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Research Article

SOME PROPERTIES FOR HIGHER ORDER COMMUTATORS OF HARDY-TYPE INTEGRAL OPERATOR ON HERZ–MORREY SPACES WITH VARIABLE EXPONENT

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ABSTRACT

In this work, the boundedness for higher order commutators of Hardy-Type integrals is obtained on Herz–Morrey spaces with variable exponent $\mathbf{M\dot{K}^{\beta(.),\lambda}_{p,q(.)}(R^n)}$ applying some properties of variable exponent.
Keywords: Variable exponent, hardy-type integral, Herz–Morrey space.

1. INTRODUCTION

The first generalization of Herz spaces in view of variable exponent is established by Izuki [9, 10]. And in 2012, Almeida and Drihem [11] discuss the boundedness of a wide class of sublinear operators on Herz spaces $K^{\beta(.),p}_{q(.)}(R^n)$ and $M\dot{K}^{\beta(.),p}_{q(.)}(R^n)$ with variable exponent $\beta(.)$ and $q(.)$.

Last twenty-five century, the variable exponent function spaces and the operator theory are applicable to the modeling for electrorheological fluids, mechanics of the continuum medium and image restoration (see for example [1–8, 20, 21, 22, 23, 24, 25] and references therein), etc.

Meanwhile, they also established Hardy–Littlewood–Sobolev theorems for fractional integrals on variable Herz spaces. In 2013, Samko [12, 13] introduced a new Herz type function spaces with variable exponent, where all the three parameters are variable, and proved the boundedness of some sublinear operators. In recently, Wu J. [14, 15] and Wu J. and Zhao W. [17] considers the boundedness for sublinear operators and commutators of fractional integrals on Herz–Morrey spaces $M\dot{K}^{\beta(.),\lambda}_{p,q(.)}(R^n)$ with variable exponent q(.) but fixed $\beta \in R$ and $p \in (0,\infty)$. The BMO space and the BMO norm are defined, respectively, as follows:

$$BMO(R^n) = \{d \in L^1_{loc}(R^n): \|d\|_{BMO(R^n)} < \infty\},$$
$$\|d\|_{BMO(R^n)} = \sup_{x\in B} \frac{1}{|B|}\int_B |d(x) - d_B|dx.$$

We will denote by $|D|$ the Lebesgue measure and by χ_D the characteristic function for a measurable set $D \subset R^n$. Given a function f, we denote the mean value of f on D by $f_D =$

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$\frac{1}{|D|}\int_D f(x)dx$. C denotes a constant that is independent of the main parameters involved but whose value may differ from line to line. For any index $1 < q(x) < \infty$, we denote by $q'(x)$ its conjugate index, namely, $q'(x) = \frac{q(x)}{q(x)-1}$. For $A \sim D$, we mean that there is a constant $C > 0$ such that $C^{-1}D \le A \le CD$.

2. PRELIMINARIES

In this section, we give the definition of Herz–Morrey spaces, and state some properties. Let Φ be a measurable set in R^n with $|\Phi| > 0$.

Let $B_t = \{x \in R^n : |x| \le 2^t\}$, $A_t = B_t \backslash B_{t-1}$ and $\chi_t = \chi_{A_t}$ for $t \in Z$.

Definition 2.1. (see [17]) Suppose that $0 \le \lambda < \infty, 0 < p < \infty$, $q(.) \in \Gamma(R^n)$ and $\beta(.): R^n \to R$ with $\beta(.) \in L^\infty(R^n)$. The variable exponent Herz–Morrey space $M\dot{K}^{\beta(.),\lambda}_{p,q(.)}(R^n)$ is defined by

$$M\dot{K}^{\beta(.),\lambda}_{p,q(.)}(R^n) = \left\{ f \in L^{q(.)}_{loc}(R^n \backslash \{0\}) : \|f\|_{M\dot{K}^{\beta(.),\lambda}_{p,q(.)}(R^n)} < \infty \right\},$$

where,

$$\|f\|_{M\dot{K}^{\beta(.),\lambda}_{p,q(.)}(R^n)} = \sup_{t_0 \in Z} 2^{-t_0\lambda} \left(\sum_{t=-\infty}^{t_0} \left\| 2^{t\beta(.)} f\chi_t \right\|^p_{L^q(R^n)} \right)^{1/p}.$$

