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Research Article

CUBIC PICTURE FUZZY SETS AND THEIR APPLICATION TO A PETROLEUM CIRCULATION CENTER EVALUATION PROBLEM

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ABSTRACT

The objective of this paper is to present novel concept of cubic picture fuzzy set to deals with uncertainties in decision making problems. In this paper, we propose the internal (external) cubic picture sets, define union and intersection under P-order & R-order and deliver some examples to support of established P-order & R-order union and intersection of internal (external) cubic picture sets. Further, we propose the series of aggregation operators to aggregate the cubic picture fuzzy information's. Then, by utilizing these operators, we develop an approach for solving the multicriteria decision-making problem and illustrate it with a numerical example of a petroleum circulation center evaluation problem to demonstrate the usage and applicability of the proposed ranking approach.
Keywords: Cubic Picture fuzzy set, P-order union and intersection, R-order union and intersection, Cubic picture fuzzy aggregation operators, decision making problem.

1. INTRODUCTION

Since, in 1965 Zadeh [44] established fuzzy set theory, it has become an essential tool to grip inaccurate and vagueness material in different areas of prevailing civilization. Such inaccuracies are associated with the membership function that belongs to [0,1]. Through membership function, we obtain information which makes possible for us to reach the conclusion. The fuzzy set theory has become a strong area of making observations in different areas like medical science, social sciences, engineering, management sciences, artificial intelligence, robotics, computer networks, decision making and so on. Due to unassociated sorts of unpredictably occurring in different areas of life like economics, engineering, medical sciences, management sciences, psychology, sociology, decision making, and fuzzy set as noted and often effective mathematical instruments have been offered to make, be moving in and grip those unpredictably.

Since the establishment of fuzzy set, several extensions have been made such as Atanassov's [6] work on intuitionistic fuzzy set (IFSs) was quite remarkable as he extended the concept of FSs by assigning non-membership degree say $N(x)$ along with membership degree say $P(x)$ with

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condition that $0 \le P(x) + N(x) \le 1$. Form last few decades, the IFS has been explored by many researchers and successfully applied to many practical fields like medical diagnosis, clustering analysis, decision making pattern recognition [4,5,7,8,12,13,33,38,39, 46,47]. Kumar and Garg [25] proposed the TOPSIS approach for intuitionistic fuzzy information. Garg and Kumar [15] proposed the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory and discussed their application in decision making problems and Garg [16] proposed some robust improved geometri c aggregation operators under interval-valued intuitionistic fuzzy environment. Strengthening the concept IFS Yager suggest Pythagorean fuzzy sets [43] which somehow enlarge the space of positive membership and negative membership by introducing some new condition that $0 \le P^2(x) + N^2(x) \le 1$.

Atanassov's [4,6] structure discourses only satisfaction and dissatisfaction degree of elements in a set which is quite insufficient as human nature has some sort of abstain and refusal issues too. Such hitches were considered by Cuong [11] and he proposed picture fuzzy sets (PFS) of the form $(P(x), I(x), N(x))$ where the elements in triplet represent satisfaction, abstain and dissatisfaction degrees respectively, under the condition that $0 \le P(x) + I(x) + N(x) \le 1$ and with refusal degree defined as $1-(P(x), I(x), N(x))$. This structure of Cuong [10] is considerably closer to human nature than that of earlier concepts and is one of the richest research area now. Singh [36] investigated the correlation coefficients for picture fuzzy set and apply the correlation coefficient to clustering analysis with picture fuzzy information. Son [37] introduce several novel fuzzy clustering algorithms on the basis of picture fuzzy sets and applications to time series forecasting and weather forecasting. Ashraf et al. [1] introduced the geometric aggregation operators for picture fuzzy information and illustrate the decision-making problems to shows the effectiveness of the proposed operators. Zeng et al. [45] proposed exponential Jensen picture fuzzy divergence measure and utilized multi-criteria group decision making problems as an application of the proposed measures. Garg [18] introduced picture fuzzy weighted averaging operator, Picture fuzzy ordered weighted averaging operator, Picture fuzzy hybrid averaging operator under picture fuzzy environment. Wei [41] introduce the cosine similarity measures for picture fuzzy information. Wang et al. [40] established the VIKOR method for picture fuzzy information's. Wei [42] proposed the picture fuzzy aggregation operators to aggregate the picture fuzzy information's. Cuong [10] proposed the picture fuzzy norm and picture fuzzy t-conorm and discussed their properties. Khan [29] proposed the generalized picture fuzzy soft sets and discussed their applications in decision making.

In 2012, Jun et al. [19] introduced cubic sets was quite remarkable, and then this notion is applied to several algebraic structures see [20-23]. Jun et al. establish cubic set with their operational properties and applied to BCK/BCI-algebras. After that Garg and Kaur [17] proposed the novel distance measures for cubic intuitionistic fuzzy sets. Kaur and Garg [26,27] proposed the novel concept of generalized cubic intuitionistic fuzzy set and introduced aggregation operators using t-norm operations and discussed their applications to group decision-making process and Kaur and Garg [28] proposed the Bonferroni mean operators under cubic intuitionistic fuzzy set environment.

After the motivation of the above discussion, Ashraf [2] extend the structure of cubic sets to the picture fuzzy sets and proposed the concept of cubic picture fuzzy set. This paper is also continuation of cubic picture fuzzy sets. In this paper, we establish the concepts of positive-internal (neutral-internal, negative-internal) cubic picture fuzzy sets and positive-external (neutral -external, negative-external) cubic picture fuzzy sets, and explore associated properties. We illustrate that the P-order union and intersection of positive-internal (neutral-internal, negative-internal) cubic picture fuzzy sets are also positive-internal (neutral-internal, negative-internal) cubic picture fuzzy sets. We deliver examples to show that the P-order union and intersection of positive-external (neutral-external, negative-external) cubic picture fuzzy sets may not be positive-external (neutral-external, negative-external) cubic picture fuzzy sets, and the R- order union and intersection of positive-internal (neutral-internal, negative-internal) cubic picture fuzzy sets may not be positive-internal (neutral-internal, negative-internal) cubic picture fuzzy sets. By providing some more conditio ns for the R- order union of two positive-internal (resp. neutral-internal and negative-internal) cubic picture fuzzy sets to be a positive-internal (resp. neutral-internal and negative-internal) cubic picture fuzzy set. Since in decision making aggregation operators play a vital role, therefore in this paper we introduced some aggregation operators based on the of the idea of t-norms and t-conorms, namely the Cubic picture fuzzy weighted average (CPFWA) operator, Cubic picture fuzzy order weighted average (CPFOWA) operator and Cubic picture fuzzy hybrid weighted average (CPFHWA) operator and established the comparison technique for ranking the alternatives.

The objectives of this paper are: (1) to introduce the Cubic picture fuzzy set (2) to define the cubic picture fuzzy numbers (CPFNs) and related basic operational identities, (3) to suggest score and accuracy functions for comparison, (4) to proposed cubic picture fuzzy aggregation operators and some debate on their operational rules, (5) to demonstrate a MADM method based on these aggregation operators under cubic picture fuzzy information,

The superfluity of this paper is planned as follows. Section "Preliminaries" gives brief reassess the initial ideas related to Intuitionistic fuzzy sets and PFSs and their properties. In next section "Cubic picture fuzzy sets and their Operations " give complete details about CPFSs and operational properties. In sections "Cubic Picture Fuzzy Weighted Averaging Aggregation Operators" proposed the aggregation operators for cubic picture fuzzy sets. In sections "MADM Method Utilizing Cubic Picture Fuzzy Aggregation Operators", Finally, based on these operators, a decision-making method has been established for ranking the alternatives by utilizing cubic picture fuzzy environment. The suggested technique has been demonstrated with an application to a petroleum circulation center evaluation problem for viewing their effectiveness as well as reliability.

2. PRELIMINARIES

The paper gives brief discussion on basic ideas associated to FS, IFS, PFS and Cubic set with their operations and operators utilizing triangular norm and conorm. Also discuss, more familiarized ideas which utilized in following analysis.

Definition 2.1. [10] Let $\psi = [0,1]$ be an unite interval. Then the mapping $T : \psi \times \psi \to \psi$ is said to be t-norm if for $x, y, z \in \psi$

- T is commutative, monotonic and associative
- $T(x,1) = x.$

Definition 2.2. [10] Let $\psi = [0,1]$ be an unite interval. Then the mapping $S : \psi \times \psi \to \psi$ is said to be t-conorm if for $x, y, z \in \psi$

- S is commutative, monotonic and associative
- $S(x,0) = x.$

Now we enlist some special types of t-norms and t-conorms [10] based on which most of the operations for FSs, IFSs, PFSs were defined. These t-norms and conorms provide a base for operations of CPFSs too

T-Norm $(T(x,y))$	T-Conorm $(S(x,y))$	proposed by
$\min(x,y)$	$\max(x,y)$	Zadeh
$x.y$	$x+y-x\cdot y$	Goguen and Bandler
$(x_1\wedge y_1, x_2\wedge y_2, x_3\vee y_3)$	$(x_1\vee y_1, x_2\wedge y_2, x_3\wedge y_3)$	B. C. Cuong
$(x_1\wedge y_1, x_2\cdot y_2, x_3\vee y_3)$	$(x_1\vee y_1, x_2\cdot y_2, x_3\wedge y_3)$	B. C. Cuong
$(x_1\cdot y_1, x_2\cdot y_2, x_3+y_3-x_3\cdot y_3)$	$(x_1+y_1-x_1\cdot y_1, x_2\cdot y_2, x_3\cdot y_3)$	B. C. Cuong

Definition 2.3. [1,10] A t-norm is an Archimedean triangular norm if is holds

(1) continuous
(2) $T(a,a)<a$ for all $a\in(0,1)$.

and called strictly Archimedean t-norm if additionally, is strictly increasing for every variable $a,b\in(0,1)$.

Definition 2.4. [1,10] A t-conorm is an Archimedean triangular conorm if is holds

(1) continuous
(2) $S(a,a)>a$ for all $a\in(0,1)$.

and called strictly Archimedean t-conorm if additionally, is strictly increasing for every variable $a,b\in(0,1)$.

There are some established strict Archimedean t-norm [31,32] that illustrate with its additive generator t as $T(a,b)=t^{-1}(t(a)+t(b))$, and same as for $S(a,b)=s^{-1}(s(a)+s(b))$ with $s(x)=t(1-x)$.

There are some established strict Archimedean t-norms [8] are,

Name	T. Norms	Additive Generators
Algebraic	$T_A(x,y)=xy$	$t(r_1)=-\log r_1$
Einstein	$T_E(x,y)=\frac{xy}{1+(1-x)(1-y)}$	$t(r_1)=\log\frac{2-r_1}{r_1}$

There are some established strict Archimedean t-conorms [8] are,

Name	T. Conorms	Additive Generators
Algebraic	$S_A(x,y)=x+y-xy$	$s(r_1)=-\log(1-r_1)$
Einstein	$S_E(x,y)=\frac{x+y}{1+xy}$	$s(r_1)=\log\frac{1+r_1}{1-r_1}$

Definition 2.5. [44] Let the universe set be $R\neq\varphi$. Then

$$A=\{\langle r, P_A(r) \mid r\in R\rangle\},$$

is said to be a fuzzy set of R, where $P_A : R\to[0,1]$ is said to be the membership degree of r in R.

Definition 2.6. [47] Let Ω be the collection of all closed subintervals of $[0,1]$ and $P_A=[P_{LA},P_{UA}]\in\Omega,$ where P_{LA} and P_{UA} are the lower extreme and the upper extreme, respectively. Let the universe set be $R\neq\varphi$. Then

$$A=\{\langle r, \pounds_{P_A}(r) \mid r\in R\rangle\},$$

is said to be an interval valued fuzzy set of $R,$ where $£_{P_A} : R \to \Omega$ is said to be the membership degree of r in R and $£_{P_A} = \left[£_{P_{LA}}, £_{P_{UA}} \right]$ is said to be an interval valued fuzzy number.

Definition 2.7. [6] Let the universe set be $R \neq \varphi.$ Then

$$A = \left\{ \left\langle r, P_A(r), N_A(r) \mid r \in R \right\rangle \right\},$$

is said to be an intuitionistic fuzzy set of R, where $P_A : R \to [0,1]$ and $N_A : R \to [0,1]$ are said to be the degree of positive membership of r in R and the negative membership degree of r in R respectively. Also P_A and N_A satisfy the following condition:

$$(\forall r \in R)\ (0 \leq P_A(r) + N_A(r) \leq 1).$$

Definition 2.8. [11] Let the universe set be $R \neq \varphi$. Then the set

$$A = \left\{ \left\langle r, P_A(r), I_A(r), N_A(r) \mid r \in R \right\rangle \right\},$$

is said to be a picture fuzzy set of R, where $P_A : R \to [0,1]$, $I_A : R \to [0,1]$ and $N_A : R \to [0,1]$ are said to be the degree of positive-membership of r in $R,$ the neutral-membership degree of r in R and the negative-membership degree of r in R respectively. Also P_A, I_A and N_A satisfy the following condition:

$$(\forall r \in R)\ \ (0 \leq P_A(r) + I_A(r) + N_A(r) \leq 1).$$

Definition 2.9. [11] Let the universe set be $R \neq \varphi.$ Then the set

$$A = \left\{ \left\langle r, £_{P_A}(r), £_{I_A}(r), £_{N_A}(r) \mid r \in R \right\rangle \right\},$$

is said to be an interval valued picture fuzzy set of $R,$ where $£_{P_A} : R \to \Omega$, $£_{I_A} : R \to \Omega$ and $£_{N_A} : R \to \Omega$ are said to be the degree of positive membership of r in R, the neutral membership degree of r in R and the negative membership degree of r in R respectively. Also $£_{P_A}$, $£_{I_A}$ and $£_{N_A}$ satisfy the following condition:

$$(\forall r \in R)\ \ (0 \leq Sup\left(£_{P_A}(r)\right) + Sup\left(£_{I_A}(r)\right) + Sup\left(£_{N_A}(r)\right) \leq 1).$$

Definition 2.10. [19] Let the universe set be $R \neq \varphi.$ Then the set

$$A = \left\{ \left\langle r, £_A(r), e_A(r) \mid r \in R \right\rangle \right\},$$

is said to be a cubic set of $R,$ where $£_A(r)$ is an interval-valued fuzzy set in R and $e_A(r)$ is a fuzzy set in $R.$ For simplicity we denoted the cubic set as, $A = \left\langle £_A, e_A \right\rangle.$

Definition 2.11. [19] Let the universe set be $R \neq \varphi.$ Then the cubic set $A = \left\langle £_A, e_A \right\rangle$ is said to be an internal cubic set. if,

$$£_A^-(r) \leq e_A(r) \leq £_A^+(r) \text{ for all } r \in R.$$

Definition 2.12. [19] Let the universe set be $R \neq \varphi.$ Then the cubic set $A = \left\langle £_A, e_A \right\rangle$ is said to be an external cubic set. if,

$$e_A(r) \notin \left(£_A^-(r), £_A^+(r)\right) \text{ for all } r \in R.$$

Example 2.13. Let the universe set be $R \neq \varphi.$ Then $A = \{\langle r, £_A(r), e_A(r) \mid r \in R \rangle\},$ be a cubic set of $R.$ If $£_A(r) = [0.3, 0.6]$ and $e_A(r) = 0.5$ for all $r \in R,$ Then the set A is called an internal cubic set. If $£_A(r) = [0.3, 0.6]$ and $e_A(r) = 0.8$ for all $r \in R,$ Then the set A is called an external cubic set. If $£_A(r) = [0.3, 0.6]$ and $e_A(r) = r$ for all $r \in R,$ Then the set A do not belong to both classes of an internal and external cubic sets.

