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Research Article

QUINTIC B-SPLINE METHOD FOR NUMERICAL SOLUTION OF THE ROSENAU-BURGERS EQUATION

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ABSTRACT

In this paper, the quintic B–spline method is employed to calculatenumerical solution of the initial-boundary value problem of Rosenau–Burgersequation. This scheme is based on the Crank–Nicolson formulation for time integration and quintic B–spline functions for space integration. The unconditional stability of the method is proved using Von–Neumann approach. A priori bound and the error estimates of the approximate solutions are discussed with a numerical example.
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1. INTRODUCTION

One of the most important nonlinear partial differential equations (PDEs) is Korteweg–de Vries (KdV) equation, which describes the vibrations of a uni–dimensional inharmonic lattice associated with the birth of the soliton. But KdV equation does not represent wave to wave interaction and wave to wall interaction. To overcome this shortcoming of the KDV equation, Rosenau [4, 5] proposed the so-called Rosenau equation:

$$u_t + u_{xxxxt} + u_x + uu_x = 0, \qquad x \in \Omega,\ t \in (0,T], \tag{1.1}$$

There are some articles and some collected works that has been focused to study the classical Rosenau equation from various points of view. M. A. Park [3] was proved he existence and the uniqueness of the solution for (1.1). However, by this time, the analytical solution for (1.1) is unknown. Since then, much work has been done on the numerical solution of (1.1) ([6, 7, 8, 9] and also the references therein). On the other hand, recently and more for the further onsideration of the nonlinear wave, by adding the viscous term u_{xx} , the Rosenau equation (1.1) leads to

$$u_t + u_{xxxxt} + u_x + uu_x - u_{xx} = 0, \qquad x \in \Omega,\ t \in (0,T], \tag{1.2}$$

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which is usually called the Rosenau–Burgers equation. There are a great of work that has been studied about the Cauchy problem of Rosenau–Burgers equation [1, 2, 10, 11]. While there are a few works that has been devoted to approximate the numerical solutions to the initial-boundary value problem of Rosenau–Burgers equation. In this paper, a B-spline algorithm based on the collocation method with trial functions taken as quintic B-spline functions over the elements will be constructed. The present algorithm will be used first to model the Rosenau–Burgers equation (1.2) and then its results will implement to approximate the numerical solution of (1.2) with the boundary conditions

$$u(x,t)=u_{xx}(x,t)=0, \quad x\in\partial\Omega,\ t\in(0,T], \tag{1.3}$$

and an initial condition

$$u(x,0)=u_0(x), \quad x\in\overline{\Omega}, \tag{1.4}$$

where $u_0(x)$ is sufficiently smooth and satisfies the compatibility condition, $\Omega=(0,L), L>0,$ and $0<T<+\infty$. For more physical significance of the Rsenaue–Burgers equation (1.2), we refer to Rosenau [1, 2, 10, 11].

The quintic B-spline basis has been used to approximate numerical solutions for some nonlinear differential equations. For instance, numerical solution of the Burger equation has been found by quintic B-spline collocation method in [1]. An algorithm based on quintic B-spline Galerkin method was devoted to obtain the solutions of the RLW equation in [2]. Numerical solutions of the KdV-Burgers equation and Korteweg–de Vries (KdV) equation was obtained using collocation of quintic B-spline interpolation functions over finite elements in [10, 11], respectively. The Kuramoto-Sivashinsky equation is also approximated by quintic B-spline in [12].

The organization of this paper is as follows. In Section 2, quintic B-spline collocation scheme is explained. In Sections 3, the quintic B-spline collocation method is applied to the Rosenau–Burgers equation (1.2). In Section 4, the stability analysis of the method is discussed. In Section 5, one examples are presented. Also the global relative error at different time is obtained for the example. A summary about overall the present work is given at the end of the paper in Section 6.

