M)_0$. First let us remark that for $0<\gamma<1$ we get for all m

$$\frac{1}{\lambda_j^{\beta}}\sum_{n\in I_j}M\left(\left|\sum_{k=1}^{\infty}b_{nk}f(|\gamma x_{k+m}|)\right|\right)\leq\frac{1}{\lambda_j^{\beta}}\sum_{n\in I_j}M\left(\left|\sum_{k=1}^{\infty}b_{nk}f(|x_{k+m}|)\right|\right).$$

Hence, if $\gamma>1$ then we may find a positive number *s* such that $\gamma<2^s$ and we obtain

$$\frac{1}{\lambda_j^{\beta}}\sum_{n\in I_j}M\left(\left|\sum_{k=1}^{\infty}b_{nk}f(|\alpha x_{k+m}|)\right|\right)$$

$$\leq\frac{1}{\lambda_j^{\beta}}\sum_{n\in I_j}M\left(d^s\left|\sum_{k=1}^{\infty}b_{nk}f(|x_{k+m}|)\right|\right),$$

$$\leq\frac{L}{\lambda_j^{\beta}}\sum_{n\in I_j}M\left(\left|\sum_{k=1}^{\infty}b_{nk}f(|x_{k+m}|)\right|\right),$$

where d and L are constant connected with the properties of f and modulus M. Finally we prove the condition (a).

(b) In the following let the numbers α, η and the elements $x, y \in \hat{V}_{\lambda}^{\beta}(B, f, M)_0$ be given. From the part (a) it follows that the following inequality is true

$$\frac{1}{\lambda_j^{\beta}} \sum_{n \in I_j} M\left(\left|\sum_{k=1}^{\infty} b_{nk} f(|\alpha x_{k+m} + \eta x_{k+m}|)\right|\right)$$

$$\leq L_1 \frac{1}{\lambda_j^{\beta}} \sum_{n \in I_j} M\left(\left|\sum_{k=1}^{\infty} b_{nk} f(|x_{k+m}|)\right|\right)$$

$$+ L_2 \frac{1}{\lambda_j^{\beta}} \sum_{n \in I_j} f\left(\left|\sum_{k=1}^{\infty} b_{nk} f(|x_{k+m}|)\right|\right),$$

where the constant L_1 and L_2 are defined as in (a). Hence $x, y \in \hat{V}_{\lambda}^{\beta}(B, f, M)$

Now we shall prove some inclusion relations

Theorem 2.2.

$$\hat{V}_{\lambda}^{\beta}(B, f) \subset \hat{V}_{\lambda}^{\beta}(B, f, M).$$

Proof. Let $x \in \hat{V}_{\lambda}^{\beta}(B, f, M)$. For a given $\varepsilon > 0$ we choose $0 < \delta < 1$ such that $f(x) < \varepsilon$ for every $x \in [0, \delta]$. We can write for all im

$$\frac{1}{\lambda_j^{\beta}} \sum_{n \in I_j} M\left(\left|\sum_{k=1}^{\infty} b_{nk} f(|x_{k+m}|)\right|\right) = S_1 + S_2,$$

where $S_1 = \frac{1}{\lambda_j^{\beta}} \sum_{n \in I_j} M\left(\left|\sum_{k=1}^{\infty} b_{nk} f(|x_{k+m}|)\right|\right)$ and this sum is taken over

$$\sum_{k=1}^{\infty} b_{nk} f(|x_{k+m}|) \leq \delta$$

and

$$S_2 = \frac{1}{\lambda_j^{\beta}} \sum_{n \in I_j} M\left(\left|\sum_{k=1}^{\infty} b_{nk} f(|x_{k+m}|)\right|\right)$$

and this sum is taken over

$$\sum_{k=1}^{\infty} b_{nk} \varphi(|x_{k+m}|) > \delta.$$

By definition of the modulus M we have $S_1 = \frac{1}{\lambda_j^{\beta}} \sum_{n \in I_j} M\left(\delta\right) = M(\delta) < \varepsilon$ and moreover

$$S_2 = M(1) \frac{1}{\delta} \frac{1}{\lambda_j^{\beta}} \sum_{n \in I_j} \sum_{k=1}^{\infty} b_{nk}(i) f(|x_{k+m}|).$$

Thus we have $x \in \hat{V}_{\lambda}^{\beta}((B, f), M)$. This completes the proof.

Theorem 2.3. *Let* M, M_1, *be modulus functions. Then we have* $\hat{V}_{\lambda}^{\beta}(B, M_1, f)_0 \subset \hat{V}_{\lambda}^{\beta}(B, f, MoM_1)_0$.

The proof is a routine verification by using standard techniques and hence is omitted.

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