3. CUBIC PICTURE FUZZY SETS AND THEIR OPERATIONS

Definition 3.1. [2] Let the universe set be $R \neq \varphi.$ Then the set

$$A = \{\langle r, £_A(r), e_A(r) \mid r \in R \rangle\},$$

is said to be a cubic picture fuzzy set of $R,$ where $£_A(r) = \{\langle r, £_{P_A}(r), £_{I_A}(r), £_{N_A}(r) \mid r \in R \rangle\}$ is an interval-valued picture fuzzy set in R and $e_A(r) = \{\langle r, P_A(r), I_A(r), N_A(r) \mid r \in R \rangle\}$ is a picture fuzzy set in $R.$ For simplicity we denoted the cubic picture fuzzy set as, $A = \langle £_A, e_A \rangle.$

Definition 3.2. Let the universe set be $R \neq \varphi.$ Then the cubic picture fuzzy set $A = \langle £_A, e_A \rangle$ in R is said to be:

(1) Positive-internal if $£_{P_A}^-(r) \leq P_A(r) \leq £_{P_A}^+(r), \ \forall \ r \in R.$

(2) Neutral-internal if $£_{I_A}^-(r) \leq I_A(r) \leq £_{I_A}^+(r), \ \forall \ r \in R.$

(3) Negative-internal if $£_{N_A}^-(r) \leq N_A(r) \leq £_{N_A}^+(r), \ \forall \ r \in R.$

If the cubic picture fuzzy set $A = \langle £_A, e_A \rangle$ in R is satisfies all the above properties, then the cubic picture fuzzy set is said to be an internal cubic picture fuzzy set in $R.$

Example 3.3. Let the universe set be $R = \{r_1, r_2, r_3\}.$ Then the pair $A = \langle £_A, e_A \rangle$ is said to be an internal cubic picture fuzzy set with the tabular representation as.

Table 1. Internal cubic picture fuzzy set $A = \langle £_A, e_A \rangle$

R	$£_A$	e_A
r_1	$([0.20, 0.33], [0.05, 0.26], [0.39, 0.41])$	$(0.25, 0.15, 0.40)$
r_2	$([0.11, 0.30], [0.30, 0.40], [0.15, 0.25])$	$(0.23, 0.39, 0.24)$
r_3	$([0.33, 0.43], [0.15, 0.28], [0.27, 0.29])$	$(0.42, 0.27, 0.28)$

Definition 3.4. Let the universe set be $R \neq \varphi.$ Then the cubic picture fuzzy set $A = \langle £_A, e_A \rangle$ in R is said to be:

(1) Positive-external if $P_A(r) \notin \left(£_{P_A}^-(r), £_{P_A}^+(r)\right), \ \forall \ r \in R.$

(2) Neutral-external if $I_A(r) \notin \left(£_{I_A}^-(r), £_{I_A}^+(r)\right), \ \forall \ r \in R.$

(3) Negative-external if $N_A(r) \notin \left(£^-_{N_A}(r), £^+_{N_A}(r)\right),\ \forall\ r \in R.$

If the cubic picture fuzzy set $A = \langle £_A, e_A \rangle$ in R is satisfies all the above properties, then the cubic picture fuzzy set is said to be an external cubic picture fuzzy set in R.

Proposition 3.5. Let the universe set be $R \neq \varphi$ and the pair $A = \langle £_A, e_A \rangle$ be a cubic picture fuzzy set in R which is not external. Then there exists $r \in R$ such that, $P_A(r) \in \left(£^-_{P_A}(r), £^+_{P_A}(r)\right)$, $I_A(r) \in \left(£^-_{I_A}(r), £^+_{I_A}(r)\right)$ and $N_A(r) \in \left(£^-_{N_A}(r), £^+_{N_A}(r)\right)$.

Proof: Straightforward.

Proposition 3.6. Let the universe set be $R \neq \varphi$ and the pair $A = \langle £_A, e_A \rangle$ be cubic picture fuzzy set in R. If the pair $A = \langle £_A, e_A \rangle$ be both positive internal and external, then $\forall\ r \in R$

$$P_A(r) \in \left\{£^-_{P_A}(r) \mid r \in R\right\} \cup \left\{£^+_{P_A}(r) \mid r \in R\right\}.$$

Proof: By utilizing the Definition 3.2, we have $£^-_{P_A}(r) \leq P_A(r) \leq £^+_{P_A}(r),\ \forall\ r \in R$ and utilizing the Definition 3.4, we have $P_A(r) \notin \left(£^-_{P_A}(r), £^+_{P_A}(r)\right),\ \forall\ r \in R$

It follows that $P_A(r) = £^-_{P_A}(r)$ or $P_A(r) = £^+_{P_A}(r)$ and so that $P_A(r) \in \left\{£^-_{P_A}(r) \mid r \in R\right\} \cup \left\{£^+_{P_A}(r) \mid r \in R\right\}$.

Similarly, we have following proposition for other membership degrees.

Proposition 3.6. Let the universe set be $R \neq \varphi$ and the pair $A = \langle £_A, e_A \rangle$ be a cubic picture fuzzy set in R. If the pair $A = \langle £_A, e_A \rangle$ be both neutral internal and external, then $\forall r \in R$

$$I_A(r) \in \left\{£^-_{I_A}(r) \mid r \in R\right\} \cup \left\{£^+_{I_A}(r) \mid r \in R\right\}.$$

Proposition 3.7. Let the universe set be $R \neq \varphi$ and the pair $A = \langle £_A, e_A \rangle$ be a cubic picture fuzzy set in R. If the pair $A = \langle £_A, e_A \rangle$ be both negative internal and external, then $\forall r \in R$

$$N_A(r) \in \left\{£^-_{N_A}(r) \mid r \in R\right\} \cup \left\{£^+_{N_A}(r) \mid r \in R\right\}.$$

Now we define some operations on the cubic picture fuzzy sets and properties of defined operations are also examined. These operations will be of great use like in defining aggregation operators for cubic picture fuzzy sets.

In our further discussion R plays the role of universal set and three pairs $A = \langle £_A, e_A \rangle, B = \langle £_B, e_B \rangle$ and $C = \langle £_C, e_C \rangle$ are the cubic picture fuzzy sets of the form $A = \langle £_A, e_A \rangle = \left\{r, \langle £_{P_A}(r), £_{I_A}(r), £_{N_A}(r) \rangle, \langle P_A(r), I_A(r), N_A(r) \rangle \mid r \in R\right\}$, with the condition that $0 \leq Sup\left[£^-_{P_A}(r), £^+_{P_A}(r)\right] + Sup\left[£^-_{I_A}(r), £^+_{I_A}(r)\right] + Sup\left[£^-_{N_A}(r), £^+_{N_A}(r)\right] \leq 1$ and $0 \leq P_A(r) + I_A(r) + N_A(r) \leq 1.$

$B = \langle £_B, e_B \rangle = \left\{r, \langle £_{P_B}(r), £_{I_B}(r), £_{N_B}(r) \rangle, \langle P_B(r), I_B(r), N_N(r) \rangle \mid r \in R\right\}$, with the condition that

$0 \leq Sup\left[£^{-}_{P_B}(r), £^{+}_{P_B}(r)\right] + Sup\left[£^{-}_{I_B}(r), £^{+}_{I_B}(r)\right] + Sup\left[£^{-}_{N_B}(r), £^{+}_{N_B}(r)\right] \leq 1$ and $0 \leq P_B(r) + I_B(r) + N_B(r) \leq 1.$

And $C = \langle £_C, e_C \rangle = \left\{r, \langle £_{P_C}(r), £_{I_C}(r), £_{N_C}(r) \rangle, \langle P_C(r), I_C(r), N_C(r) \rangle \mid r \in R\right\},$ with the condition that $0 \leq Sup\left[£^{-}_{P_C}(r), £^{+}_{P_C}(r)\right] + Sup\left[£^{-}_{I_C}(r), £^{+}_{I_C}(r)\right] + Sup\left[£^{-}_{N_C}(r), £^{+}_{N_C}(r)\right] \leq 1$ and $0 \leq P_C(r) + I_C(r) + N_C(r) \leq 1.$

Definition 3.8. Let for two cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle,$ we define equality as

$$A = B \Leftrightarrow £_A = £_B \text{ and } e_A = e_B.$$

Definition 3.9. Let for two cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle,$ we define P-order as

$$A \subseteq_P B \Leftrightarrow £_A \subseteq £_B \text{ and } e_A \leq e_B.$$

Definition 3.10. Let for two cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle,$ we define R-order as

$$A \subseteq_R B \Leftrightarrow £_A \subseteq £_B \text{ and } e_A \geq e_B.$$

Definition 3.11. Let $A_j = \left\langle £_{A_j}, e_{A_j} \right\rangle$ be collection of cubic picture fuzzy sets in $R \neq 0.$ Then we define P-order union as

$\overset{P}{\bigcup_{j\in I}} A_j = \left(\bigcup_{j\in I} £_{A_j}, \bigvee_{j\in I} e_{A_j} \right),$ where I be an index set and

$$\bigcup_{j\in I} £_{A_j} = \left\{ \left\langle r; \left(\bigcup_{j\in I} £_{P_{A_j}} \right)(r), \left(\bigcup_{j\in I} £_{I_{A_j}} \right)(r), \left(\bigcup_{j\in I} £_{N_{A_j}} \right)(r) \right\rangle \mid r \in R \right\},$$

$$\bigvee_{j\in I} e_{A_j} = \left\{ \left\langle r; \left(\bigvee_{j\in I} P_{A_j} \right)(r), \left(\bigvee_{j\in I} I_{A_j} \right)(r), \left(\bigvee_{j\in I} N_{A_j} \right)(r) \right\rangle \mid r \in R \right\}.$$

Definition 3.12. Let $A_j = \left\langle £_{A_j}, e_{A_j} \right\rangle$ be collection of cubic picture fuzzy sets in $R \neq 0.$ Then we define R-order union as $\overset{R}{\bigcup_{j\in I}} A_j = \left(\bigcup_{j\in I} £_{A_j}, \bigwedge_{j\in I} e_{A_j} \right),$ where

$$\bigcup_{j\in I} £_{A_j} = \left\{ \left\langle r; \left(\bigcup_{j\in I} £_{P_{A_j}} \right)(r), \left(\bigcup_{j\in I} £_{I_{A_j}} \right)(r), \left(\bigcup_{j\in I} £_{N_{A_j}} \right)(r) \right\rangle \mid r \in R \right\}, \text{ and}$$

$$\bigwedge_{j\in I} e_{A_j} = \left\{ \left\langle r; \left(\bigwedge_{j\in I} P_{A_j} \right)(r), \left(\bigwedge_{j\in I} I_{A_j} \right)(r), \left(\bigwedge_{j\in I} N_{A_j} \right)(r) \right\rangle \mid r \in R \right\}.$$

Definition 3.13. Let $A_j = \left\langle £_{A_j}, e_{A_j} \right\rangle$ be collection of cubic picture fuzzy sets in $R \neq 0.$ Then we define P-order intersection as $\overset{P}{\bigcap_{j\in I}} A_j = \left(\bigcap_{j\in I} £_{A_j}, \bigwedge_{j\in I} e_{A_j} \right),$ where

$$\bigcap_{j\in I} £_{A_j} = \left\{ \left\langle r; \left(\bigcap_{j\in I} £_{P_{A_j}} \right)(r), \left(\bigcap_{j\in I} £_{I_{A_j}} \right)(r), \left(\bigcap_{j\in I} £_{N_{A_j}} \right)(r) \right\rangle \mid r \in R \right\}, \text{ and}$$

$$\bigwedge_{j\in I} e_{A_j} = \left\{ \left\langle r; \left(\bigwedge_{j\in I} P_{A_j} \right)(r), \left(\bigwedge_{j\in I} I_{A_j} \right)(r), \left(\bigwedge_{j\in I} N_{A_j} \right)(r) \right\rangle \mid r \in R \right\}.$$

Definition 3.14. Let $A_j = \left\langle £_{A_j}, e_{A_j} \right\rangle$ be collection of cubic picture fuzzy sets in $R \neq 0.$ Then we define R-order intersection as $\overset{P}{\bigcap_{j\in I}} A_j = \left(\bigcap_{j\in I} £_{A_j}, \bigvee_{j\in I} e_{A_j} \right),$ where

$$\bigcap_{j\in I} £_{A_j} = \left\{ \left\langle r; \left(\bigcap_{j\in I} £_{P_{A_j}} \right)(r), \left(\bigcap_{j\in I} £_{I_{A_j}} \right)(r), \left(\bigcap_{j\in I} £_{N_{A_j}} \right)(r) \right\rangle \mid r \in R \right\}, \text{ and}$$

$$\bigvee_{j\in I} e_{A_j} = \left\{ \left\langle r; \left(\bigvee_{j\in I} P_{A_j} \right)(r), \left(\bigvee_{j\in I} I_{A_j} \right)(r), \left(\bigvee_{j\in I} N_{A_j} \right)(r) \right\rangle \mid r \in R \right\}.$$

The complement of $A = \langle £_A, e_A \rangle$ is define as $A^c = \left\langle £^c_A, e^c_A \right\rangle$ be also cubic picture fuzzy set, where $£^c_A = \left\{ \left\langle r, £_{P_A}(r), £_{I_A}(r), £_{N_A}(r) \right\rangle \mid r \in R \right\}$ be an complement of interval picture fuzzy set and $e^c_A = \left\{ \left\langle r, P^c_A(r), I^c_A(r), N^c_A(r) \right\rangle \mid r \in R \right\}$ be an complement of picture fuzzy set in $R.$ Obviously, $\left(A^c_j\right)^c = A_j,$ $\left(\overset{P}{\bigcup_{j\in I}} A_j \right)^c = \overset{P}{\bigcap_{j\in I}} A^c_j,$ $\left(\overset{P}{\bigcap_{j\in I}} A_j \right)^c = \overset{P}{\bigcup_{j\in I}} A^c_j,$ $\left(\overset{R}{\bigcup_{j\in I}} A_j \right)^c = \overset{R}{\bigcap_{j\in I}} A^c_j$ and $\left(\overset{R}{\bigcap_{j\in I}} A_j \right)^c = \overset{R}{\bigcup_{j\in I}} A^c_j.$

Proposition 3.15. Let for any cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$, $B = \langle £_B, e_B \rangle$ and $C = \langle £_C, e_C \rangle$ in $R.$ For P-order we have

(1) If $A \subseteq_p B$ and $B \subseteq_p C$ then $A \subseteq_p C.$

(2) If $A \subseteq_p B$ then $B^C \subseteq_p A^C.$

(3) If $A \subseteq_p B$ and $A \subseteq_p C$ then $A \subseteq_p B \overset{P}{\bigcap} C.$

(3) If $A \subseteq_p B$ and $C \subseteq_p B$ then $A \overset{P}{\bigcup} C \subseteq_P B.$

Proposition 3.16. Let for any cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$, $B = \langle £_B, e_B \rangle$ and $C = \langle £_C, e_C \rangle$ in $R.$ For R-order we have

(1) If $A \subseteq_R B$ and $B \subseteq_R C$ then $A \subseteq_R C.$

(2) If $A \subseteq_R B$ then $B^C \subseteq_R A^C.$

(3) If $A \subseteq_R B$ and $A \subseteq_R C$ then $A \subseteq_R B \overset{R}{\bigcap} C.$

(3) If $A \subseteq_R B$ and $C \subseteq_R B$ then $A \overset{R}{\bigcup} C \subseteq_R B.$

Theorem 3.17. Let the universe set be $R \neq \varphi$ and the pair $A = \langle £_A, e_A \rangle$ be a cubic picture fuzzy set in R. If the pair $A = \langle £_A, e_A \rangle$ be a positive internal (resp. positive external), then the complement $A^c = \langle £_A^c, e_A^c \rangle$ of $A = \langle £_A, e_A \rangle$ be a positive internal (resp. positive external) cubic picture fuzzy set in R.