2. DESCRIPTION OF THE QUINTIC B-SPLINE METHOD

The solution domain $0\le x\le 1$ is partitioned in to a mesh of uniform length $h=x_{i+1}-x_i$, by the knots x_i where $i=0,1,2,\cdots,N$ such that $0=x_0<x_1<\cdots<x_{n-1}<x_N=1$. Our numerical treatment for Rosenau-Burgers equation using the collocation method with quintic B-spline is to find an approximate solution $U_N(x,t)$ to the exact solution $u(x,t)$ in the form:

$$U_N(x,t)=\sum_{i=-2}^{N+2}\delta_i(t)B_i(x), \tag{2.1}$$

where $\delta_i(t)$ are time-dependent quantities to be determined from the boundary conditions and collocation form of the differential equations, and $B_i(x)$ are the quintic B-spline basis functions at knots, given by [13, 14, 15].

$$B_i(x)=\frac{1}{h^5}\begin{cases}(x-x_{i-3})^5, & x\in[x_{i-3},x_{i-2})\\ (x-x_{i-3})^5-6(x-x_{i-2})^5, & x\in[x_{i-2},x_{i-1})\\ (x-x_{i-3})^5-6(x-x_{i-2})^5+15(x-x_{i-1})^5, & x\in[x_{i-1},x_i)\\ (x_{i+3}-x)^5-6(x_{i+2}-x)^5+15(x_{i-1}-x)^5, & x\in[x_i,x_{i+1})\\ (x_{i+3}-x)^5-6(x_{i+2}-x)^5, & x\in[x_{i+1},x_{i+2})\\ (x_{i+3}-x)^5, & x\in[x_{i+2},x_{i+3})\\ 0, & \text{otherwise}\end{cases}, \tag{2.2}$$

where $\{B_{-2},B_{-1},B_0,B_1,B_2,...,B_{N+1},B_{N+2}\}$ forms a basis over the region $0\le x\le 1$. Each quintic B-spline covers six elements so that an element is covered by six quintic B-splines [16]. Over the element $[x_m,x_{m+1}]$ the variation of the function $U(x,t)$ is formed from

$$U(x,t)=\sum_{j=m-2}^{m+3}\delta_j(t)B_j(x), \tag{2.3}$$

In terms of a local coordinate system ξ given by $h\xi=x-x_m$, where $h=x_{m+1}-x_m$ and $0\le\xi\le 1$, expressions for the element splines are [10]

$$\begin{aligned}B_{m-2}(x)&=1-5\xi+10\xi^2-10\xi^3+5\xi^4-\xi^5,\\ B_{m-1}(x)&=26-50\xi+20\xi^2+20\xi^3-20\xi^4+5\xi^5,\\ B_m(x)&=66-60\xi^2+30\xi^4-10\xi^5,\\ B_{m+1}(x)&=26+50\xi+20\xi^2-20\xi^3-20\xi^4+10\xi^5,\\ B_{m+2}(x)&=1+5\xi+10\xi^2+10\xi^3+5\xi^4-5\xi^5,\\ B_{m+3}(x)&=\xi^5.\end{aligned} \tag{2.4}$$

Using approximate function (2.1) and quintic spline (2.2), the approximate values at the knots of $U(x)$ and its derivatives up to fourth order are determined in terms of the time parameters δ_m as

$$\begin{aligned}U_m&=\delta_{m+2}+26\delta_{m+1}+66\delta_m+26\delta_{m-1}+\delta_{m-2},\\ hU'_m&=5(\delta_{m+2}+10\delta_{m+1}-10\delta_{m-1}-\delta_{m-2}),\\ h^2U''_m&=20(\delta_{m+2}+2\delta_{m+1}-6\delta_m+2\delta_{m-1}+\delta_{m-2}),\\ h^3U'''_m&=60(\delta_{m+2}-2\delta_{m+1}+2\delta_{m-1}-\delta_{m-2}),\\ h^4U^{(iv)}_m&=120(\delta_{m+2}-4\delta_{m+1}+6\delta_m-4\delta_{m-1}+\delta_{m-2}),\end{aligned} \tag{2.5}$$

where dashes represent differentiation with respect to space variable.