Compare the variable exponent Herz–Morrey space $M\dot{K}^{\beta(.),\lambda}_{p,q(.)}(R^n)$ with the variable exponent Herz space [11] $\dot{K}^{\beta(.),p}_{q(.)}(R^n)$, where

$$\dot{K}^{\beta(.),p}_{q(.)}(R^n) = \left\{ f \in L^{q(.)}_{loc}(R^n \backslash \{0\}) : \sum_{t=-\infty}^{\infty} \left\| 2^{t\beta(.)} f\chi_t \right\|^p_{L^q(R^n)} < \infty \right\}.$$

Obviously, $M\dot{K}^{\beta(.),\lambda}_{p,q(.)}(R^n) = \dot{K}^{\beta(.),p}_{q(.)}(R^n)$. When $\beta(.)$ is constant, we have $M\dot{K}^{\beta(.),\lambda}_{p,q(.)}(R^n) = M\dot{K}^{\beta,\lambda}_{p,q(.)}(R^n)$ (see [18]). If both $\beta(.)$ and $q(.)$ are constants, and $\lambda = 0$, then $M\dot{K}^{\beta(.),\lambda}_{p,q(.)}(R^n) = \dot{K}^{\beta,p}_{q}(R^n)$ are classical Herz spaces. When $\lambda = 0$ and $\beta(.)$ is a constant, we can see that our result below generalizes the result in the setting of the variable exponent Herz space, which proved by Izuki in [16]. So in this work, we only give the result when $\lambda > 0$.

Proposition 2.2. (see [17]) Suppose that $q_1(.) \in \Gamma(R^n) \cap \mathcal{T}(R^n)$, $0 < 1 < \frac{n}{q_1(.)}$ and define the variable exponent $q_2(.)$ by

$$\frac{1}{q_1(.)} - \frac{1}{q_2(.)} = \frac{1}{n} \quad (1)$$

Then we have $\|Hf\|_{L^{q_2}(R^n)} \le C\|f\|_{L^{q_1}(R^n)}$ for all $f \in L^{q_1}(R^n)$.

Lemma 2.3. b $\in BMO(R^n), k \in N$ and $i, j \in Z$ with $i < j$. Then we have

$$C^{-1}\|d\|^k_{BMO(R^n)} \le \sup_B \frac{1}{\|\chi_B\|_{L^q(R^n)}} \left\| \chi_B (d - d_B)^k \right\|_{L^q(R^n)} \le C\|d\|^k_{BMO(R^n)},$$

$$\left\| \chi_{B_j} \left(d - d_{B_i}\right)^k \right\|_{L^q(R^n)} \le C(j-i)^k \|b\|^k_{BMO(R^n)} \left\| \chi_{B_j} \right\|_{L^q(R^n)}.$$

The above result is proved by Izuki [9]. We remark that Lemma 2.3 is a generalization of well-known properties for BMO spaces.

Proposition 2.4. (see [17]) Let $q(.) \in \Gamma(R^n), p \in (0, \infty)$ and $0 \le \lambda < \infty$. If real-valued function $\beta(.) \in L^\infty(R^n) \cap \mathcal{T}^{loc}_0(R^n) \cap \mathcal{T}^{loc}_\infty(R^n)$, then