Proof: If the pair $A = \langle £_A, e_A \rangle$ be a positive internal (resp. positive external) cubic picture fuzzy set in R. Then by the Definition 3.2, we have $£_{P_A}^-(r) \leq P_A(r) \leq £_{P_A}^+(r) \left(resp.,\ P_A(r) \notin \left(£_{P_A}^-(r), £_{P_A}^+(r) \right) \right), \forall r \in R$

this implies that $\forall r \in R$, $1 - £_{P_A}^+(r) \leq 1 - P_A(r) \leq 1 - £_{P_A}^-(r)$ $\left(resp.,\ 1 - P_A(r) \notin \left(1 - £_{P_A}^+(r), 1 - £_{P_A}^-(r) \right) \right)$.

Therefore $A^c = \langle £_A^c, e_A^c \rangle$ be a positive internal (resp. positive external) cubic picture fuzzy set in R.

Similarly,

Theorem 3.18. Let the universe set be $R \neq \varphi$ and the pair $A = \langle £_A, e_A \rangle$ be a cubic picture fuzzy set in R. If the pair $A = \langle £_A, e_A \rangle$ be an neutral internal (resp. neutral external), then the complement $A^c = \langle £_A^c, e_A^c \rangle$ of $A = \langle £_A, e_A \rangle$ be an neutral internal (resp. neutral external) cubic picture fuzzy set in R.

Theorem 3.19. Let the universe set be $R \neq \varphi$ and the pair $A = \langle £_A, e_A \rangle$ be a cubic picture fuzzy set in R. If the pair $A = \langle £_A, e_A \rangle$ be an negative internal (resp. negative external), then the complement $A^c = \langle £_A^c, e_A^c \rangle$ of $A = \langle £_A, e_A \rangle$ be an negative internal (resp. negative external) cubic picture fuzzy set in R.

Corollary 3.20. Let the universe set be $R \neq \varphi$ and the pair $A = \langle £_A, e_A \rangle$ be a cubic picture fuzzy set in R. If the pair $A = \langle £_A, e_A \rangle$ be an internal (resp., external), then the complement $A^c = \langle £_A^c, e_A^c \rangle$ of $A = \langle £_A, e_A \rangle$ be an internal (resp., external) cubic picture fuzzy set in R.

Theorem 3.21. Let $A_j = \langle £_{A_j}, e_{A_j} \rangle$ be the collection of positive internal cubic picture fuzzy sets in $R \neq 0$. Then the P-order union and intersection of $A_j = \langle £_{A_j}, e_{A_j} \rangle$ is also be a positive internal cubic picture fuzzy sets in R.

Proof: Since, $A_j = \langle £_{A_j}, e_{A_j} \rangle$ be collection of positive internal cubic picture fuzzy sets in R, we have by Definition 3.2, $£_{P_{A_j}}^-(r) \leq P_{A_j}(r) \leq £_{P_{A_j}}^+(r), \forall r \in R,\ j \in I$.

Then, it follows that

$\left(\bigcup_{j\in I} \text{£}_{P_{A_j}}\right)^{-}(r) \leq \left(\bigvee_{j\in I} P_{A_j}\right)(r) \leq \left(\bigcup_{j\in I} \text{£}_{P_{A_j}}\right)^{+}(r),$ and

$\left(\bigcap_{j\in I} \text{£}_{P_{A_j}}\right)^{-}(r) \leq \left(\bigwedge_{j\in I} P_{A_j}\right)(r) \leq \left(\bigcap_{j\in I} \text{£}_{P_{A_j}}\right)^{+}(r).$

Therefore

$\overset{P}{\bigcup_{j\in I}} A_j = \left(\bigcup_{j\in I} \text{£}_{A_j}, \bigvee_{j\in I} e_{A_j}\right)$ and $\overset{P}{\bigcap_{j\in I}} A_j = \left(\bigcap_{j\in I} \text{£}_{A_j}, \bigwedge_{j\in I} e_{A_j}\right)$ are positive internal cubic picture fuzzy sets in R.

Similarly,

Theorem 3.22. Let $A_j = \left\langle \text{£}_{A_j}, e_{A_j} \right\rangle$ be the collection of neutral internal cubic picture fuzzy sets in $R \neq 0$. Then the P-order union and intersection of $A_j = \left\langle \text{£}_{A_j}, e_{A_j} \right\rangle$ are also neutral internal cubic picture fuzzy sets in R.

Theorem 3.23. Let $A_j = \left\langle \text{£}_{A_j}, e_{A_j} \right\rangle$ be the collection of negative internal cubic picture fuzzy sets in $R \neq 0$. Then the P-order union and intersection of $A_j = \left\langle \text{£}_{A_j}, e_{A_j} \right\rangle$ are also negative internal cubic picture fuzzy sets in R.

Corollary 3.24. Let $A_j = \left\langle \text{£}_{A_j}, e_{A_j} \right\rangle$ be collection of internal cubic picture fuzzy sets in $R \neq 0$. Then the P-order union and intersection of $A_j = \left\langle \text{£}_{A_j}, e_{A_j} \right\rangle$ are also internal cubic picture fuzzy sets in R.

In below example we seen that every P-order union and intersection of negative external (resp., positive external, neutral external) cubic picture fuzzy sets may not be negative external (resp., positive external, neutral external) cubic picture fuzzy sets.

Example 3.25. Let for two cubic picture fuzzy sets $A = \left\langle \text{£}_A, e_A \right\rangle$ and $B = \left\langle \text{£}_B, e_B \right\rangle$ in R, define as

$A = \left\{ r, \left\langle [0.32, 0.53], [0.09, 0.15], [0.22, 0.31] \right\rangle, \left\langle 0.48, 0.13, 0.36 \right\rangle \mid r \in [0,1] \right\}$ and

$B = \left\{ r, \left\langle [0.29, 0.43], [0.24, 0.38], [0.06, 0.18] \right\rangle, \left\langle 0.37, 0.29, 0.33 \right\rangle \mid r \in [0,1] \right\}.$

Then A and B are negative external cubic picture fuzzy sets in $[0,1]$, and $A \overset{p}{\bigcup} B = \left(\text{£}_A \cup \text{£}_B, e_A \vee e_B \right)$ with

$\text{£}_A \cup \text{£}_B = \left\{ r, \left\langle [0.32, 0.53], [0.24, 0.38], [0.22, 0.31] \right\rangle \mid r \in [0,1] \right\}$ and

$e_A \vee e_B = \left\{ r, \left\langle 0.48, 0.29, 0.36 \right\rangle \mid r \in [0,1] \right\}$

is not negative external cubic picture fuzzy sets in $[0,1]$.

also, $A\overset{p}{\bigcap}B=\left(£_A\cap £_B, e_A\wedge e_B\right)$ with $£_A\cap £_B=\left\{r,\left\langle[0.29,0.43],[0.09,0.15],[0.06,0.18]\right\rangle \mid r\in[0,1]\right\}$

$e_A\wedge e_B=\left\{r,\left\langle 0.37,0.13,0.33\right\rangle \mid r\in[0,1]\right\}$ is not negative external cubic picture fuzzy sets in [0,1].

Example 3.26. Let the universe set be $R=\{r_1,r_2,r_3\}$. Then the pairs $A=\left\langle £_A,e_A\right\rangle$ and $B=\left\langle £_B,e_B\right\rangle$ are said to be a cubic picture fuzzy set with the tabular representation as (below).

cubic picture fuzzy set A

R	$£_A$	e_A
r_1	$([0.20,0.30],[0.15,0.26],[0.39,0.41])$	$(0.51,0.08,0.40)$
r_2	$([0.14,0.28],[0.29,0.39],[0.18,0.28])$	$(0.13,0.43,0.24)$
r_3	$([0.41,0.43],[0.09,0.18],[0.27,0.33])$	$(0.26,0.27,0.28)$

cubic picture fuzzy set B

R	$£_B$	e_B
r_1	$([0.20,0.33],[0.05,0.16],[0.39,0.41])$	$(0.35,0.19,0.40)$
r_2	$([0.11,0.29],[0.30,0.40],[0.15,0.25])$	$(0.32,0.29,0.24)$
r_3	$([0.33,0.39],[0.15,0.23],[0.27,0.29])$	$(0.42,0.27,0.28)$

$A\overset{p}{\bigcup}B=\left(£_A\cup £_B, e_A\vee e_B\right)$

R	$£_A\cup £_B$	$e_A\vee e_B$
r_1	$([0.20,0.33],[0.15,0.26],[0.39,0.41])$	$(0.51,0.19,0.40)$
r_2	$([0.14,0.29],[0.30,0.40],[0.18,0.28])$	$(0.32,0.43,0.24)$
r_3	$([0.41,0.43],[0.15,0.23],[0.27,0.33])$	$(0.42,0.27,0.28)$

$A\overset{p}{\bigcap}B=\left(£_A\cap £_B, e_A\wedge e_B\right)$

R	$£_A\cap £_B$	$e_A\wedge e_B$
r_1	$([0.20,0.30],[0.05,0.16],[0.39,0.41])$	$(0.35,0.08,0.40)$
r_2	$([0.11,0.28],[0.29,0.39],[0.15,0.25])$	$(0.13,0.29,0.24)$
r_3	$([0.33,0.39],[0.09,0.18],[0.27,0.29])$	$(0.26,0.27,0.28)$

Then $A=\left\langle £_A,e_A\right\rangle$ and $B=\left\langle £_B,e_B\right\rangle$ are both positive and neutral external cubic picture fuzzy sets in $[0,1]$, and tabular representation of $A\overset{p}{\bigcup}B$, $A\overset{p}{\bigcap}B$ are neither an positive external nor neutral external cubic picture fuzzy sets.

Similarly, we seen that every P-order union and intersection of positive internal (resp., neutral internal, negative internal) cubic picture fuzzy sets may not be positive internal (resp., neutral internal, negative internal) cubic picture fuzzy sets.

Example 3.27. Let for two cubic picture fuzzy sets $A=\langle \pounds_A, e_A\rangle$ and $B=\langle \pounds_B, e_B\rangle$ in R, define as

$$A=\left\{r,\left\langle [0.32,0.53],[0.09,0.15],[0.22,0.31]\right\rangle,\left\langle 0.48,0.18,0.33\right\rangle \mid r\in[0,1]\right\} \text{ and}$$

$$B=\left\{r,\left\langle [0.29,0.43],[0.24,0.35],[0.06,0.18]\right\rangle,\left\langle 0.37,0.37,0.33\right\rangle \mid r\in[0,1]\right\}.$$

Then A and B are positive internal cubic picture fuzzy sets in $[0,1]$, and $A\overset{p}{\bigcup}B=\left(\pounds_A\cup\pounds_B, e_A\vee e_B\right)$ with

$$\pounds_A\cup\pounds_B=\left\{r,\left\langle [0.32,0.53],[0.24,0.35],[0.22,0.31]\right\rangle \mid r\in[0,1]\right\},$$

$$e_A\vee e_B=\left\{r,\left\langle 0.48,0.37,0.33\right\rangle \mid r\in[0,1]\right\}$$

is not a positive internal cubic picture fuzzy set in $[0,1]$.

also, $A\overset{p}{\bigcap}B=\left(\pounds_A\cap\pounds_B, e_A\wedge e_B\right)$ with

$$\pounds_A\cap\pounds_B=\left\{r,\left\langle [0.29,0.43],[0.09,0.15],[0.06,0.18]\right\rangle \mid r\in[0,1]\right\}$$

$e_A\wedge e_B=\left\{r,\left\langle 0.37,0.18,0.33\right\rangle \mid r\in[0,1]\right\}$ is not a positive internal cubic picture fuzzy set in $[0,1]$.

Theorem 3.28. Let the universe set be $R\neq\varphi$ and for two positive internal cubic picture fuzzy sets $A=\langle \pounds_A, e_A\rangle$ and $B=\langle \pounds_B, e_B\rangle$ in R. Then for all $r\in R$ such that

$$\max\left\{\pounds^-_{P_A}(r),\pounds^-_{P_B}(r)\right\}\leq\left(P_A\wedge P_B\right)(r).$$

Then R-order union of A and B is a positive internal cubic picture fuzzy set in R.

Proof: Since $A=\langle \pounds_A, e_A\rangle$ and $B=\langle \pounds_B, e_B\rangle$ are two positive internal cubic picture fuzzy sets in R. Then by Definition 3.2 we have $\forall r\in R$, $\pounds^-_{P_A}(r)\leq P_A(r)\leq\pounds^+_{P_A}(r)$, $\pounds^-_{P_B}(r)\leq P_B(r)\leq\pounds^+_{P_B}(r)$, so $\left(P_A\wedge P_B\right)(r)\leq\left(\pounds_{P_A}\cup\pounds_{P_B}\right)^+(r)$,

it follows that

$$\left(\pounds_{P_A}\cup\pounds_{P_B}\right)^-(r)=\max\left\{\pounds^-_{P_A}(r),\pounds^-_{P_B}(r)\right\}\leq\left(P_A\wedge P_B\right)(r)\leq\left(\pounds_{P_A}\cup\pounds_{P_B}\right)^+(r).$$

Hence

$A\overset{R}{\bigcup}B=\left(\pounds_A\cup\pounds_B, e_A\wedge e_B\right)$ is a R-order union of A and B is a positive internal cubic picture fuzzy set in R.