3. SOLUTION OF ROSENAU–BURGERS EQUATION

The Rosenau–Burgers equation can be rewritten as

$$(u+u_{xxxx})_t+uu_x+u_x-u_{xx}=0, \quad (x,t)\in[0,1]\times(0,T] \tag{3.1}$$

with the boundary conditions

$$\begin{aligned}&u(0,t)=u(1,t)=0,\\&u_{xx}(0,t)=u_{xx}(1,t)=0,\end{aligned} \tag{3.2}$$

and initial condition

$$u(x,0)=u_0(x), \tag{3.3}$$

We discrete the time derivative of Eq. (3.1) by a first order accurate forward difference formula and apply the θ-weighted scheme, $(0\le\theta\le1)$, to the space derivative at two adjacent time levels to obtain the equation

$$\begin{aligned}&\frac{\left(U^{n+1}+(U_{xxxx})^{n+1}\right)-\left(U^{n}+(U_{xxxx})^{n}\right)}{k}+\theta\left\{(UU_x)^{n+1}+(U_x)^{n+1}-(U_{xx})^{n+1}\right\}\\&+(1-\theta)\left\{(UU_x)^{n}+(U_x)^{n}-(U_{xx})^{n}\right\}=0,\end{aligned} \tag{3.4}$$

where k is time step and the superscripts n and $n+1$ are successive time levels. In this work we take $\theta=\frac{1}{2}$, . Hence, Eq. (3.4) takes the form

$$\begin{aligned}&\frac{\left(U^{n+1}+(U_{xxxx})^{n+1}\right)-\left(U^{n}+(U_{xxxx})^{n}\right)}{k}+\frac{(UU_x)^{n+1}+(UU_x)^{n}}{2}+\frac{(U_x)^{n+1}+(U_x)^{n}}{2}\\&-\frac{(U_{xx})^{n+1}+(U_{xx})^{n}}{2}=0,\end{aligned} \tag{3.5}$$

The nonlinear term in Eq. (3.5) is approximated by the following formula based on Taylor series:

$$(UU_x)^{n+1}=U^{n+1}(U_x)^{n}+U^{n}(U_x)^{n+1}-(UU_x)^{n}, \tag{3.6}$$

Putting values from Eq. (3.6) in Eq. (3.5) we get,

$$\begin{aligned}&\frac{\left(U^{n+1}+(U_{xxxx})^{n+1}\right)-\left(U^{n}+(U_{xxxx})^{n}\right)}{k}+\frac{U^{n+1}(U_x)^{n}+U^{n}(U_x)^{n+1}}{2}+\frac{(U_x)^{n+1}+(U_x)^{n}}{2}\\&-\frac{(U_{xx})^{n+1}+(U_{xx})^{n}}{2}=0,\end{aligned} \tag{3.7}$$

Rearranging the terms and simplifying we get,

$$U^{n+1}+(U_{xxxx})^{n+1}+\frac{k}{2}\left\{U^{n+1}(U_x)^n+U^n(U_x)^{n+1}+(U_x)^{n+1}-(U_{xx})^{n+1}\right\}$$
$$=U^n+(U_{xxxx})^n-\frac{k}{2}(U_x)^n+\frac{k}{2}(U_{xx})^n, \tag{3.8}$$

Substituting the approximate solution U for u and putting the values of the nodal values U, its derivatives using Eqs. (2.5) at the knots in Eq. (3.8) yields the following difference equation with the variables δ_i and for $m=0,1,2,...,N$:

$$C_2\delta_{m+2}^{n+1}+C_1\delta_{m+1}^{n+1}+C_0\delta_m^{n+1}+C_{-1}\delta_{m-1}^{n+1}+C_{-2}\delta_{m-2}^{n+1}=\overline{C}_2\delta_{m+2}^n+\overline{C}_1\delta_{m+1}^n+\overline{C}_0\delta_m^n+\overline{C}_{-1}\delta_{m-1}^n+\overline{C}_{-2}\delta_{m-2}^n, \tag{3.9}$$