$$\|f\|_{M\dot{K}^{\beta(.),\lambda}_{p,q(.)}(R^n)} = \sup_{t_0\in Z} 2^{-t_0\lambda}\left(\sum_{t=-\infty}^{t_0}\left\|2^{t\beta(.)}f\chi_t\right\|^p_{L^q(R^n)}\right)^{1/p}$$
$$\approx max\left\{\sup_{\substack{t_0<0\\ t_0\in Z}} 2^{-t_0\lambda}\left(\sum_{t=-\infty}^{\tilde{t}_1} 2^{t\beta(0)p}\|f\chi_t\|^p_{L^q(R^n)}\right)^{1/p},\right.$$
$$\left.\sup_{\substack{t_0\geq 0\\ t_0\in Z}}\left(2^{-t_0\lambda}\left(\sum_{t=-\infty}^{\tilde{t}_2} 2^{t\beta(0)p}\|f\chi_t\|^p_{L^q(R^n)}\right)^{1/p} + 2^{-t_0\lambda}\left(\sum_{t=0}^{\tilde{t}_3} 2^{t\beta_\infty p}\|f\chi_t\|^p_{L^q(R^n)}\right)^{1/p}\right)\right\}$$
where $\tilde{t}_1 = \tilde{t}_3 = t_0, \tilde{t}_2 = -1$.

Lemma 2.5. (see [8, 19] Generalized Hölder's inequality) Suppose that $q(.) \in \Gamma(R^n)$, then for any $f \in L^{q(.)}(R^n)$ and any $g \in L^{q'(.)}(R^n)$, we have

$$\int_{R^n}|f(x)g(x)|dx \leq C_q\|f\|_{L^{q(.)}(R^n)}\|g\|_{L^{q'(.)}(R^n)}$$

where $C_q = 1 + \frac{1}{q^-} - \frac{1}{q^+}$.

3. RESULT AND DISCUSSION

Theorem 3.1. Suppose that $q_1(.) \in \Gamma(R^n) \cap \mathcal{T}^{log}(R^n)$. Define the variable exponent $q_2(.)$ by (1). Let $k \in N, 0 < p_1 \leq p_2 < \infty, 0 < \alpha < \frac{n}{(q_1)_+}, \lambda \geq 0$ and $\beta(.) \in L^\infty(R^n)$ be log-Hölder continuous both at the origin and at infinity, with $\lambda - n\delta_2 < \beta(0) \leq \beta_\infty < \lambda + n\delta_1$, where $\delta_1 \in \left(0, \frac{1}{(q_1')_+}\right)$ and $\delta_2 \in \left(0, \frac{1}{(q_2)_+}\right)$ are the constants appearing in (See [17, Remark 2.13]). Then for all $d \in BMO(R^n)$, we obtain

$$\left\|H^m_{\alpha,d}(f)\right\|_{M\dot{K}^{\beta(.),\lambda}_{p_2,q_2}(R^n)} \leq C\|d\|^k_{BMO(R^n)}\|f\|_{M\dot{K}^{\beta(.),\lambda}_{p_1,q_1}(R^n)}.$$

Proof of Theorem 3.1. If $\beta(.)$ be constant exponent, then the above result can be founded in [15]. When $\lambda = 0$, the above result is also valid. For any $f \in M\dot{K}^{\beta(.),\lambda}_{p_1,q_1}(R^n)$ and $d \in BMO(R^n)$. If we denote $f_j = f\chi_j = \chi_{A_j}$ for each $j \in Z$, then we can write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

Because of $0 < \frac{p_1}{p_2} \leq 1$, we apply inequality

$$\left(\sum_{i=-\infty}^{\infty}|a_i|\right)^{\frac{p_1}{p_2}} \leq \sum_{i=-\infty}^{\infty}|a_i|^{\frac{p_1}{p_2}}, \tag{2}$$