Similarly,

Theorem 3.29. Let the universe set be $R\neq\varphi$ and for two neutral internal cubic picture fuzzy sets $A=\langle \pounds_A, e_A\rangle$ and $B=\langle \pounds_B, e_B\rangle$ in R. Then for all $r\in R$ such that

$$\max\left\{\pounds^-_{I_A}(r),\pounds^-_{I_B}(r)\right\}\leq\left(I_A\wedge I_B\right)(r).$$

Then R-order union of A and B is a neutral internal cubic picture fuzzy set in R.

Theorem 3.30. Let the universe set be $R \neq \varphi$ and for two negative internal cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ in R. Then for all $r \in R$ such that

$$\max\left\{£^{-}_{N_A}(r), £^{-}_{N_B}(r)\right\} \leq \left(N_A \wedge N_B\right)(r).$$

Then R-order union of A and B is a negative internal cubic picture fuzzy set in R.

Corollary 3.31. Let the universe set be $R \neq \varphi$ and for two internal cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ in R. If R-order union of A and B is satisfies all conditions of positive, neutral and negative internal cubic picture fuzzy sets. Then R-order union of A and B is said to be internal cubic picture fuzzy set in R.

Theorem 3.32. Let the universe set be $R \neq \varphi$ and for two positive internal cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ in R. Then for all $r \in R$ such that

$$\left(P_A \vee P_B\right)(r) \leq \min\left\{£^{+}_{P_A}(r), £^{+}_{P_B}(r)\right\}.$$

Then R-order intersection of A and B is a positive internal cubic picture fuzzy set in R.

Proof: Since $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ are two positive internal cubic picture fuzzy sets in R. Then by Definition 3.2 we have $\forall\ r \in R$, $£^{-}_{P_A}(r) \leq P_A(r) \leq £^{+}_{P_A}(r)$, $£^{-}_{P_B}(r) \leq P_B(r) \leq £^{+}_{P_B}(r)$, so $\left(£_{P_A} \cap £_{P_B}\right)^{-}(r) \leq \left(P_A \vee P_B\right)(r)$,

it follows that

$$\left(£_{P_A} \cap £_{P_B}\right)^{-}(r) \leq \left(P_A \vee P_B\right)(r) \leq \min\left\{£^{+}_{P_A}(r), £^{+}_{P_B}(r)\right\} = \left(£_{P_A} \cap £_{P_B}\right)^{+}(r).$$

Hence

$A \overset{R}{\bigcap} B = \left(£_A \cap £_B, e_A \vee e_B\right)$ is a R-order intersection of A and B is a positive internal cubic picture fuzzy set in R.

Similarly,

Theorem 3.33. Let the universe set be $R \neq \varphi$ and for two neutral internal cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ in R. Then for all $r \in R$ such that

$$\left(I_A \vee I_B\right)(r) \leq \min\left\{£^{+}_{I_A}(r), £^{+}_{I_B}(r)\right\}.$$

Then R-order intersection of A and B is a neutral internal cubic picture fuzzy set in R.

Theorem 3.34. Let the universe set be $R \neq \varphi$ and for two negative internal cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ in R. Then for all $r \in R$ such that

$$\left(N_A \vee N_B\right)(r) \leq \min\left\{£^{+}_{N_A}(r), £^{+}_{N_B}(r)\right\}.$$

Then R-order intersection of A and B is a negative internal cubic picture fuzzy set in R.

Corollary 3.35. Let the universe set be $R \neq \varphi$ and for two internal cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ in R. If R-order intersection of A and B is satisfies all conditions of positive, neutral and negative internal cubic picture fuzzy sets. Then R-order intersection of A and B is said to be internal cubic picture fuzzy set in R.

Definition 3.36. Let the universe set be $R \neq \varphi$ and the pair $A = \langle £_A, e_A \rangle$ be a cubic picture fuzzy set in $R.$ The compliment is denoted and defined as

$$A^c = \langle £_A^c, e_A^c \rangle = \left\{ r, \langle £_{N_A}(r), £_{I_A}(r), £_{P_A}(r) \rangle, \langle N_A(r), I_A(r), P_A(r) \rangle \mid r \in R \right\}.$$

Example 3.37. Let the universe set be $R = \{ r_1, r_2, r_3 \}$. Then the compliment of a cubic picture fuzzy set is calculate as, with the tabular representation as (below).

cubic picture fuzzy set A

R	$£_A$	e_A
r_1	$([0.20, 0.30], [0.15, 0.26], [0.39, 0.41])$	$(0.51, 0.08, 0.40)$
r_2	$([0.14, 0.28], [0.29, 0.39], [0.18, 0.28])$	$(0.13, 0.43, 0.24)$
r_3	$([0.41, 0.43], [0.09, 0.18], [0.27, 0.33])$	$(0.26, 0.27, 0.28)$

compliment of a cubic picture fuzzy set A

R	$£_A^c$	e_A^c
r_1	$([0.39, 0.41], [0.15, 0.26], [0.20, 0.30])$	$(0.40, 0.08, 0.51)$
r_2	$([0.18, 0.28], [0.29, 0.39], [0.14, 0.28])$	$(0.24, 0.43, 0.13)$
r_3	$([0.27, 0.33], [0.09, 0.18], [0.41, 0.43])$	$(0.28, 0.27, 0.26)$

Definition 3.38. Let the universe set be $R \neq \varphi$. $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ be any two CPFNs in R and $\tau \geq 0.$ Then the operations of CPFNs based on strict Archimedean triangular norm and conorm can be defined and denotes as

$$(1)\ \langle £_A, e_A \rangle \oplus \langle £_B, e_B \rangle = \left\{ \begin{array}{c} \left\{ \begin{array}{c} \begin{bmatrix} s^{-1}\left(s\left(£_{P_A}^{-} \right) + s\left(£_{P_B}^{-} \right) \right), \\ s^{-1}\left(s\left(£_{P_A}^{+} \right) + s\left(£_{P_B}^{+} \right) \right) \end{bmatrix}, \\ \begin{bmatrix} t^{-1}(t(£_{I_A}^{-}) + t(£_{I_B}^{-})), \\ t^{-1}(t(£_{I_A}^{+}) + t(£_{I_B}^{+})) \end{bmatrix}, \\ \begin{bmatrix} t^{-1}(t(£_{N_A}^{-}) + t(£_{N_B}^{-})), \\ t^{-1}(t(£_{N_A}^{+}) + t(£_{N_B}^{+})) \end{bmatrix} \end{array} \right\}, \\ \left\{ \begin{array}{c} s^{-1}\left(s\left(P_A \right) + s\left(P_B \right) \right),\ t^{-1}(t(I_A) + t(I_B)), \\ t^{-1}(t(N_A) + t(N_B)) \end{array} \right\} \end{array} \right\},$$

$$(2)\quad \tau\cdot\langle £_A, e_A\rangle = \left\{\begin{matrix} \left\{\begin{matrix} \begin{bmatrix} s^{-1}(\tau s(£^-_{P_A})), \\ s^{-1}(\tau s(£^+_{P_A})) \end{bmatrix}, \\ \begin{bmatrix} t^{-1}(\tau t(£^-_{I_A})), \\ t^{-1}(\tau t(£^+_{I_A})) \end{bmatrix}, \\ \begin{bmatrix} t^{-1}(\tau t(£^-_{N_A})), \\ t^{-1}(\tau t(£^+_{N_A})) \end{bmatrix} \end{matrix}\right\}, \\ \left\{\begin{matrix} s^{-1}(\tau s(P_A)), t^{-1}(\tau t(I_A)), \\ t^{-1}(\tau t(N_A)) \end{matrix}\right\} \end{matrix}\right\},$$

If we allocate a specific value to generators t & s, then this particular operations of CPFNs will be attained. If $t(r_1) = -\log(r_1)$ and $s(r_1) = -\log(1-r_1)$ then

$$(3)\quad \langle £_A, e_A\rangle \oplus \langle £_B, e_B\rangle = \left\{\begin{matrix} \left\{\begin{matrix} \begin{bmatrix} £^-_{P_A} + £^-_{P_B} - £^-_{P_A}\cdot £^-_{P_B}, \\ £^+_{P_A} + £^+_{P_B} - £^+_{P_A}\cdot £^+_{P_B} \end{bmatrix}, \\ \left[£^-_{I_A}\cdot £^-_{I_B},\ £^+_{I_A}\cdot £^+_{I_B} \right], \\ \left[£^-_{N_A}\cdot £^-_{N_B},\ £^+_{N_A}\cdot £^+_{N_B} \right] \end{matrix}\right\}, \\ \left\{ P_A + P_B - P_A\cdot P_B,\ I_A\cdot I_B,\ N_A\cdot N_B \right\} \end{matrix}\right\}$$

$$(4)\quad \tau\cdot\langle £_A, e_A\rangle = \left\{\begin{matrix} \left\{\begin{matrix} \left[1-(1-£^-_{P_A})^\tau,\ 1-(1-£^+_{P_A})^\tau\right], \\ \left[(£^-_{I_A})^\tau,\ (£^+_{I_A})^\tau\right], \\ \left[(£^-_{N_A})^\tau,\ (£^+_{N_A})^\tau\right] \end{matrix}\right\}, \\ \left\{1-(1-P_A)^\tau,\ (I_A)^\tau, (N_A)^\tau\right\} \end{matrix}\right\}.$$

3.1. Comparison Rules for CPFNs

In this section we introduced some functions which play important role for the ranking of CPFNs are described as.

Definition 3.39. Let $A = \langle £_A, e_A\rangle$ be any CPFN. Then

(1) $sc(A) = \frac{\left(£^-_{P_A} + £^+_{P_A} - £^-_{I_A} - £^+_{I_A} - £^-_{N_A} - £^+_{N_A}\right) + \left(P_{e_k} + 1 - I_{e_k} + 1 - N_{e_k}\right)}{6}$

$= \frac{1}{6}\left(2 + £^-_{P_A} + £^+_{P_A} - £^-_{I_A} - £^+_{I_A} - £^-_{N_A} - £^+_{N_A} + P_{e_k} - I_{e_k} - N_{e_k}\right)$ which denoted as score function.

(2) $ac(A) = \frac{\left(£^-_{P_A} + £^+_{P_A} + £^-_{N_A} + £^+_{N_A}\right) + \left(P_{e_k} + N_{e_k}\right)}{6}$ which denoted as accuracy function.

Utilizing Definition 3.39, technique for equating the CPFNs can be described as

Definition 3.40. Let $A = \langle £_A, e_A\rangle$ and $B = \langle £_B, e_B\rangle$ be any two CPFNs in R. Then by using the Definition 3.39, equating technique can be described as,

(a) If $sc(A) > sc(B)$, then $A > B$.

(b) If $sc(A) = sc(B)$, and $ac(A) > ac(B)$, then $A > B$.

(c) If $sc(A) = sc(B)$, $ac(A) = ac(B)$ then $A = B$.

4. CUBIC PICTURE FUZZY WEIGHTED AVERAGING AGGREGATION OPERATORS

Definition 4.1. Let $A_k = \left\langle \pounds_{A_k}, e_{A_k} \right\rangle, \left(k \in N\right)$ be any collection of CPFNs and $CPFWA : CPFN^n \to CPFN$, then CPFWA describe as,

$$CPFWA\left(A_1, A_2, ..., A_n\right) = \sum_{k=1}^{n} \tau_k A_k,$$

In which $\tau = \left\{\tau_1, \tau_2, ..., \tau_n\right\}$ be the weight vector of $A_k = \left\langle \pounds_{A_k}, e_{A_k} \right\rangle$, $k \in N$ with $\tau_k \geq 0$ and $\sum_{k=1}^{n} \tau_k = 1$.

Theorem 4.2. Let $A_k = \left\langle \pounds_{A_k}, e_{A_k} \right\rangle, \left(k \in N\right)$ be any collection of CPFNs. Then by utilizing the Definition 3.38 and operational properties of CPFNs, we can obtain the following outcome.

$$CPFWA\left(A_1, A_2, ..., A_n\right) = \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left[s^{-1}\left(\sum_{k=1}^{n} \tau_k s\left(\pounds^{-}_{P_{A_k}}\right)\right),\ s^{-1}\left(\sum_{k=1}^{n} \tau_k s\left(\pounds^{+}_{P_{A_k}}\right)\right) \right], \\ \left[t^{-1}\left(\sum_{k=1}^{n} \tau_k t\left(\pounds^{-}_{I_{A_k}}\right)\right),\ t^{-1}\left(\sum_{k=1}^{n} \tau_k t\left(\pounds^{+}_{I_{A_k}}\right)\right) \right], \\ \left[t^{-1}\left(\sum_{k=1}^{n} \tau_k t\left(\pounds^{-}_{N_{A_k}}\right)\right),\ t^{-1}\left(\sum_{k=1}^{n} \tau_k t\left(\pounds^{+}_{P_{A_k}}\right)\right) \right] \end{array} \right\}, \\ \left\{ \begin{array}{c} s^{-1}\left(\sum_{k=1}^{n} \tau_k s\left(P_{A_k}\right)\right), t^{-1}\left(\sum_{k=1}^{n} \tau_k t\left(I_{A_k}\right)\right), \\ t^{-1}\left(\sum_{k=1}^{n} \tau_k t\left(N_{A_k}\right)\right) \end{array} \right\} \end{array} \right\}.$$

If we allocate value to generators t and s as in Algebraic strict Archimedean t-norm and t-conorm, then this particular operation of CPFNs will be attained as,

$$CPFWA\left(A_1, A_2, ..., A_n\right) = \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left[\begin{array}{c} 1 - \Pi_{k=1}^{n}(1 - \pounds^{-}_{P_{A_k}})^{\tau_k}, \\ 1 - \Pi_{k=1}^{n}(1 - \pounds^{+}_{P_{A_k}})^{\tau_k} \end{array} \right], \\ \left[\Pi_{k=1}^{n}(\pounds^{-}_{I_{A_k}})^{\tau_k},\ \Pi_{k=1}^{n}(\pounds^{+}_{I_{A_k}})^{\tau_k} \right], \\ \left[\Pi_{k=1}^{n}(\pounds^{-}_{N_{A_k}})^{\tau_k},\ \Pi_{k=1}^{n}(\pounds^{+}_{N_{A_k}})^{\tau_k} \right] \end{array} \right\}, \\ \left\{ \begin{array}{c} 1 - \Pi_{k=1}^{n}(1 - P_{A_k})^{\tau_k}, \Pi_{k=1}^{n}(I_{A_k})^{\tau_k}, \\ \Pi_{k=1}^{n}(N_{A_k})^{\tau_k} \end{array} \right\} \end{array} \right\}.$$

Where $\tau=\{\tau_1,\tau_2,...,\tau_n\}$ be the weight vector of $A_k=\langle £_{A_k}, e_{A_k}\rangle,\ k\in N$ with $\tau_k\geq 0$ and $\sum_{k=1}^{n}\tau_k=1.$