where

$$\begin{aligned}
C_2&=\left(1+\frac{k}{2}U_x^n\right)+\frac{120}{h^4}+\frac{5}{2}\frac{k}{h}(U^n+1)-10\frac{k}{h^2},\\
C_1&=26\left(1+\frac{k}{2}U_x^n\right)-4\left(\frac{120}{h^4}\right)+25\frac{k}{h}(U^n+1)-20\frac{k}{h^2},\\
C_0&=66\left(1+\frac{k}{2}U_x^n\right)+6\left(\frac{120}{h^4}\right)+60\frac{k}{h^2},\\
C_{-1}&=26\left(1+\frac{k}{2}U_x^n\right)-4\left(\frac{120}{h^4}\right)-25\frac{k}{h}(U^n+1)-20\frac{k}{h^2},\\
C_{-2}&=\left(1+\frac{k}{2}U_x^n\right)+\frac{120}{h^4}-\frac{5}{2}\frac{k}{h}(U^n+1)-10\frac{k}{h^2},
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\overline{C}_2&=1+\frac{120}{h^4}-\frac{5}{2}\frac{k}{h}+10\frac{k}{h^2},\\
\overline{C}_1&=26-4\left(\frac{120}{h^4}\right)-25\frac{k}{h}+20\frac{k}{h^2},\\
\overline{C}_0&=66+6\left(\frac{120}{h^4}\right)-66\frac{k}{h^2},\\
\overline{C}_{-1}&=26-4\left(\frac{120}{h^4}\right)+25\frac{k}{h}+20\frac{k}{h^2},\\
\overline{C}_{-2}&=1+\frac{120}{h^4}+\frac{5}{2}\frac{k}{h}+10\frac{k}{h^2},
\end{aligned} \tag{3.11}$$

The systems (3.10)-(3.11) consists of $(N+1)$ linear equations in $(N+5)$ unknowns

$$(\delta_{-2},\delta_{-1},\delta_0,\delta_1,\delta_2,...,\delta_{N-1},\delta_N,\delta_{N+1},\delta_{N+2})^T,$$

To obtain a unique solution to the systems (3.10) and (3.11), four additional constraints are required. These are obtained from the boundary conditions (3.2). Imposition of the boundary conditions enables us to eliminate the parameters δ_{-2}, δ_{-1} and $\delta_{N+1}, \delta_{N+2}$ from the system. In order to eliminate the parameters δ_{-2}, δ_{-1} and $\delta_{N+1}, \delta_{N+2}$ from the system (3.9), we have used the boundary conditions

$$u(x_0, t) = u(x_N, t) = 0,$$
$$u_{xx}(x_0, t) = u_{xx}(x_N, t) = 0,$$

Expanding u in terms of approximate quintic B-spline formula from (2.5) at $x_0 = 0,$ and putting $m = 0$ in (2.5) we get,

$$\begin{aligned} &\delta_2 + 26\delta_1 + 66\delta_0 + 26\delta_{-1} + \delta_{-2} = 0, \\ &\delta_2 + 2\delta_1 - 6\delta_0 + 2\delta_{-1} + \delta_{-2} = 0, \end{aligned} \tag{3.11}$$

then

$$\begin{aligned} &\delta_{-1} = -3\delta_0 - \delta_1, \\ &\delta_{-2} = 12\delta_0 - \delta_2, \end{aligned} \tag{3.12}$$

Similarly at $x_N = 1,$ putting $m = N$ in (2.5) we get,

$$\begin{aligned} &\delta_{N+2} + 26\delta_{N+1} + 66\delta_N + 26\delta_{N-1} + \delta_{N-2} = 0, \\ &\delta_{N+2} + 2\delta_{N+1} - 6\delta_N + 2\delta_{N-1} + \delta_{N-2} = 0, \end{aligned} \tag{3.13}$$

where leads to

$$\begin{aligned} &\delta_{N+1} = -3\delta_N - \delta_{N-1}, \\ &\delta_{N+2} = 12\delta_N - \delta_{N-2}, \end{aligned} \tag{3.14}$$

Eliminating parameters δ_{-2}, δ_{-1} and $\delta_{N+1}, \delta_{N+2}$, the system (3.9) is reduced to a penta-diagonal system of $(N+1)$ linear equations with $(N+1)$ unknowns, given by $AX_{n+1} = \overline{A}X_n$ where

$$X_{n+1} = (\delta_0^{n+1}, \delta_1^{n+1}, \delta_2^{n+1}, ..., \delta_{N-1}^{n+1}, \delta_N^{n+1})^T,$$
$$X_n = (\delta_0^n, \delta_1^n, \delta_2^n, ..., \delta_{N-1}^n, \delta_N^n)^T,$$