and Proposition 2.4 , we obtain

$$\left\|H^k_{\alpha,d}(f)\right\|^{p_1}_{M\dot{K}^{\beta(.),\lambda}_{p_2,q_2}(R^n)} = \sup_{t_0\in Z} 2^{-t_0\lambda p_1}\left(\sum_{t=-\infty}^{t_0}\left\|2^{t\beta(.)}H^k_{\alpha,d}(f)\chi_k\right\|^{p_2}_{L^{q_2}(R^n)}\right)^{p_1/p_2}$$
$$\approx \max\left\{\sup_{\substack{t_0\in Z\\ t_0<0}} 2^{-t_0\lambda p_1}\left(\sum_{t=-\infty}^{t_0} 2^{t\beta(0)p_2}\left\|H^k_{\alpha,d}(f)\chi_t\right\|^{p_2}_{L^{q_2}(R^n)}\right)^{p_1/p_2},\right.$$
$$\left.\sup_{\substack{t_0\in Z\\ t_0\geq 0}} 2^{-t_0\lambda p_1}\left[\left(\sum_{t=-\infty}^{-1} 2^{t\beta(0)p_2}\|k(f)\chi_t\|^{p_2}_{L^{q_2}(R^n)}\right)^{p_1/p_2} + \left(\sum_{t=0}^{t_0} 2^{t\beta_\infty p_2}\left\|H^k_{\alpha,d}(f)\chi_k\right\|^{p_2}_{L^{q_2}(R^n)}\right)^{p_1/p_2}\right]\right\}$$
$$\leq \max\left\{\sup_{\substack{t_0\in Z\\ t_0<0}} 2^{-t_0\lambda p_1}\left(\sum_{t=-\infty}^{t_0} 2^{t\beta(0)p_1}\left\|H^k_{\alpha,d}(f)\chi_t\right\|^{p_1}_{L^{q_2}(R^n)}\right)^{p_1/p_2},\right.$$
$$\left.\sup_{\substack{t_0\in Z\\ t_0\geq 0}} 2^{-t_0\lambda p_1}\left[\left(\sum_{t=-\infty}^{-1} 2^{t\beta(0)p_1}\left\|H^k_{\alpha,d}(f)\chi_t\right\|^{p_1}_{L^{q_2}(R^n)}\right)^{p_1/p_2} + \left(\sum_{t=0}^{t_0} 2^{t\beta_\infty p_1}\left\|H^k_{\alpha,d}(f)\chi_t\right\|^{p_1}_{L^{q_2}(R^n)}\right)^{p_1/p_2}\right]\right\}$$
$$\equiv \max\{E_1, E_2 + E_3\},$$

where

$$E_1 = \sup_{\substack{t_0 \in Z \\ t_0 < 0}} 2^{-t_0 \lambda p_1} \left(\sum_{t=-\infty}^{t_0} 2^{t\beta(0)p_1} \left\| H_{\alpha,d}^k(f)\chi_t \right\|_{L^{q_2}(R^n)}^{p_1} \right)$$
$$E_2 = \sup_{\substack{t_0 \in Z \\ t_0 \geq 0}} 2^{-t_0 \lambda p_1} \left(\sum_{t=-\infty}^{-1} 2^{t\beta(0)p_1} \left\| H_{\alpha,d}^k(f)\chi_t \right\|_{L^{q_2}(R^n)}^{p_1} \right)$$
$$E_3 = \sup_{\substack{t_0 \in Z \\ t_0 \geq 0}} 2^{-t_0 \lambda p_1} \left(\sum_{t=0}^{t_0} 2^{t\beta_\infty p_1} \left\| H_{\alpha,d}^k(f)\chi_t \right\|_{L^{q_2}(R^n)}^{p_1} \right)$$