Proof: We done the prove by utilizing the technique of mathematical induction. Therefore, we follow as

(a) For $n=2$, since

$$\tau_1 A_1=\left\{\begin{array}{c}\left\{\begin{array}{c}\left[1-(1-£_{P_{A_1}}^{-})^{\tau_1}, 1-(1-£_{P_{A_1}}^{+})^{\tau_1}\right],\\ \left[(£_{I_{A_1}}^{-})^{\tau_1},\ (£_{I_{A_1}}^{+})^{\tau_1}\right],\ \left[(£_{N_{A_1}}^{-})^{\tau_1},\ (£_{N_{A_1}}^{+})^{\tau_1}\right]\end{array}\right\},\\ \left\{1-(1-P_{A_1})^{\tau_1},(I_{A_1})^{\tau_1},(N_{A_1})^{\tau_1}\right\}\end{array}\right\}$$

and

$$\tau_2 A_2=\left\{\begin{array}{c}\left\{\begin{array}{c}\left[1-(1-£_{P_{A_2}}^{-})^{\tau_2}, 1-(1-£_{P_{A_2}}^{+})^{\tau_2}\right],\\ \left[(£_{I_{A_2}}^{-})^{\tau_2},\ (£_{I_{A_2}}^{+})^{\tau_2}\right],\ \left[(£_{N_{A_2}}^{-})^{\tau_2},\ (£_{N_{A_2}}^{+})^{\tau_2}\right]\end{array}\right\},\\ \left\{1-(1-P_{A_2})^{\tau_2},(I_{A_2})^{\tau_2},(N_{A_2})^{\tau_2}\right\}\end{array}\right\}$$

Then

$$CPFWA(A_1,A_2)=\tau_1A_1+\tau_2A_2$$

$$=\left\{\begin{array}{c}\left\{\begin{array}{c}\left[1-(1-£_{P_{A_1}}^{-})^{\tau_1}, 1-(1-£_{P_{A_1}}^{+})^{\tau_1}\right],\\ \left[(£_{I_{A_1}}^{-})^{\tau_1},\ (£_{I_{A_1}}^{+})^{\tau_1}\right],\\ \left[(£_{N_{A_1}}^{-})^{\tau_1},\ (£_{N_{A_1}}^{+})^{\tau_1}\right]\end{array}\right\},\\ \left\{1-(1-P_{A_1})^{\tau_1},(I_{A_1})^{\tau_1},(N_{A_1})^{\tau_1}\right\}\end{array}\right\}+\left\{\begin{array}{c}\left\{\begin{array}{c}\left[1-(1-£_{P_{A_2}}^{-})^{\tau_2}, 1-(1-£_{P_{A_2}}^{+})^{\tau_2}\right],\\ \left[(£_{I_{A_2}}^{-})^{\tau_2},\ (£_{I_{A_2}}^{+})^{\tau_2}\right],\\ \left[(£_{N_{A_2}}^{-})^{\tau_2},\ (£_{N_{A_2}}^{+})^{\tau_2}\right]\end{array}\right\},\\ \left\{1-(1-P_{A_2})^{\tau_2},(I_{A_2})^{\tau_2},(N_{A_2})^{\tau_2}\right\}\end{array}\right\}$$

$$=\left\{\begin{array}{c}\left\{\begin{array}{c}\left[\begin{array}{c}1-(1-£_{P_{A_1}}^{-})^{\tau_1},\\ 1-(1-£_{P_{A_1}}^{+})^{\tau_1}\end{array}\right]+\left[\begin{array}{c}1-(1-£_{P_{A_2}}^{-})^{\tau_2},\\ 1-(1-£_{P_{A_2}}^{+})^{\tau_2}\end{array}\right]-\\ \left[\begin{array}{c}1-(1-£_{P_{A_1}}^{-})^{\tau_1},\\ 1-(1-£_{P_{A_1}}^{+})^{\tau_1}\end{array}\right].\left[\begin{array}{c}1-(1-£_{P_{A_2}}^{-})^{\tau_2},\\ 1-(1-£_{P_{A_2}}^{+})^{\tau_2}\end{array}\right],\\ \left[(£_{I_{A_1}}^{-})^{\tau_1},\ (£_{I_{A_1}}^{+})^{\tau_1}\right]\cdot\left[(£_{I_{A_2}}^{-})^{\tau_2},\ (£_{I_{A_2}}^{+})^{\tau_2}\right],\\ \left[(£_{N_{A_1}}^{-})^{\tau_1},\ (£_{N_{A_1}}^{+})^{\tau_1}\right]\cdot\left[(£_{N_{A_2}}^{-})^{\tau_2},\ (£_{N_{A_2}}^{+})^{\tau_2}\right]\end{array}\right\},\\ \left\{\begin{array}{c}1-(1-P_{A_1})^{\tau_1}+1-(1-P_{A_2})^{\tau_2}-\\ \left(1-(1-P_{A_1})^{\tau_1}\right)\cdot\left(1-(1-P_{A_2})^{\tau_2}\right),\\ (I_{A_1})^{\tau_1}\cdot(I_{A_2})^{\tau_2},(N_{A_1})^{\tau_1}\cdot(N_{A_2})^{\tau_2}\end{array}\right\}\end{array}\right\}=\left\{\begin{array}{c}\left\{\begin{array}{c}\left[\begin{array}{c}1-(1-£_{P_{A_1}}^{-})^{\tau_1}(1-£_{P_{A_2}}^{-})^{\tau_2},\\ 1-(1-£_{P_{A_1}}^{+})^{\tau_1}(1-£_{P_{A_2}}^{+})^{\tau_2}\end{array}\right],\\ \left[\begin{array}{c}(£_{I_{A_1}}^{-})^{\tau_1}\cdot(£_{I_{A_2}}^{-})^{\tau_2},\\ (£_{I_{A_1}}^{+})^{\tau_1}\cdot(£_{I_{A_2}}^{+})^{\tau_2}\end{array}\right],\\ \left[\begin{array}{c}(£_{N_{A_1}}^{-})^{\tau_1}\cdot(£_{N_{A_2}}^{-})^{\tau_2},\\ (£_{N_{A_1}}^{+})^{\tau_1}\cdot(£_{N_{A_2}}^{+})^{\tau_2}\end{array}\right]\end{array}\right\},\\ \left\{\begin{array}{c}1-(1-P_{A_1})^{\tau_1}(1-P_{A_2})^{\tau_2},\\ (I_{A_1})^{\tau_1}\cdot(I_{A_2})^{\tau_2},(N_{A_1})^{\tau_1}\cdot(N_{A_2})^{\tau_2}\end{array}\right\}\end{array}\right\}$$

$$= \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left[\begin{array}{c} 1-\Pi_{k=1}^{2}(1-£^{-}_{P_{A_k}})^{\tau_k}, \\ 1-\Pi_{k=1}^{2}(1-£^{+}_{P_{A_k}})^{\tau_k} \end{array} \right], \left[\begin{array}{c} \Pi_{k=1}^{2}(£^{-}_{I_{A_k}})^{\tau_k}, \\ \Pi_{k=1}^{2}(£^{+}_{I_{A_k}})^{\tau_k} \end{array} \right], \\ \left[\begin{array}{c} \Pi_{k=1}^{2}(£^{-}_{N_{A_k}})^{\tau_k}, \\ \Pi_{k=1}^{2}(£^{+}_{N_{A_k}})^{\tau_k} \end{array} \right] \end{array} \right\}, \\ \left\{ 1-\Pi_{k=1}^{2}(1-P_{A_k})^{\tau_k}, \Pi_{k=1}^{2}(I_{A_k})^{\tau_k}, \Pi_{k=1}^{2}(N_{A_k})^{\tau_k} \right\} \end{array} \right\}$$

(b) Suppose that outcome is true for $n = z$ that is,

$$CPFWA\left(A_1, A_2, ..., A_z\right) = \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left[\begin{array}{c} 1-\Pi_{k=1}^{z}(1-£^{-}_{P_{A_k}})^{\tau_k}, \\ 1-\Pi_{k=1}^{z}(1-£^{+}_{P_{A_k}})^{\tau_k} \end{array} \right], \\ \left[\begin{array}{c} \Pi_{k=1}^{z}(£^{-}_{I_{A_k}})^{\tau_k}, \\ \Pi_{k=1}^{z}(£^{+}_{I_{A_k}})^{\tau_k} \end{array} \right], \\ \left[\begin{array}{c} \Pi_{k=1}^{z}(£^{-}_{N_{A_k}})^{\tau_k}, \\ \Pi_{k=1}^{z}(£^{+}_{N_{A_k}})^{\tau_k} \end{array} \right] \end{array} \right\}, \\ \left\{ \begin{array}{c} 1-\Pi_{k=1}^{z}(1-P_{A_k})^{\tau_k}, \\ \Pi_{k=1}^{z}(I_{A_k})^{\tau_k}, \Pi_{k=1}^{z}(N_{A_k})^{\tau_k} \end{array} \right\} \end{array} \right\},$$

(c) Now we must prove that outcome is true for $n = z+1,$ by utilizing the (a) & (b) we have

$$CPFWA\left(A_1, A_2, ..., A_z, A_{z+1}\right) = \sum_{k=1}^{z}\tau_k A_k + \tau_{z+1}A_{z+1}$$

$$= \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left[\begin{array}{c} 1-\Pi_{k=1}^{z}(1-£^{-}_{P_{A_k}})^{\tau_k}, \\ 1-\Pi_{k=1}^{z}(1-£^{+}_{P_{A_k}})^{\tau_k} \end{array} \right], \\ \left[\Pi_{k=1}^{z}(£^{-}_{I_{A_k}})^{\tau_k},\ \Pi_{k=1}^{z}(£^{+}_{I_{A_k}})^{\tau_k} \right], \\ \left[\Pi_{k=1}^{z}(£^{-}_{N_{A_k}})^{\tau_k}, \Pi_{k=1}^{z}(£^{+}_{N_{A_k}})^{\tau_k} \right] \end{array} \right\}, \\ \left\{ \begin{array}{c} 1-\Pi_{k=1}^{z}(1-P_{A_k})^{\tau_k}, \Pi_{k=1}^{z}(I_{A_k})^{\tau_k}, \\ \Pi_{k=1}^{z}(N_{A_k})^{\tau_k} \end{array} \right\} \end{array} \right\} + \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left[\begin{array}{c} 1-(1-£^{-}_{P_{A_{z+1}}})^{\tau_{z+1}}, \\ 1-(1-£^{+}_{P_{A_{z+1}}})^{\tau_{z+1}} \end{array} \right], \\ \left[(£^{-}_{I_{A_{z+1}}})^{\tau_{z+1}},\ (£^{+}_{I_{A_{z+1}}})^{\tau_{z+1}} \right], \\ \left[(£^{-}_{N_{A_{z+1}}})^{\tau_{z+1}},\ (£^{+}_{N_{A_{z+1}}})^{\tau_{z+1}} \right] \end{array} \right\}, \\ \left\{ \begin{array}{c} 1-(1-P_{A_{z+1}})^{\tau_{z+1}}, (I_{A_{z+1}})^{\tau_{z+1}}, \\ (N_{A_{z+1}})^{\tau_{z+1}} \end{array} \right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left\{ \begin{array}{c} \begin{bmatrix} 1-\Pi_{k=1}^{z}(1-\pounds_{P_{A_k}}^{-})^{\tau_k}, \\ 1-\Pi_{k=1}^{z}(1-\pounds_{P_{A_k}}^{+})^{\tau_k} \end{bmatrix} + \begin{bmatrix} 1-(1-\pounds_{P_{A_{z+1}}}^{-})^{\tau_{z+1}}, \\ 1-(1-\pounds_{P_{A_{z+1}}}^{+})^{\tau_{z+1}} \end{bmatrix} - \\ \begin{bmatrix} 1-\Pi_{k=1}^{z}(1-\pounds_{P_{A_k}}^{-})^{\tau_k}, \\ 1-\Pi_{k=1}^{z}(1-\pounds_{P_{A_k}}^{+})^{\tau_k} \end{bmatrix} \cdot \begin{bmatrix} 1-(1-\pounds_{P_{A_{z+1}}}^{-})^{\tau_{z+1}}, \\ 1-(1-\pounds_{P_{A_{z+1}}}^{+})^{\tau_{z+1}} \end{bmatrix}, \\ \left[\Pi_{k=1}^{z}(\pounds_{I_{A_k}}^{-})^{\tau_k},\ \Pi_{k=1}^{z}(\pounds_{I_{A_k}}^{+})^{\tau_k}\right] \cdot \left[(\pounds_{I_{A_{z+1}}}^{-})^{\tau_{z+1}},\ (\pounds_{I_{A_{z+1}}}^{+})^{\tau_{z+1}}\right], \\ \left[\Pi_{k=1}^{z}(\pounds_{N_{A_k}}^{-})^{\tau_k},\ \Pi_{k=1}^{z}(\pounds_{N_{A_k}}^{+})^{\tau_k}\right] \cdot \left[(\pounds_{N_{A_{z+1}}}^{-})^{\tau_{z+1}}, (\pounds_{N_{A_{z+1}}}^{+})^{\tau_{z+1}}\right] \end{array} \right\}, \\ \left\{ \begin{array}{c} 1-\Pi_{k=1}^{z}(1-P_{A_k})^{\tau_k} + 1-(1-P_{A_{z+1}})^{\tau_{z+1}} - \\ 1-\Pi_{k=1}^{z}(1-P_{A_k})^{\tau_k} \cdot 1-(1-P_{A_{z+1}})^{\tau_{z+1}}, \\ \Pi_{k=1}^{z}(I_{A_k})^{\tau_k} \cdot (I_{A_{z+1}})^{\tau_{z+1}}, \Pi_{k=1}^{z}(N_{A_k})^{\tau_k} \cdot (N_{A_{z+1}})^{\tau_{z+1}} \end{array} \right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left\{ \begin{array}{c} \begin{bmatrix} 1-\Pi_{k=1}^{z}(1-\pounds_{P_{A_k}}^{-})^{\tau_k}(1-\pounds_{P_{A_{z+1}}}^{-})^{\tau_{z+1}}, \\ 1-\Pi_{k=1}^{z}(1-\pounds_{P_{A_k}}^{+})^{\tau_k}(1-\pounds_{P_{A_{z+1}}}^{+})^{\tau_{z+1}} \end{bmatrix}, \\ \begin{bmatrix} \Pi_{k=1}^{z}(\pounds_{I_{A_k}}^{-})^{\tau_k} \cdot (\pounds_{I_{A_{z+1}}}^{-})^{\tau_{z+1}}, \\ \Pi_{k=1}^{z}(\pounds_{I_{A_k}}^{+})^{\tau_k} \cdot (\pounds_{I_{A_{z+1}}}^{+})^{\tau_{z+1}} \end{bmatrix}, \\ \begin{bmatrix} \Pi_{k=1}^{z}(\pounds_{N_{A_k}}^{-})^{\tau_k} \cdot (\pounds_{N_{A_{z+1}}}^{-})^{\tau_{z+1}}, \\ \Pi_{k=1}^{z}(\pounds_{N_{A_k}}^{+})^{\tau_k} \cdot (\pounds_{N_{A_{z+1}}}^{+})^{\tau_{z+1}} \end{bmatrix} \end{array} \right\}, \\ \left\{ \begin{array}{c} 1-\Pi_{k=1}^{z}(1-P_{A_k})^{\tau_k}(1-P_{A_{z+1}})^{\tau_{z+1}}, \\ \Pi_{k=1}^{z}(I_{A_k})^{\tau_k} \cdot (I_{A_{z+1}})^{\tau_{z+1}}, \Pi_{k=1}^{z}(N_{A_k})^{\tau_k} \cdot (N_{A_{z+1}})^{\tau_{z+1}} \end{array} \right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left\{ \begin{array}{c} \begin{bmatrix} 1-\Pi_{k=1}^{z+1}(1-\pounds_{P_{A_k}}^{-})^{\tau_k}, \\ 1-\Pi_{k=1}^{z+1}(1-\pounds_{P_{A_k}}^{+})^{\tau_k} \end{bmatrix}, \\ \left[\Pi_{k=1}^{z+1}(\pounds_{I_{A_k}}^{-})^{\tau_k},\ \Pi_{k=1}^{z+1}(\pounds_{I_{A_k}}^{+})^{\tau_k}\right], \\ \left[\Pi_{k=1}^{z+1}(\pounds_{N_{A_k}}^{-})^{\tau_k},\ \Pi_{k=1}^{z+1}(\pounds_{N_{A_k}}^{+})^{\tau_k}\right] \end{array} \right\}, \\ \left\{ \begin{array}{c} 1-\Pi_{k=1}^{z+1}(1-P_{A_k})^{\tau_k}, \Pi_{k=1}^{z+1}(I_{A_k})^{\tau_k}, \\ \Pi_{k=1}^{z+1}(N_{A_k})^{\tau_k} \end{array} \right\} \end{array} \right\},$$