where T stands for transpose. The coefficient matrix A is given by

$$A=\begin{pmatrix} 12C_{-2}-3C_{-1}+C_0 & C_1-C_0 & C_2-C_{-2} & 0 & 0 & 0 & 0 \\ C_0-3C_{-2} & C_0-C_{-2} & C_1 & C_2 & 0 & 0 & 0 \\ C_{-2} & C_{-1}0 & C_0 & C_1 & C_2 & 0 & 0 \\ 0 & C_{-2} & C_{-1} & C_0 & C_1 & C_2 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & C_{-2} & C_{-1} & C_0 & C_1 & C_2 \\ 0 & 0 & 0 & C_{-2} & C_{-1} & C_0-C_2 & C_1-3C_2 \\ 0 & 0 & 0 & 0 & C_{-2}-C_2 & C_0-C_1 & C_0-3C_1+12C_2 \end{pmatrix}$$

where C_{-2},C_{-1},C_0,C_1 and C_2 are given in (3.10), and the coefficient matrix $\overline{A}$, is

$$\overline{A}=\begin{pmatrix} 12\overline{C}_{-2}-3\overline{C}_{-1}+\overline{C}_0 & \overline{C}_1-\overline{C}_0 & \overline{C}_2-\overline{C}_{-2} & 0 & 0 & 0 & 0 \\ \overline{C}_0-3\overline{C}_{-2} & \overline{C}_0-\overline{C}_{-2} & \overline{C}_1 & \overline{C}_2 & 0 & 0 & 0 \\ \overline{C}_{-2} & \overline{C}_{-1} & \overline{C}_0 & \overline{C}_1 & \overline{C}_2 & 0 & 0 \\ 0 & \overline{C}_{-2} & \overline{C}_{-1} & \overline{C}_0 & \overline{C}_1 & \overline{C}_2 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \overline{C}_{-2} & \overline{C}_{-1} & \overline{C}_0 & \overline{C}_1 & \overline{C}_2 \\ 0 & 0 & 0 & \overline{C}_{-2} & \overline{C}_{-1} & \overline{C}_0-\overline{C}_2 & \overline{C}_1-3\overline{C}_2 \\ 0 & 0 & 0 & 0 & \overline{C}_{-2}-\overline{C}_2 & \overline{C}_0-\overline{C}_1 & \overline{C}_0-3\overline{C}_1+12\overline{C}_2 \end{pmatrix}$$

where $\overline{C}_{-2},\overline{C}_{-1},\overline{C}_0,\overline{C}_1$ and $\overline{C}_2$ are also given in (3.10). This penta-diagonal system can be solved by a modified form of Thomas algorithm. The time evolution of the approximate solution $U_N(x,t)$ is determined by the time evolution of the vector X_N^n which is found repeatedly by solving the recurrence relation, once the initial vectors X_N^0 have been computed from the initial and boundary conditions.

3.1. The initial state

The initial vector X_N^0 can be determined from the initial condition $u(x,0)=u_0(x)$ which gives $(N+1)$ equation in $(N+5)$ unknowns. For the determination of the unknowns relations at the knot are used the boundary conditions (3.2).

The initial vector is then determined as the solution of the matrix equation $A_N^0 X_N^0 = u_0(x)$, where

$$A_N^0 = \begin{pmatrix} 54 & 60 & 6 & 0 & 0 & 0 & 0 \\ \frac{101}{4} & \frac{135}{2} & \frac{105}{4} & 1 & 0 & 0 & 0 \\ 1 & 26 & 66 & 26 & 1 & 0 & 0 \\ 0 & 1 & 26 & 66 & 26 & 1 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 1 & 26 & 66 & 26 & 1 \\ 0 & 0 & 0 & 1 & \frac{105}{4} & \frac{135}{2} & \frac{101}{4} \\ 0 & 0 & 0 & 0 & 6 & 60 & 54 \end{pmatrix},$$

$$X_N^0 = (\delta_0^0, \delta_1^0, \delta_2^0, \cdots, \delta_{N-1}^0, \delta_N^0)^T,$$

$$u_0(x) = (u_0(x_0), u_0(x_1), \cdots, u_0(x_{N-1}), u_0(x_N))^T,$$

where $u_0(x_i), i = 0,1,2,....N$ can be obtained by initial condition (3.3).