It is not difficult to find that the estimate of E_1 is analogous to that of E_2; therefore, the estimates for E_1 and E_3 will be considered here. To E_1, we have

$$E_1 = \sup_{\substack{t_0 \in Z \\ t_0 < 0}} 2^{-t_0 \lambda p_1} \left(\sum_{t=-\infty}^{t_0} 2^{t\beta(0)p_1} \left\| H_{\alpha,d}^k(f)\chi_t \right\|_{L^{q_2}(R^n)}^{p_1} \right)$$
$$\leq C \sup_{\substack{t_0 \in Z \\ t_0 < 0}} 2^{-t_0 \lambda p_1} \left(\sum_{t=-\infty}^{t_0} 2^{t\beta(0)p_1} \left(\sum_{j=-\infty}^{t-2} \left\| H_{\alpha,d}^k(f)\chi_j \right\|_{L^{q_2}(R^n)} \right)^{p_1} \right)$$
$$+ C \sup_{\substack{t_0 \in Z \\ t_0 < 0}} 2^{-t_0 \lambda p_1} \left(\sum_{t=-\infty}^{t_0} 2^{t\beta(0)p_1} \left(\sum_{j=t-1}^{t+1} \left\| H_{\alpha,d}^k(f)\chi_j \right\|_{L^{q_2}(R^n)} \right)^{p_1} \right)$$
$$+ C \sup_{\substack{t_0 \in Z \\ t_0 < 0}} 2^{-t_0 \lambda p_1} \left(\sum_{t=-\infty}^{t_0} 2^{t\beta(0)p_1} \left(\sum_{j=t+2}^{\infty} \left\| H_{\alpha,d}^k(f)\chi_j \right\|_{L^{q_2}(R^n)} \right)^{p_1} \right)$$
$$\equiv C(E_{11} + E_{12} + E_{13}).$$

First we estimate E_{12}. Using [17, Propositions 2.9] and Proposition 2.4, we have

$$E_{12} = C \sup_{\substack{t_0 \in Z \\ t_0 < 0}} 2^{-t_0 \lambda p_1} \left(\sum_{t=-\infty}^{t_0} 2^{t\beta(0)p_1} \left(\sum_{j=t-1}^{t+1} \left\| H_{\alpha,d}^k(f)\chi_j \right\|_{L^{q_2}(R^n)} \right)^{p_1} \right)$$
$$\leq C \|d\|_{BMO(R^n)}^{k,p_1} \|f\|_{M\dot{K}_{p_1,q_1}^{\beta(\cdot),\lambda}(R^n)}^{p_1}.$$

For E_{11}. Note that when $x \in A_k$, $j \leq t-2$, and $y \in A_j$. Therefore, using the generalized Hölder's inequality (see Lemma 2.5), we have

$$\left| H_{\alpha,d}^k(f_j)(x)\chi_t(x) \right|$$
$$\leq C 2^{t(\alpha-n)} \|f_j\|_{L^{q_1}(R^n)} \left(\left| d(x) - d_{B_j} \right|^k \|\chi_j\|_{L^{q'_1(\cdot)}(R^n)} + \left\| \left(d - d_{B_j} \right)^k \chi_j \right\|_{L^{q'_1(\cdot)}(R^n)} \right) \chi_t(x).$$

Thus, from Lemma 2.3, and note that $\|\chi_i\|_{L^{p(\cdot)}(R^n)} \leq \|\chi_{B_i}\|_{L^{p(\cdot)}(R^n)}$, it follows that

$$\left\| H_{\alpha,d}^k(f_j)\chi_t \right\|_{L^{q_2}(R^n)}$$
$$\leq C 2^{t(\alpha-n)} (t-j)^k \|d\|_{BMO(R^n)}^k \|f_j\|_{L^{q_1}(R^n)} \left\| \chi_{B_j} \right\|_{L^{q'_1(\cdot)}(R^n)} \|\chi_{B_t}\|_{L^{q_2(\cdot)}(R^n)} \tag{3}$$

Note that $\chi_{B_t}(x) \leq C 2^{-t\alpha} H_\alpha(\chi_{B_t})(x)$ (see [19]), by Proposition 2.2 and [17, Lemma 2.12], we obtain

$$\|\chi_{B_t}\|_{L^{q_2(\cdot)}(R^n)} \leq C 2^{-t\alpha} \|\chi_{B_t}\|_{L^{q_1(\cdot)}(R^n)} \tag{4}$$