i.e., Outcome is satisfying for $n = z+1$. Thus, outcome is satisfied for whole n. Therefore

$$CPFWA(A_1, A_2, ..., A_n) = \left\{ \begin{array}{c} \left\{ \begin{array}{c} \begin{bmatrix} 1-\Pi_{k=1}^{n}(1-£_{P_{A_k}}^{-})^{\tau_k}, \\ 1-\Pi_{k=1}^{n}(1-£_{P_{A_k}}^{+})^{\tau_k} \end{bmatrix}, \\ \left[\Pi_{k=1}^{n}(£_{I_{A_k}}^{-})^{\tau_k},\ \Pi_{k=1}^{n}(£_{I_{A_k}}^{+})^{\tau_k} \right], \\ \left[\Pi_{k=1}^{n}(£_{N_{A_k}}^{-})^{\tau_k},\ \Pi_{k=1}^{n}(£_{N_{A_k}}^{+})^{\tau_k} \right] \end{array} \right\}, \\ \left\{ \begin{array}{c} 1-\Pi_{k=1}^{n}(1-P_{A_k})^{\tau_k}, \Pi_{k=1}^{n}(I_{A_k})^{\tau_k}, \\ \Pi_{k=1}^{n}(N_{A_k})^{\tau_k} \end{array} \right\} \end{array} \right\}.$$

which done the proof.

Remark: There are some properties which are fulfilled by the CPFWA operator obviously.

(1) Idempotency: Let $A_k = \langle £_{A_k}, e_{A_k} \rangle,\ k \in N$ be any collection of CPFNs. If all of $A_k = \langle £_{A_k}, e_{A_k} \rangle,\ k \in N$ are identical. Then there is

$$CPFWA(A_1, A_2, ..., A_n) = A.$$

(2) Boundedness: Let $A_k = \langle £_{A_k}, e_{A_k} \rangle,\ k \in N$ be any collection of CPFNs. Let $A_k^- = \left\{ \left\{ \min_k £_{P_{A_k}}, \min_k £_{I_{A_k}}, \max_k £_{N_{A_k}} \right\}, \left\{ \min_k P_{A_k}, \min_k I_{A_k}, \max_k N_{A_k} \right\} \right\}$ and $A_k^+ = \left\{ \left\{ \max_k £_{P_{A_k}}, \min_k £_{I_{A_k}}, \min_k £_{N_{A_k}} \right\}, \left\{ \max_k P_{A_k}, \min_k I_{A_k}, \min_k N_{A_k} \right\} \right\}$ for whole $k \in N$, therefore

$$A_k^- \subseteq CPFWA(A_1, A_2, ..., A_n) = A_k^+.$$

(3) Monotonically: Let $B_k = \langle P_{b_k}, I_{b_k}, N_{b_k} \rangle,\ k \in N$ be any collection of CPFNs. If it satisfies that $A_k \subseteq B_k$ for whole, $k \in N$, then

$$CPFWA(A_1, A_2, ..., A_n) = CPFWA(B_1, B_2, ..., B_n).$$

Example 4.3. Suppose that $A_1 = \langle £_{A_1}, e_{A_1} \rangle$, $A_2 = \langle £_{A_2}, e_{A_2} \rangle$, $A_3 = \langle £_{A_3}, e_{A_3} \rangle$ and $A_4 = \langle £_{A_4}, e_{A_4} \rangle$ are cubic picture fuzzy sets in R, define as

$A_1 = \{\langle [0.32, 0.53], [0.09, 0.15], [0.22, 0.31] \rangle, \langle 0.48, 0.13, 0.36 \rangle\}$,

$A_2 = \{\langle [0.29, 0.43], [0.24, 0.38], [0.06, 0.18] \rangle, \langle 0.37, 0.29, 0.33 \rangle\}$,

$A_3 = \{\langle [0.22, 0.40], [0.12, 0.10], [0.28, 0.39] \rangle, \langle 0.44, 0.23, 0.31 \rangle\}$,

$A_4 = \{\langle [0.19, 0.23], [0.17, 0.33], [0.39, 0.42] \rangle, \langle 0.52, 0.08, 0.38 \rangle\}$ and

$(0.2, 0.3, 0.1, 0.4)^T$ are the weight vector of $A_k\ (k = 1, 2, 3, 4)$. Then

$$CPFWA\left(A_1, A_2, A_{3,} A_4\right) = \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left[\begin{array}{c} 1-\left(1-0.32\right)^{0.2}\left(1-0.29\right)^{0.3}\left(1-0.22\right)^{0.1}\left(1-0.19\right)^{0.4}, \\ 1-\left(1-0.53\right)^{0.2}\left(1-0.43\right)^{0.3}\left(1-0.40\right)^{0.1}\left(1-0.23\right)^{0.4} \end{array} \right], \\ \left[\begin{array}{c} \left(0.09\right)^{0.2}\left(0.24\right)^{0.3}\left(0.12\right)^{0.1}\left(0.17\right)^{0.4}, \\ \left(0.15\right)^{0.2}\left(0.38\right)^{0.3}\left(0.10\right)^{0.1}\left(0.33\right)^{0.4} \end{array} \right], \\ \left[\begin{array}{c} \left(0.22\right)^{0.2}\left(0.06\right)^{0.3}\left(0.28\right)^{0.1}\left(0.39\right)^{0.4}, \\ \left(0.31\right)^{0.2}\left(0.18\right)^{0.3}\left(0.39\right)^{0.1}\left(0.42\right)^{0.4} \end{array} \right] \end{array} \right\}, \\ \left\{ \begin{array}{c} 1-\left(1-0.48\right)^{0.2}\left(1-0.37\right)^{0.3}\left(1-0.44\right)^{0.1}\left(1-0.52\right)^{0.4}, \\ \left(0.13\right)^{0.2}\left(0.29\right)^{0.3}\left(0.23\right)^{0.1}\left(0.08\right)^{0.4}, \\ \left(0.36\right)^{0.2}\left(0.33\right)^{0.3}\left(0.31\right)^{0.1}\left(0.38\right)^{0.4} \end{array} \right\} \end{array} \right\}$$

$$= \left\{ \left\langle [0.2509, 0.3782], [0.1603, 0.2609], [0.1918, 0.3042] \right\rangle, \left\langle 0.4625, 0.1441, 0.3530 \right\rangle \right\}$$

Definition 4.4. Let $A_k = \left\langle £_{A_k}, e_{A_k} \right\rangle,\ k \in N$ be a collection of CPFNs, and $CPFOWA : CPFN^n \to CPFN$, then the CPFOWA describe as,

$$CPFOWA_w(A_1, A_2, ..., A_n) = \sum_{k=1}^{n} \tau_k A_{\eta(k)},$$

with dimensions n, where kth biggest weighted value is $A_{\eta(k)}$ consequently by total order $A_{\eta(1)} \geq A_{\eta(2)} \geq ... \geq A_{\eta(n)}$. $\tau = \{\tau_1, \tau_2, ..., \tau_n\}$ are the weight vectors such that $\tau_k \geq 0\ (k \in N)$ and $\Sigma_{k=1}^{n} \tau_k = 1$.

Theorem 4.5. Let $A_k = \left\langle £_{A_k}, e_{A_k} \right\rangle,\ k \in N$ be a collection of CPFNs. Then by utilizing the Definition 4.4 and operational properties of CPFNs, we can obtain the following outcome.

$$CPFOWA_w(A_1, A_2, ..., A_n) = \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left[s^{-1}\left(\sum_{k=1}^{n} \tau_k s\left(£^{-}_{P_{A_{\eta(k)}}} \right) \right),\ s^{-1}\left(\sum_{k=1}^{n} \tau_k s\left(£^{+}_{P_{A_{\eta(k)}}} \right) \right) \right], \\ \left[t^{-1}\left(\sum_{k=1}^{n} \tau_k t\left(£^{-}_{I_{A_{\eta(k)}}} \right) \right),\ t^{-1}\left(\sum_{k=1}^{n} \tau_k t\left(£^{+}_{I_{A_{\eta(k)}}} \right) \right) \right], \\ \left[t^{-1}\left(\sum_{k=1}^{n} \tau_k t\left(£^{-}_{N_{A_{\eta(k)}}} \right) \right),\ t^{-1}\left(\sum_{k=1}^{n} \tau_k t\left(£^{+}_{P_{A_{\eta(k)}}} \right) \right) \right] \end{array} \right\}, \\ \left\{ \begin{array}{c} s^{-1}\left(\sum_{k=1}^{n} \tau_k s\left(P_{A_{\eta(k)}} \right) \right), t^{-1}\left(\sum_{k=1}^{n} \tau_k t\left(I_{A_{\eta(k)}} \right) \right), \\ t^{-1}\left(\sum_{k=1}^{n} \tau_k t\left(N_{A_{\eta(k)}} \right) \right) \end{array} \right\} \end{array} \right\}.$$

Where $\tau=\{\tau_1,\tau_2,...,\tau_n\}$ are the weight vector of $A_k=\langle £_{A_k},e_{A_k}\rangle,\ k\in N$ with $\tau_k\geq 0$ and $\sum_{k=1}^{n}\tau_k=1.$

If we allocate value to generators t and s as in Algebraic strict Archimedean t-norm and t-conorm, then this particular operation of CPFNs will be attained as,

$$CPFOWA(A_1,A_2,...,A_n)=\left\{\begin{array}{l}\left\{\begin{array}{c}\begin{bmatrix}1-\Pi_{k=1}^{n}(1-£^{-}_{P_{A\eta(k)}})^{\tau_k},\\ 1-\Pi_{k=1}^{n}(1-£^{+}_{P_{A_{\eta(k)}}})^{\tau_k}\end{bmatrix},\\ \left[\Pi_{k=1}^{n}(£^{-}_{I_{A_{\eta(k)}}})^{\tau_k},\ \Pi_{k=1}^{n}(£^{+}_{I_{A_{\eta(k)}}})^{\tau_k}\right],\\ \left[\Pi_{k=1}^{n}(£^{-}_{N_{A_{\eta(k)}}})^{\tau_k},\ \Pi_{k=1}^{n}(£^{+}_{N_{A_{\eta(k)}}})^{\tau_k}\right]\end{array}\right\},\\ \left\{\begin{array}{c}1-\Pi_{k=1}^{n}(1-P_{A_{\eta(k)}})^{\tau_k},\Pi_{k=1}^{n}(I_{A_{\eta(k)}})^{\tau_k},\\ \Pi_{k=1}^{n}(N_{A_{\eta(k)}})^{\tau_k}\end{array}\right\}\end{array}\right\}.$$

Where $A_{\eta(k)}$ is kth largest value consequently by total order $A_{\eta(1)}\geq A_{\eta(2)}\geq...\geq A_{\eta(n)}$.

Proof: Theorem 4.5 take the form by utilized the technique of mathematical induction on n, and procedure is eliminating here.

Remark: There are some properties which are fulfilled by the CPFOWA operator obviously.