4. STABILITY OF THE PROPOSED SCHEME

The Von–Neumann stability method (Fourier mode method) is used for the stability of scheme developed in the previous section. To apply this method, we have linearized the non-linear term UU_x by considering U as a constant in (3.9), therefore $U_x, U_{xx}, ... = 0.$

Theorem 4.1. *The quintic B-spline method (3.8) for the solution of Rosenau-Burgers equation (3.1) is unconditionally stable.*

Proof. We implement the Von–Neumann stability method (Fourier mode method) in which the growth factor of a typical Fourier mode is defined as $\delta_m^n = \xi^n \exp(i\rho m h)$, where ρ and h are the mode number and element size, respectively, and $i = \sqrt{-1}$. Now substituting δ_m^n into linearized form of (3.9), the formulae (3.9) leads to

$$\xi\{C_2 e^{2i\rho h} + C_1 e^{i\rho h} + C_0 + C_{-1} e^{-i\rho h} + C_{-2} e^{-2i\rho h}\} = \overline{C}_2 e^{2i\rho h} + \overline{C}_1 e^{i\rho h} + \overline{C}_0 + \overline{C}_{-1} e^{-i\rho h} + \overline{C}_{-2} e^{-2i\rho h}, \quad (4.1)$$

Here C_j and $\overline{C}_j$, for $j = -2,-1,0,1,2$ have their predefined definition given in (3.10)-(3.11). Set $X = \frac{120}{h^4}$, $Y = \frac{k}{2} U_x^n$, $Z = \frac{5}{2}\frac{k}{h} U^n$ and $W = \frac{5}{2}\frac{k}{h}$. Simplifying Eq. (4.1), we get

$$\xi = \frac{a - ib_1}{a + ib_2}, \quad (4.2)$$

where

$$a = (2+2X)\cos(2\rho h) + (52-8X)\cos(\rho h) + 66+6X,$$
$$b_1 = 2W\sin(2\rho h) + 20W\sin(\rho h), \tag{4.3}$$
$$b_2 = 2(Z+W)\sin(2\rho h) + 20(Z+W)\sin(\rho h),$$

From (4.3), we get

$$b_2 = b_1 + 2Z\sin(2\rho h) + 20Z\sin(\rho h),$$

therefore $a^2 + b_1^2 \le a^2 + b_2^2$. This implies $\|\xi\| \le 1$, which is the condition for scheme to be unconditionally stable.

5. NUMERICAL COMPUTATIONS

Consider the following initial-boundary problem of Rosenau–Burgers equation [17]

$$(u + u_{xxxx})_t + uu_x + u_x - u_{xx} = 0, \qquad (x,t) \in [0,1]\times(0,T], \tag{5.1}$$

with the boundary conditions

$$\begin{aligned} u(0,t) = u(1,t) = 0, \\ u_{xx}(0,t) = u_{xx}(1,t) = 0, \end{aligned} \qquad t \in [0,T], \tag{5.2}$$

and initial condition

$$u(x,0) = x^4(1-x^4), \qquad x \in [0,1], \tag{5.3}$$

We divide the domain $[0,1]$ into $N_i = 5,10,20,40,80$ intervals with each of equal intervals h_i, respect to $k = \frac{1}{20}$, , where $h_i = \frac{1}{N_i}$, for $i = 1,2,3,4,5.$

Since we do not know the exact solution of (5.1)-(5.3), a comparison between the numerical solutions on a coarse mesh and those on a refine mesh is made [13]. Since the numerical solution U_{h_i} of Quintic B-spline collocation method (2.5) is zero at boundaries $x = 0,1$, we can compute ratios of convergence at each time step n, by the following relation

$$R_h^n = \frac{\left\| U_h^n - U_{\frac{h}{2}}^n \right\| + \left\| \Delta_h \left(U_h^n - U_{\frac{h}{2}}^n\right) \right\|}{\left\| U_{\frac{h}{2}}^n - U_{\frac{h}{4}}^n \right\| + \left\| \Delta_h \left(U_{\frac{h}{2}}^n - U_{\frac{h}{4}}^n\right) \right\|},$$

where $\Delta_h v_i^n = \dfrac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{h^2}$. The maximum time step size used in all calculations is $k = \dfrac{1}{20}$. The average ratio of convergence R_h^n, based on both infinite norm and L^2-norm is

$R_{av} = \dfrac{1}{M}\sum_{n=1}^{M} R_h^n$, on $0 \le x \le 1$ and $0 \le t \le 1$ are given in Tables 1 and 2. Here U_h^n is a numerical solution of (5.1) at $t_n = nk$ with step size h, which shown in the Figures 1, 2, 3 and Figure 4.