Using [17, Lemma 2.12], [17, Remark 2.13] and (4), we have

$$2^{t(\alpha-n)} \left\| \chi_{B_j} \right\|_{L^{q'_1(\cdot)}(R^n)} \|\chi_{B_t}\|_{L^{q_2(\cdot)}(R^n)} \leq C \left\| \chi_{B_j} \right\|_{L^{q'_1(\cdot)}(R^n)} \|\chi_{B_t}\|_{L^{q'_1(\cdot)}(R^n)}^{-1}$$
$$= C \frac{\left\| \chi_{B_j} \right\|_{L^{q'_1(\cdot)}(R^n)}}{\|\chi_{B_t}\|_{L^{q'_1(\cdot)}(R^n)}} \leq C 2^{(j-t)n\delta_1}. \tag{5}$$

On the other hand, note the following fact ($\tilde{t}_j$, (j= 1, 2, 3) come from Proposition 2.4)

i. ($\tilde{t}_j < 0$, (j = 1, 2, 3)

$$\|f_j\|_{L^{q_1}(R^n)} \leq C 2^{j(\lambda - \beta(0))} \|f\|_{M\dot{K}_{p_1,q_1}^{\beta(\cdot),\lambda}(R^n)} \tag{6}$$

ii. $(\tilde{t}_j \geq 0$, (j = 1, 2, 3)

$$\|f_j\|_{L^{q_1}(R^n)} \leq C2^{j(\lambda-\beta_\infty)}\|f\|_{M\dot{K}^{\beta(.),\lambda}_{p_1,q_1}(R^n)} \tag{7}$$

Definition 2.1, Proposition 2.4 and the condition of $\beta(.)$, are used in above facts. Thus, combining (3), (5) and (6), and using $\beta(0) < \beta_\infty < \lambda + n\delta_1$, it follows that

$$E_{11} = C \sup_{\substack{t_0\in Z\\ t_0<0}} 2^{-t_0\lambda p_1}\left(\sum_{t=-\infty}^{t_0} 2^{t\beta(0)p_1}\left(\sum_{j=-\infty}^{t-2}\|H^k_{\alpha,d}(f)\chi_t\|_{L^{q_2}(R^n)}\right)^{p_1}\right) \leq C\|d\|^{k,p_1}_{BMO(R^n)}\|f\|^{p_1}_{M\dot{K}^{\beta(.),\lambda}_{p_1,q_1}(R^n)}$$

Now, let us turn to estimate for E_{13}. Note that when $x \in A_k$, $j \geq t+2$, and $y \in A_j$. Therefore, using the generalized Hölder's inequality (see Lemma 2.5), we have

$$|H^k_{\alpha,d}(f_j)(x)\chi_t(x)| \leq C2^{j(\alpha-n)}\|f_j\|_{L^{q_1}(R^n)}\left(|d(x)-d_{B_t}|^k\|\chi_j\|_{L^{q'_1(.)}(R^n)} + \left\|(d-d_{B_t})^k\chi_j\right\|_{L^{q'_1(.)}(R^n)}\right)\chi_t(x).$$

Using Lemma 2.3, it follows that

$$\|H^k_{\alpha,d}(f_j)\chi_t\|_{L^{q_2}(R^n)} \leq C2^{j(\alpha-n)}(j-t)^k\|d\|^k_{BMO(R^n)}\|f_j\|_{L^{q_1}(R^n)}\left\|\chi_{B_j}\right\|_{L^{q'_1(.)}(R^n)}\|\chi_{B_t}\|_{L^{q_2(.)}(R^n)}. \tag{8}$$

Note that $\chi_{B_j}(x) \leq C2^{-j\alpha}H_\alpha(\chi_{B_j})(x)$ (see [19]), by Proposition 2.2 and [17, Lemma 2.12], we obtain

$$\left\|\chi_{B_j}\right\|_{L^{q_2(.)}(R^n)} \leq C2^{-j(\alpha-n)}\left\|\chi_{B_j}\right\|^{-1}_{L^{q'_1(.)}(R^n)}$$