(1) Idempotency: Let $A_k=\langle £_{A_k},e_{A_k}\rangle,\ k\in N$ be any collection of CPFNs. If all of $A_k=\langle £_{A_k},e_{A_k}\rangle,\ k\in N$ are identical. Then there is

$$CPFOWA(A_1,A_2,...,A_n)=A.$$

(2) Boundedness: Let $A_k=\langle £_{A_k},e_{A_k}\rangle, k\in N$ be any collection of CPFNs. Let $A_k^-=\left\{\left\{\min_k £_{P_{A_k}},\min_k £_{I_{A_k}},\max_k £_{N_{A_k}}\right\},\left\{\min_k P_{A_k},\min_k I_{A_k},\max_k N_{A_k}\right\}\right\}$ and $A_k^+=\left\{\left\{\max_k £_{P_{A_k}},\min_k £_{I_{A_k}},\min_k £_{N_{A_k}}\right\},\left\{\max_k P_{A_k},\min_k I_{A_k},\min_k N_{A_k}\right\}\right\}$ for whole $k\in N,$ therefore

$$A_k^-\subseteq CPFOWA(A_1,A_2,...,A_n)=A_k^+.$$

(3) Monotonically: Let $B_k=\langle P_{b_k},I_{b_k},N_{b_k}\rangle,\ k\in N$ be any collection of CPFNs. If it satisfies that $A_k\subseteq B_k$ for whole, $k\in N,$ then

$$CPFOWA(A_1,A_2,...,A_n)=CPFOWA(B_1,B_2,...,B_n).$$

Example 4.6. Suppose that $A_1=\langle £_{A_1},e_{A_1}\rangle$, $A_2=\langle £_{A_2},e_{A_2}\rangle$, $A_3=\langle £_{A_3},e_{A_3}\rangle$ and $A_4=\langle £_{A_4},e_{A_4}\rangle$ are cubic picture fuzzy sets in $R,$ define as

$$A_1=\{\langle[0.32,0.53],[0.09,0.15],[0.22,0.31]\rangle,\langle 0.48,0.13,0.36\rangle\},$$

$A_2 = \{\langle[0.29, 0.43], [0.24, 0.38], [0.06, 0.18]\rangle, \langle 0.37, 0.29, 0.33\rangle\}$,

$A_3 = \{\langle[0.22, 0.40], [0.12, 0.10], [0.28, 0.39]\rangle, \langle 0.44, 0.23, 0.31\rangle\}$,

$A_4 = \{\langle[0.19, 0.23], [0.17, 0.33], [0.39, 0.42]\rangle, \langle 0.52, 0.08, 0.38\rangle\}$ and $(0.2, 0.3, 0.1, 0.4)^T$ are the weight vector of $A_k\ (k = 1, 2, 3, 4)$. Then utilizing Definition 3.39 to calculate the score values of $A_k\ (k = 1, 2, 3, 4)$.

$$sc(A_1) = \frac{(2 + 0.32 + 0.53 - 0.09 - 0.15 - 0.22 - 0.31 + 0.48 - 0.13 - 0.36)}{6} = 0.345$$

$$sc(A_2) = \frac{(2 + 0.29 + 0.43 - 0.24 - 0.38 - 0.06 - 0.18 + 0.37 - 0.29 - 0.33)}{6} = 0.268$$

$$sc(A_3) = \frac{(2 + 0.22 + 0.40 - 0.12 - 0.10 - 0.28 - 0.39 + 0.44 - 0.23 - 0.31)}{6} = 0.271$$

$$sc(A_4) = \frac{(2 + 0.19 + 0.23 - 0.17 - 0.33 - 0.39 - 0.42 + 0.52 - 0.08 - 0.38)}{6} = 0.195$$

Therefore

$$sc(A_1) > sc(A_3) > sc(A_2) > sc(A_4).$$

Then

$$A_1 > A_3 > A_2 > A_4$$

and hence

$A_{\eta(1)} = A_1 = \{\langle[0.32, 0.53], [0.09, 0.15], [0.22, 0.31]\rangle, \langle 0.48, 0.13, 0.36\rangle\}$,

$A_{\eta(2)} = A_3 = \{\langle[0.22, 0.40], [0.12, 0.10], [0.28, 0.39]\rangle, \langle 0.44, 0.23, 0.31\rangle\}$,

$A_{\eta(3)} = A_2 = \{\langle[0.29, 0.43], [0.24, 0.38], [0.06, 0.18]\rangle, \langle 0.37, 0.29, 0.33\rangle\}$,

$A_{\eta(4)} = A_4 = \{\langle[0.19, 0.23], [0.17, 0.33], [0.39, 0.42]\rangle, \langle 0.52, 0.08, 0.38\rangle\}$. Then

$$CPFOWA\left(A_{\eta(1)},A_{\eta(2)},A_{\eta(3)},A_{\eta(4)}\right)=\left\{\begin{array}{c}\left\{\begin{array}{c}\left[\begin{array}{c}1-(1-0.32)^{0.2}(1-0.22)^{0.3}(1-0.29)^{0.1}(1-0.19)^{0.4},\\ 1-(1-0.53)^{0.2}(1-0.40)^{0.3}(1-0.43)^{0.1}(1-0.23)^{0.4}\end{array}\right],\\ \left[\begin{array}{c}(0.09)^{0.2}(0.12)^{0.3}(0.24)^{0.1}(0.17)^{0.4},\\ (0.15)^{0.2}(0.10)^{0.3}(0.38)^{0.1}(0.33)^{0.4}\end{array}\right],\\ \left[\begin{array}{c}(0.22)^{0.2}(0.28)^{0.3}(0.06)^{0.1}(0.39)^{0.4},\\ (0.31)^{0.2}(0.39)^{0.3}(0.18)^{0.1}(0.42)^{0.4}\end{array}\right]\end{array}\right\},\\ \left\{\begin{array}{c}1-(1-0.48)^{0.2}(1-0.44)^{0.3}(1-0.37)^{0.1}(1-0.52)^{0.4},\\ (0.13)^{0.2}(0.23)^{0.3}(0.29)^{0.1}(0.08)^{0.4},\\ (0.36)^{0.2}(0.31)^{0.3}(0.33)^{0.1}(0.38)^{0.4}\end{array}\right\}\end{array}\right\}.$$

$$=\left\{\left\langle[0.2367,0.3718],[0.1395,0.1998],[0.2611,0.3551]\right\rangle,\left\langle 0.4750,0.1376,0.3486\right\rangle\right\}.$$

Definition 4.7. Let $A_k=\left\langle £_{A_k},e_{A_k}\right\rangle,\ k\in N$ be a collection of CPFNs, and $CPFHWA\ :\ CPFN^n\rightarrow CPFN,$ then the CPFHWA describe as,

$$CPFHWA_{\tau,\omega}(A_1,A_2,...A_n)=\sum_{k=1}^{n}\tau_k A'_{\eta(k)},$$

with dimensions n, where kth biggest weighted value is $A_{\eta(k)}$ and $A'_k\left(A'_k=n\tau_k A_k,\ k\in N\right),\ \tau=\{\tau_1,\tau_2,...,\tau_n\}$ are the weight vectors such that $\tau_k\geq 0\ (k\in N)$ and $\Sigma_{k=1}^{n}\tau_k=1$. Also $\omega=(\omega_1,\omega_2,...,\omega_n)$ are the associated weight vector such that $\omega_k\geq 0$ $(k\in N)$ and $\Sigma_{k=1}^{n}\omega_k=1$, and balancing coefficient is n.

Theorem 4.8. Let $A_k=\left\langle £_{A_k},e_{A_k}\right\rangle,\ k\in N$ be a collection of CPFNs. Then by utilizing the definition 4.7 and operational properties of CPFNs, we can obtain the following outcome.

$$CPFHWA_w(A_1,A_2,...A_n)=\left\{\begin{array}{c}\left\{\begin{array}{c}\left[s^{-1}\left(\sum_{k=1}^{n}\tau_k s\left(£^-_{P_{A'_{\eta(k)}}}\right)\right),\ s^{-1}\left(\sum_{k=1}^{n}\tau_k s\left(£^+_{P_{A'_{\eta(k)}}}\right)\right)\right],\\ \left[t^{-1}\left(\sum_{k=1}^{n}\tau_k t\left(£^-_{I_{A'_{\eta(k)}}}\right)\right),\ t^{-1}\left(\sum_{k=1}^{n}\tau_k t\left(£^+_{I_{A'_{\eta(k)}}}\right)\right)\right],\\ \left[t^{-1}\left(\sum_{k=1}^{n}\tau_k t\left(£^-_{N_{A'_{\eta(k)}}}\right)\right),\ t^{-1}\left(\sum_{k=1}^{n}\tau_k t\left(£^+_{P_{A'_{\eta(k)}}}\right)\right)\right]\end{array}\right\},\\ \left\{\begin{array}{c}s^{-1}\left(\sum_{k=1}^{n}\tau_k s\left(P_{A'_{\eta(k)}}\right)\right),t^{-1}\left(\sum_{k=1}^{n}\tau_k t\left(I_{A'_{\eta(k)}}\right)\right),\\ t^{-1}\left(\sum_{k=1}^{n}\tau_k t\left(N_{A'_{\eta(k)}}\right)\right)\end{array}\right\}\end{array}\right\}$$

Where $\tau=\{\tau_1,\tau_2,...,\tau_n\}$ be the weight vector of $A_k=\left\langle £_{A_k}, e_{A_k}\right\rangle,\ k\in N$ with $\tau_k\geq 0$ and $\sum_{k=1}^{n}\tau_k=1.$

If we allocate value to generators t and s as in Algebraic strict Archimedean t-norm and t-conorm, then this particular operations of CPFNs will be attained as,

$$CPFHWA_w(A_1,A_2,...A_n)=\left\{\begin{array}{c}\left\{\begin{array}{c}\left[\begin{array}{c}1-\Pi_{k=1}^{n}(1-£^{-}_{P_{A'_{\eta(k)}}})^{\omega_k},\\ 1-\Pi_{k=1}^{n}(1-£^{+}_{P_{A'_{\eta(k)}}})^{\omega_k}\end{array}\right],\\ \left[\Pi_{k=1}^{n}(£^{-}_{I_{A'_{\eta(k)}}})^{\omega_k},\ \Pi_{k=1}^{n}(£^{+}_{I_{A'_{\eta(k)}}})^{\omega_k}\right],\\ \left[\Pi_{k=1}^{n}(£^{-}_{N_{A'_{\eta(k)}}})^{\omega_k},\ \Pi_{k=1}^{n}(£^{+}_{N_{A'_{\eta(k)}}})^{\omega_k}\right]\end{array}\right\},\\ \left\{\begin{array}{c}1-\Pi_{k=1}^{n}(1-P_{A'_{\eta(k)}})^{\omega_k},\Pi_{k=1}^{n}(I_{A'_{\eta(k)}})^{\omega_k},\\ \Pi_{k=1}^{n}(N_{A'_{\eta(k)}})^{\omega_k}\end{array}\right\}\end{array}\right\}.$$

Where $A_{\eta(k)}$ is kth biggest value consequently by total order $A_{\eta(1)}\geq A_{\eta(2)}\geq...\geq A_{\eta(n)}$ and kth biggest weighted value is $A'_{\eta(k)},\ A'_k\left(A'_k=n\tau_k A_k,\ k\in N\right).$

Proof: Theorem 4.8 take the form by utilized the technique of mathematical induction on n. Procedure is eliminating here.

Remark: There are some properties which are fulfilled by the CPFHWA operator obviously.

(1) Idempotency: Let $A_k=\left\langle £_{A_k}, e_{A_k}\right\rangle,\ k\in N$ be any collection of CPFNs. If all of $A_k=\left\langle £_{A_k}, e_{A_k}\right\rangle,\ k\in N$ are identical. Then there is

$$CPFHWA\left(A_1,A_2,...,A_n\right)=A.$$

(2) Boundedness: Let $A_k=\left\langle £_{A_k}, e_{A_k}\right\rangle,\ k\in N$ be any collection of CPFNs. Let $A_k^-=\left\{\left\{\min_k £_{P_{A_k}},\min_k £_{I_{A_k}},\max_k £_{N_{A_k}}\right\},\left\{\min_k P_{A_k},\min_k I_{A_k},\max_k N_{A_k}\right\}\right\}$ and $A_k^+=\left\{\left\{\max_k £_{P_{A_k}},\min_k £_{I_{A_k}},\min_k £_{N_{A_k}}\right\},\left\{\max_k P_{A_k},\min_k I_{A_k},\min_k N_{A_k}\right\}\right\}$ for whole $k\in N,$ therefore

$$A_k^-\subseteq CPFHWA\left(A_1,A_2,...,A_n\right)=A_k^+.$$

(3) Monotonically: Let $B_k=\left\langle P_{b_k}, I_{b_k}, N_{b_k}\right\rangle,\ k\in N$ be any collection of CPFNs. If it satisfies that $A_k\subseteq B_k$ for whole, $k\in N,$ then

$$CPFHWA\left(A_1,A_2,...,A_n\right)=CPFHWA\left(B_1,B_2,...,B_n\right).$$

Example 4.9. Suppose that $A_1 = \langle £_{A_1}, e_{A_1} \rangle$, $A_2 = \langle £_{A_2}, e_{A_2} \rangle$, $A_3 = \langle £_{A_3}, e_{A_3} \rangle$ and $A_4 = \langle £_{A_4}, e_{A_4} \rangle$ are cubic picture fuzzy sets in R, define as

$$A_1 = \{\langle [0.32, 0.53], [0.09, 0.15], [0.22, 0.31] \rangle, \langle 0.48, 0.13, 0.36 \rangle\},$$

$$A_2 = \{\langle [0.19, 0.23], [0.17, 0.33], [0.39, 0.42] \rangle, \langle 0.52, 0.08, 0.38 \rangle\},$$

$$A_3 = \{\langle [0.29, 0.43], [0.11, 0.14], [0.06, 0.08] \rangle, \langle 0.32, 0.14, 0.09 \rangle\},$$

$$A_4 = \{\langle [0.22, 0.40], [0.12, 0.10], [0.28, 0.39] \rangle, \langle 0.44, 0.23, 0.31 \rangle\}$$

and let $\tau = (0.35, 0.25, 0.10, 0.30)^T$ are the weight vector of $A_k\ (k = 1,2,3,4)$. If we utilize the CPFHWA operator to aggregate the given data, then we first determine the weighting vector $\omega = (0.224, 0.236, 0.304, 0.236)^T$ associated weight vector of $A_k'\ (k = 1,2,3,4)$. Then by the Definition 4.4, we have

$$A_1' = \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left[1-(1-0.32)^{4\times 0.35}, 1-(1-0.53)^{4\times 0.35}\right], \\ \left[(0.09)^{4\times 0.35}, (0.15)^{4\times 0.35}\right], \\ \left[(0.22)^{4\times 0.35}, (0.31)^{4\times 0.35}\right] \end{array} \right\}, \\ \left\{1-(1-0.48)^{4\times 0.35}, (0.13)^{4\times 0.35}, (0.36)^{4\times 0.35}\right\} \end{array} \right\}$$

$$= \{\{[0.417, 0.652], [0.034, 0.070], [0.120, 0.194]\}, \{0.599, 0.057, 0.239\}\}$$

Similarly, we can calculate

$$A_2' = \{\{[0.190, 0.230], [0.170, 0.330], [0.390, 0.420]\}, \{0.520, 0.080, 0.380\}\}$$

$$A_3' = \{\{[0.128, 0.201], [0.413, 0.455], [0.324, 0.364]\}, \{0.142, 0.455, 0.381\}\}$$

$$A_4' = \{\{[0.257, 0.458], [0.078, 0.063], [0.217, 0.323]\}, \{0.501, 0.171, 0.245\}\}$$

Then utilizing Definition 3.39 to calculate the score values of $A_k'\ (k = 1,2,3,4)$;

$$sc(A_1') = 0.492, sc(A_2') = 0.195,$$
$$sc(A_3') = 0.013, sc(A_4') = 0.353$$

Therefore

$$sc(A_1') > sc(A_4') > sc(A_2') > sc(A_3').$$

Then

$$A_1' > A_4' > A_2' > A_3'$$

and hence

$A'_{\eta(1)} = A'_1 = \{\{[0.417, 0.652], [0.034, 0.070], [0.120, 0.194]\}, \{0.599, 0.057, 0.239\}\},$

$A'_{\eta(2)} = A'_4 = \{\{[0.257, 0.458], [0.078, 0.063], [0.217, 0.323]\}, \{0.501, 0.171, 0.245\}\},$

$A'_{\eta(3)} = A'_2 = \{\{[0.190, 0.230], [0.170, 0.330], [0.390, 0.420]\}, \{0.520, 0.080, 0.380\}\},$