Table 1. The ratios of convergence R_h^n, based on infinite norm when $k = \dfrac{1}{20}$.

		R_h^n			
n	t	$h = \frac{1}{5}$	$h = \frac{1}{10}$	$h = \frac{1}{20}$	$h = \frac{1}{40}$
2	0.1	5.1154	4.1669	4.0346	4.0079
4	0.2	5.1226	4.1668	4.0343	4.0078
6	0.3	5.1299	4.1667	4.0340	4.0076
8	0.4	5.1372	4.1665	4.0336	4.0074
10	0.5	5.1446	4.1664	4.0333	4.0072
12	0.6	5.1520	4.1663	4.0330	4.0070
14	0.7	5.1594	4.1662	4.0327	4.0068
16	0.8	5.1669	4.1661	4.0324	4.0066
18	0.9	5.1743	4.1660	4.0321	4.0064
20	1	5.1818	4.1659	4.0318	4.0062
R_{av}		5.1484	4.1664	4.0332	4.0071

Table 2. The ratios of convergence R_h^n, based on L^2-norm when $k = \dfrac{1}{20}$.

		R_h^n			
n	t	$h = \frac{1}{5}$	$h = \frac{1}{10}$	$h = \frac{1}{20}$	$h = \frac{1}{40}$
2	0.1	4.9909	4.1757	4.0393	4.0094
4	0.2	4.9939	4.1752	4.0388	4.0091
6	0.3	4.9969	4.1747	4.0384	4.0088
8	0.4	4.9999	4.1742	4.0379	4.0085
10	0.5	5.0025	4.1737	4.0374	4.0082
12	0.6	5.0057	4.1732	4.0369	4.0079
14	0.7	5.0087	4.1728	4.0365	4.0076
16	0.8	5.0116	4.1723	4.0360	4.0073
18	0.9	5.0145	4.1718	4.0356	4.0070
20	1	5.0174	4.1714	4.0351	4.0067
R_{av}		5.0042	4.1735	4.0372	4.0081

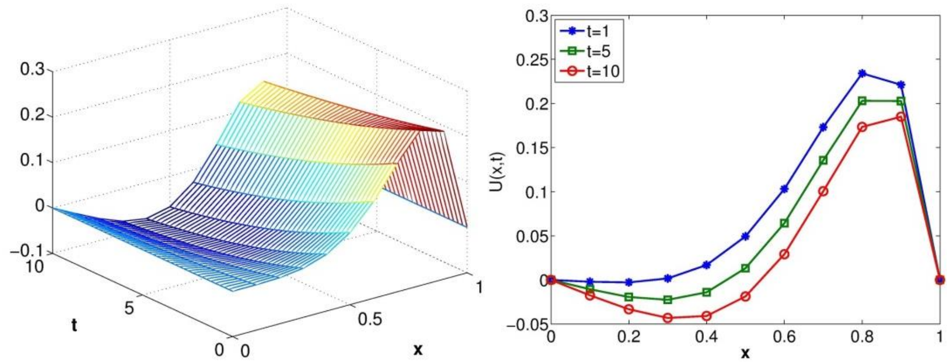

Figure 1. The approximation solution (left) and its concentration (right) of solution $U(x,t)$, for $h=\frac{1}{5}$ and $k=\frac{1}{20}$ plotted as a function of $x=0:h:1$ and $t=0:k:10$.