Thus, we have

$$2^{j(\alpha-n)}\left\|\chi_{B_j}\right\|_{L^{q'_1(.)}(R^n)} \leq C\left\|\chi_{B_j}\right\|^{-1}_{L^{q_2(.)}(R^n)} \tag{9}$$

Using [17, Lemma 2.12; Remark 2.13] and (9), we have

$$2^{j(\alpha-n)}\left\|\chi_{B_j}\right\|_{L^{q'_1(.)}(R^n)}\|\chi_{B_t}\|_{L^{q_2(.)}(R^n)} \leq C\left\|\chi_{B_j}\right\|^{-1}_{L^{q_2(.)}(R^n)}\|\chi_{B_t}\|_{L^{q_2(.)}(R^n)} \leq C\frac{\|\chi_{B_t}\|_{L^{q_2(.)}(R^n)}}{\left\|\chi_{B_j}\right\|_{L^{q_2(.)}(R^n)}} \leq C2^{(t-j)n\delta_2} \tag{10}$$

Thus, combining (6), (8) and (10), and using $\lambda - n\delta_2 < \beta(0) < \beta_\infty$, it follows that

$$E_{13} = \sup_{\substack{t_0\in Z\\ t_0<0}} 2^{-t_0\lambda p_1}\left(\sum_{t=-\infty}^{t_0} 2^{t\beta(0)p_1}\left(\sum_{j=t+2}^{\infty}\|H^k_{\alpha,d}(f)\chi_t\|_{L^{q_2}(R^n)}\right)^{p_1}\right) \leq C\|d\|^{k,p_1}_{BMO(R^n)}\|f\|^{p_1}_{M\dot{K}^{\beta(.),\lambda}_{p_1,q_1}(R^n)}.$$

Combining the estimates for E_{11}, E_{12} and E_{13} yields

$$E_1 \leq C\|d\|^{k,p_1}_{BMO(R^n)}\|f\|^{p_1}_{M\dot{K}^{\beta(.),\lambda}_{p_1,q_1}(R^n)}.$$

For E_3, similar to the estimate of E_1, using Propositions 2.2-2.4, Lemmas 2.3–2.5, [17, Lemma 2.12; Remark 2.13], (2)—(5), (7)—(10), we have

$$E_3 \leq C\|d\|^{k,p_1}_{BMO(R^n)}\|f\|^{p_1}_{M\dot{K}^{\beta(.),\lambda}_{p_1,q_1}(R^n)}.$$

Joint the estimates for E_1, E_2 and E_3, we obtain

$$\|H^k_{\alpha,d}(f)\|_{M\dot{K}^{\beta(.),\lambda}_{p_2,q_2}(R^n)} \leq C\|d\|^k_{BMO(R^n)}\|f\|_{M\dot{K}^{\beta(.),\lambda}_{p_1,q_1}(R^n)}.$$

This finishes the proof of Theorem 3.1.

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REFERENCES

[1] Ruzicka, M., (2000) Electrorheological Fluids: Modeling and Mathematical Theory. *Lecture Notes in Mathematics*, vol. 1748. Springer, New York.

[2] Diening, L., Ruzicka, M., (2003), Calder´on–Zygmund operators on generalized Lebesgue spaces Lp(·) and problems related to fluid dynamics. *Journal fur die reine und angewandte Mathematic* 563, 197–220.

[3] Levine, S., Chen, Y., Stanich, J.,(2004), Image restoration via nonstandard diffusion. *Technical Report*: 04-01, Department of Mathematics and Computer Science, Duquesne University.

[4] Chen, Y., Levine, S., Rao, M.,(2006), Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.* 66(4), 1383–1406.