$A'_{\eta(4)} = A'_3 = \{\{[0.128, 0.201], [0.413, 0.455], [0.324, 0.364]\}, \{0.142, 0.455, 0.381\}\}.$ Then

$$CPFHOWA\left(A'_{\eta(1)}, A'_{\eta(2)}, A'_{\eta(3)}, A'_{\eta(4)}\right) = \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left[\begin{array}{c} 1-(1-0.417)^{0.224}(1-0.257)^{0.236}(1-0.190)^{0.304}(1-0.128)^{0.236}, \\ 1-(1-0.652)^{0.224}(1-0.458)^{0.236}(1-0.230)^{0.304}(1-0.201)^{0.236} \end{array} \right], \\ \left[\begin{array}{c} (0.034)^{0.224}(0.078)^{0.236}(0.170)^{0.304}(0.413)^{0.236}, \\ (0.070)^{0.224}(0.063)^{0.236}(0.330)^{0.304}(0.455)^{0.236} \end{array} \right], \\ \left[\begin{array}{c} (0.120)^{0.224}(0.217)^{0.236}(0.390)^{0.304}(0.324)^{0.236}, \\ (0.194)^{0.224}(0.323)^{0.236}(0.420)^{0.304}(0.364)^{0.236} \end{array} \right] \end{array} \right\}, \\ \left\{ \begin{array}{c} 1-(1-0.599)^{0.224}(1-0.501)^{0.236}(1-0.520)^{0.304}(1-0.142)^{0.236}, \\ (0.057)^{0.224}(0.171)^{0.236}(0.080)^{0.304}(0.455)^{0.236}, \\ (0.239)^{0.224}(0.245)^{0.236}(0.380)^{0.304}(0.381)^{0.236} \end{array} \right\} \end{array} \right\}.$$

$$= \{\langle [0.249, 0.401], [0.121, 0.170], [0.249, 0.321] \rangle, \langle 0.466, 0.133, 0.308 \rangle\}.$$

5. MADM METHOD UTILIZING CUBIC PICTURE FUZZY AGGREGATION OPERATORS

This section proposes the technique to solve the MADM problems by utilizing the Cubic picture fuzzy aggregation operators. For a MADM problem, assuming that $C = \{c_1, c_2, ..., c_m\}$ be any finite collection of m alternatives, $G = \{g_1, g_2, ..., g_n\}$ be any finite collection of n attributes. Let

$$A = [A_{jk}] = \left[\left\langle £_{A_{jk}}, e_{A_{jk}} \right\rangle\right]_{m \times n} = \begin{array}{c} \\ c_1 \\ c_2 \\ \vdots \\ c_m \end{array} \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ \left\langle £_{A_{11}}, e_{A_{11}} \right\rangle & \left\langle £_{A_{12}}, e_{A_{12}} \right\rangle & \cdots & \left\langle £_{A_{1n}}, e_{A_{1n}} \right\rangle \\ \left\langle £_{A_{21}}, e_{A_{21}} \right\rangle & \left\langle £_{A_{22}}, e_{A_{22}} \right\rangle & \cdots & \left\langle £_{A_{2n}}, e_{A_{2n}} \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle £_{A_{m1}}, e_{A_{m1}} \right\rangle & \left\langle £_{A_{m2}}, e_{A_{m2}} \right\rangle & \cdots & \left\langle £_{A_{mn}}, e_{A_{mn}} \right\rangle \end{pmatrix}$$

be the decision matrices (DMs), where $\left\langle £_{A_{jk}}, e_{A_{jk}} \right\rangle$ are the collection of CPFNs and represents the evaluation information of every alternative $c_i\,(i = 1, 2, ..., m)$ on attribute $g_i\,(i = 1, 2, ..., n)$.

If $\tau = \{\tau_1, \tau_2, ..., \tau_n\}$ are the weight vector of attribute, with $\tau_k \geq 0$ and $\sum_{k=1}^{n} \tau_k = 1$. Then, listed below the main technique of handling the MADM problems:

STEP-1. Normalized the given Decision Matrix.

Normalize these collective information decision matrices by transforming the rating values of cost type into benefit type, if any, by using the normalization formula:

$$A_{jk}^{c} = \left\langle £_{A_{jk}}^{c}, e_{A_{jk}}^{c} \right\rangle = \left\{ r, \left\langle £_{N_{A_{jk}}}(r), £_{I_{A_{jk}}}(r), £_{P_{A_{jk}}}(r) \right\rangle, \left\langle N_{A_{jk}}(r), I_{A_{jk}}(r), P_{A_{jk}}(r) \right\rangle \mid r \in R \right\}.$$

Where A_{jk}^{c} is the complement of A_{jk}.

STEP-2.

We accumulate the CPFNs for every decision maker which are given. Aggregate the cubic picture fuzzy numbers by utilizing the proposed cubic picture fuzzy weighted averaging aggregation operator separately for each alternative as follows

$$r_i = CPFWA(\sigma_{i1}, \sigma_{i2}, ..., \sigma_{in}) = \left\{ \begin{array}{l} s^{-1}\left(\tau_1 s\left(P_{\sigma_{i1}}\right) + \tau_2 s\left(P_{\sigma_{i2}}\right) + ... + \tau_n s\left(P_{\sigma_{in}}\right)\right), \\ t^{-1}\left(\tau_1 t\left(I_{\sigma_{i1}}\right) + \tau_2 t\left(I_{\sigma_{i2}}\right) + ... + \tau_n t\left(I_{\sigma_{in}}\right)\right), \\ t^{-1}\left(\tau_1 t\left(N_{\sigma_{i1}}\right) + \tau_2 t\left(N_{\sigma_{i2}}\right) + ... + \tau_n t\left(N_{\sigma_{in}}\right)\right) \end{array} \right\}$$

STEP-3.

Compute the score of the aggregated CPFN r_i, using Definition 3.39.

$$sc(A) = \frac{\left(£_{P_A}^{-} + £_{P_A}^{+} - £_{I_A}^{-} - £_{I_A}^{+} - £_{N_A}^{-} - £_{N_A}^{+}\right) + \left(P_{e_k} + 1 - I_{e_k} + 1 - N_{e_k}\right)}{6}$$

STEP-4.

According to Definition 3.40, ranks to alternatives $a_i (i = 1, 2, ..., m)$ with the order of their score and accuracy value to choose the best one option.

5.1. An Application to a Petroleum Circulation Center Evaluation Problem

In this section, the proposed ranking method is applied to deal with circulation center evaluation problem. Consider a committee of decision-maker to perform the evaluation and to select the most suitable circulation center, among the three circulation centers G_1, G_2 and G_3. The decision maker evaluates the circulation center according to four attributes, which are given as follows:

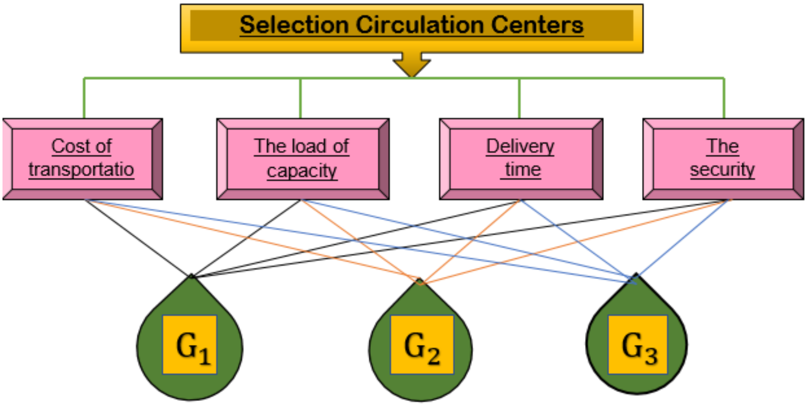
1) Cost of transportation (C_1),

2) the load of capacity (C_2),

3) the satisfy demand with minimum delay (C_3), and

4) the security (C_4).

The hierarchy structure of the decision problem is shown in Fig. (below)



and the average weights of the attributes from the decision-making committee that are $\tau = (0.224, 0.236, 0.304, 0.236)^T$. Now we enlist the suitability ratings of alternatives versus the four attributes are presented. According to the suitability ratings of three alternatives, G_1, G_2 and G_3 under four attributes C_1, C_2, C_3 and C_4 can be obtained as shown in below table.

petroleum circulation center information

	(C_1)
G_1	⟨([0.32, 0.53], [0.09, 0.15], [0.22, 0.31]), (0.48, 0.13, 0.36)⟩
G_2	⟨([0.19, 0.23], [0.17, 0.33], [0.39, 0.42]), (0.52, 0.08, 0.38)⟩
G_3	⟨([0.22, 0.40], [0.12, 0.10], [0.28, 0.39]), (0.44, 0.23, 0.31)⟩
	(C_2)
G_1	⟨([0.20, 0.30], [0.05, 0.16], [0.39, 0.41]), (0.35, 0.08, 0.40)⟩
G_2	⟨([0.11, 0.28], [0.29, 0.39], [0.15, 0.25]), (0.13, 0.29, 0.24)⟩
G_3	⟨([0.33, 0.39], [0.09, 0.18], [0.27, 0.29]), (0.26, 0.27, 0.28)⟩
	(C_3)
G_1	⟨([0.20, 0.30], [0.15, 0.26], [0.39, 0.41]), (0.51, 0.08, 0.40)⟩
G_2	⟨([0.14, 0.28], [0.29, 0.39], [0.18, 0.28]), (0.13, 0.43, 0.24)⟩
G_3	⟨([0.41, 0.43], [0.09, 0.18], [0.27, 0.33]), (0.26, 0.27, 0.28)⟩
	(C_4)
G_1	⟨([0.39, 0.41], [0.15, 0.26], [0.20, 0.30]), (0.40, 0.08, 0.51)⟩
G_2	⟨([0.18, 0.28], [0.29, 0.39], [0.14, 0.28]), (0.24, 0.43, 0.13)⟩
G_3	⟨([0.27, 0.33], [0.09, 0.18], [0.41, 0.43]), (0.28, 0.27, 0.26)⟩

STEP-1.

Since the attributes are uniform so there is no need to normalize.

STEP-2.

Operate the aggregation operator to evaluate DMs of cubic picture fuzzy information.

Case:1 (Utilizing CPFWA)

	petroleum circulation center information
G_1	$\{\langle[0.276, 0.385],[0.103, 0.0.204],[0.293, 0.357]\rangle,\langle 0.443, 0.089, 0.413\rangle\}$
G_2	$\{\langle[0.154, 0.269],[0.257, 0.375],[0.193, 0.298]\rangle,\langle 0.262, 0.268, 0.230\rangle\}$
G_3	$\{\langle[0.319, 0.391],[0.095, 0.157],[0.300, 0.353]\rangle,\langle 0.309, 0.260, 0.281\rangle\}$

Case:2 (Utilizing CPFOWA)

	petroleum circulation center information
G_1	$\{\langle[0.367, 0.440],[0.101, 0.188],[0.240, 0.308]\rangle,\langle 0.349, 0.172, 0.341\rangle\}$
G_2	$\{\langle[0.221, 0.330],[0.097, 0.171],[0.366, 0.410]\rangle,\langle 0.408, 0.135, 0.341\rangle\}$
G_3	$\{\langle[0.154, 0.269],[0.257, 0.375],[0.193, 0.298]\rangle,\langle 0.262, 0.268, 0.230\rangle\}$

Case:3 (Utilizing CPFHWA)

	petroleum circulation center information
G_1	$\{\langle[0.289, 0.261],[0.173, 0.276],[0.342, 0.413]\rangle,\langle 0.288, 0.272, 0.436\rangle\}$
G_2	$\{\langle[0.168, 0.264],[0.180, 0.260],[0.360, 0.407]\rangle,\langle 0.339, 0.211, 0.448\rangle\}$
G_3	$\{\langle[0.119, 0.207],[0.350, 0.473],[0.305, 0.410]\rangle,\langle 0.220, 0.346, 0.347\rangle\}$

STEP-3.

Now we find out the score and accuracy function respectively by utilizing the Definition 3.39 as,

Case:1

$sc(G_1) = 0.274$
$sc(G_2) = 0.177$
$sc(G_3) = 0.262$

Case:2

$sc(G_1) = 0.301$
$sc(G_2) = 0.239$
$sc(G_3) = 0.177$

Case:3

$sc(G_1) = 0.171$
$sc(G_2) = 0.116$
$sc(G_3) = 0.052$

STEP-4.

Now rank the all the alternative by utilizing the comparison technique in Definition 3.40 for aggregation operator tabular representation is given below,

Calculation result of each petroleum circulation center		
	Score	Ranking
CPFWA	$sc(G_1) > sc(G_3) > sc(G_2)$	$G_1 > G_3 > G_2$
CPFOWA	$sc(G_1) > sc(G_2) > sc(G_3)$	$G_1 > G_2 > G_3$
CPFHOWA	$sc(G_1) > sc(G_2) > sc(G_3)$	$G_1 > G_2 > G_3$

5.2. Advantages and Comparison of the Proposed Cubic Picture Fuzzy Sets

(1) Although to solve MADM problem in different areas, PFS theory has been profitably tested, some situations in real life PFSs are not applicable. In those situations, we used the CPFSs, which are an extension of IFSs, so the CPFS is most often better than the IFS and PFS. In some situations, the PFS cannot solve a problem which CPFS can solve, e.g., if a DM gives information about positive, neutral and negative membership grades in interval and in single value, then this type of problem is only valid for the CPFS. Particularly, CPFS is more capable of handling uncertain problems.

(2) In this paper, we use the cubic picture fuzzy information and proposed a technique to making decisions in complex real-life problems. Numerical examples proposed in this paper cannot be handled with pre-existing structures such as the fuzzy sets, intuitionistic fuzzy sets, cubic sets and picture fuzzy sets. So, our proposed technique is a generalization of the pre-existing structure of fuzzy sets to deal with real-life decision-making problems more smoothly.

6. CONCLUSION

Aggregation of information is an important technique to tackles an MADM problems. Many aggregation operators for different sets have been introduced. In this Paper, we have established the cubic picture fuzzy sets, a generalization of fuzzy sets whose combination of IVPFSs and PFSs. Cuong's construction of picture fuzzy sets is of prodigious reputation but decision makers are somehow restricting in assigning values due to the condition on $P(x)$ $I(x)$ and $N(x)$. Dealing with such kind of circumstances, we propose a new structure by defining cubic picture fuzzy sets. We have investigated some basic operations and properties and proposed an extension principle of CPFS. Since in decision making problems aggregation operators play a vital role, therefore in this paper we introduced some aggregation operators based on the of the idea of t-norms and t-conorms, namely the Cubic picture fuzzy weighted average (CPFWA) operator, Cubic picture fuzzy order weighted average (CPFOWA) operator and Cubic picture fuzzy hybrid order weighted average (CPFHOWA) operator. After that, an approach for ranking the cubic picture fuzzy numbers is proposed to illustrate the usage, applicability, and advantages of the proposed ranking approach, it has been applied for evaluating the circulation center of petroleum as an applicable problem. We have outlined a practical example about of an arrangement of drive frameworks to check the created approach and to show its common uses and adaptability as compare to conventional methods.

Compliance with Ethical Standards

Conflict of Interest: The authors declare that they have no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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