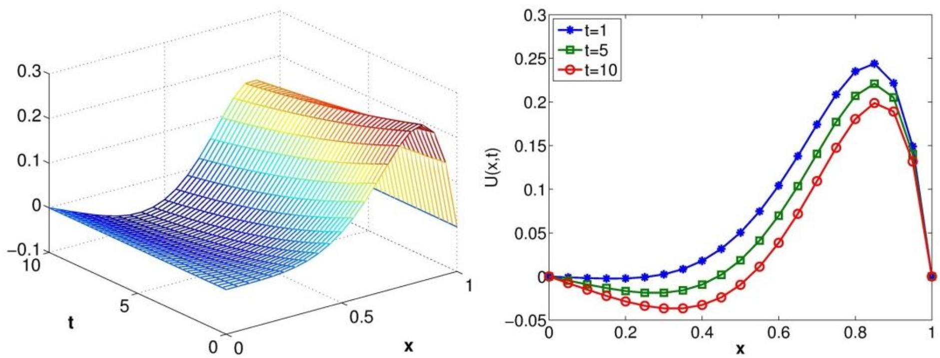

Figure 2. The approximation solution (left) and its concentration (right) of solution $U(x,t)$, for $h=\frac{1}{10}$ and $k=\frac{1}{20}$ plotted as a function of $x=0:h:1$ and $t=0:k:10$.

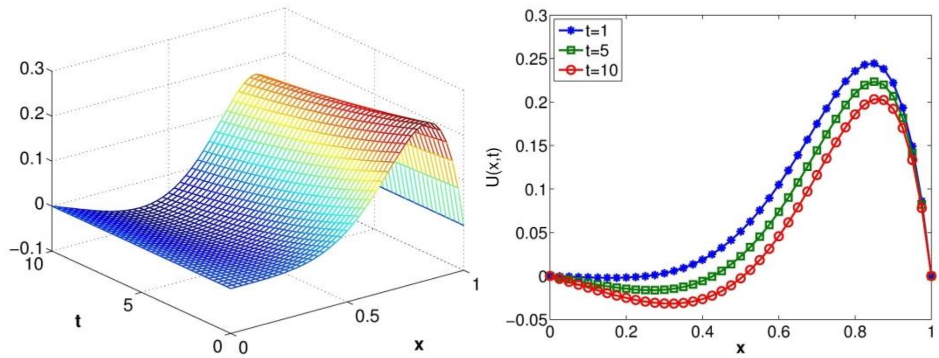

Figure 3. The approximation solution (left) and its concentration (right) of solution $U(x,t)$, for $h=\frac{1}{20}$ and $k=\frac{1}{20}$ plotted as a function of $x=0:h:1$ and $t=0:k:10$.

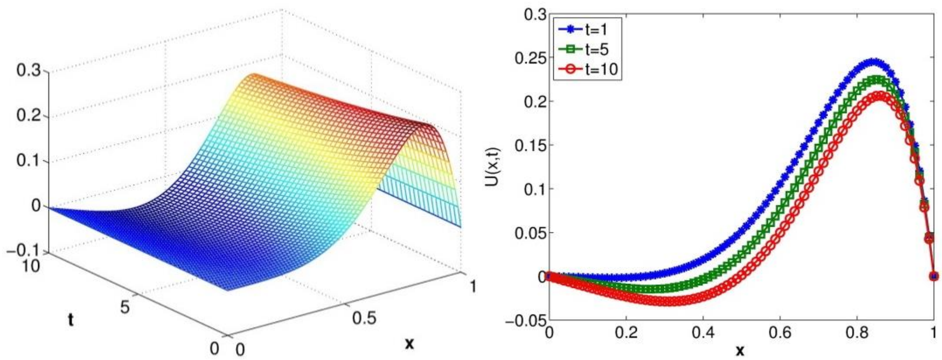

Figure 4. The approximation solution (left) and its concentration (right) of solution $U(x,t)$, for $h=\frac{1}{40}$ and $k=\frac{1}{20}$ plotted as a function of $x=0:h:1$ and $t=0:k:10$.

6. CONCLUSIONS

In this paper, a numerical algorithm for the nonlinear Rosenau–Burgers equation is proposed using a collocation method with the quintic B–spline functions. This scheme is based on the Crank–Nicolson formulation for time integration and quintic B–spline functions for space integration. By the application point of view the quintic B–spline method considered in this work is simple and straight forward. The algorithm described above works for a large class of linear and nonlinear problems. The solution obtained is presented graphically at various time steps which show the same characteristics as given in the literature. Since we do not know the exact solution of the nonlinear (KdV–like) Rosenau–Burgers equation, a comparison between the numerical solutions on a coarse mesh and those on a refine mesh is made. The ratios of convergence R_h^n, based on infinite norm and L^2-norm, mentioned in the Tables 1 and 2, show that the simulating results are in excellent agreement with the analytical solutions.

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