[5] Li, F., Li, Z., Pi, L.,(2010), Variable exponent functionals in image restoration. Appl. Math. Comput. 216(3), 870–882.

[6] Harjulehto, P., H¨ast¨o, P., Latvala, V., et al.,(2013), Critical variable exponent functional in image restoration. *Appl. Math. Lett.* 26(1), 56–60.

[7] Sakharkar, M., Sarde, M.,(2014), Image denoising based on variable exponent using adaptive Perona Malik model. *Int. J. Eng. Sci. Invent*. 3(6), 34–37.

[8] Kovacik, O., Rakosnık, J.,(1991), On spaces Lp(x) and Wk,p(x). *Czechoslov. Math*. 41(116), 592–618.

[9] Izuki, M.,(2010), Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization. *Anal. Math*. 36, 33–5.

[10] Izuki, M.,(2009), Herz and amalgam spaces with variable exponent, the Haar wavelets and greediness of the wavelet system. *East J. Approx*. 15(1), 87–109.

[11] Almeida, A., Drihem, D.,(2012), Maximal, potential and singular type operators on Herz spaces with variable exponents. *J. Math. Anal. Appl*. 39(2), 781–795.

[12] Samko, S.,(2013),Variable exponent Herz spaces. *Mediterr. J. Math*. 10(4), 2007–2025.

[13] Samko, S.,(2013), Erratum to "Variable exponent Herz spaces". *Mediterr. J. Math*. doi:10.1007/s00009-013-0285-X, 2013. *Mediterr. J. Math*. 10(4), 2027–2030.

[14] Wu, J.,(2014), Boundedness of some sublinear operators on Herz–Morrey spaces with variable exponent. *Georgian Math. J*. 21(1), 101–111.

[15] Wu, J.,(2014), Boundedness for commutators of fractional integrals on Herz–Morrey spaces with variable exponent. *Kyoto J. Math*. 54(3), 483–495.

[16] Izuki, M.,(2010), Commutators of fractional integrals on Lebesgue and Herz spaces with variable exponent. *Rendiconti del Circolo Matematico di Palermo* 59,461–472.

[17] Wu, J and Zhao, W.,(2017), Boundedness for Higher Order Commutatorsof Fractional Integrals on Variable Exponent Herz–Morrey Spaces, *Mediterr. J. Math*. 14:198.

[18] Wu, J.L.,(2014), Boundedness for fractional Hardy-type operator on Herz–Morrey spaces with variable exponent. Bull. *Korean Math. Soc*. 51(2), 423–435.

[19] Izuki, M.,(2010), Fractional integrals on Herz–Morrey spaces with variable exponent. *Hiroshima Math. J*. 40, 343–355.

[20] Akin, L.,(2018), A Characterization of Approximation of Hardy Operators in VLS, *Celal Bayar University Journal of Science* Volume 14, Issue 3, p 333-336.

[21] Akin, L.,(2019), Compactness of Fractional Maximal Operator in Weighted and Variable Exponent Spaces, *Erzincan University, Journal of Science and Technology*, Volume12, issue:1, pp:185-190.
[22] Mamedov, F., Zeren, Y. and Akin, L.,(2017), Compactification of Weighted Hardy Operator in Variable Exponent Lebesgues Spaces, *Asian Journal of Mathematics and Computer Research* 17(1): 38-47.
[23] Akin, L.,(2018), On Two Weight Criterions for The Hardy Littlewood Maximal Operator in BFS, *Asian Journal of Science and Technology* Vol. 09, Issue, 05, pp.8085-8089, May.
[24] Akin, L.,(2019), On Some Properties of Integral-Type Operator in Weighted Herz Spaces with Variable Exponent Lebesgue Spaces, *Universal Journal of Mathematics and Applications*, 2 (3), 148-151.
[25] Pişkin, E.,(2018) Finite time blow up of solutions for a strongly damped nonlinear Klein-Gordon equation with variable exponents, *Honam Mathematical J.,* 40(4), 771-783.